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Lipschitz estimates for systems with ellipticity conditions at infinity

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Abstract In the general vector-valued case $N \geq 1$, we prove the Lipschitz continuity of local minimizers to some integrals of the calculus of variations of the form $\int_\Omega g(x, |Du|)\,\mathrm{d}x$, with p, q-growth conditions only for $|Du| \to +\infty$ and without further structure conditions on the integrand $g = g(x, |Du|)$. We apply the regularity results to weak solutions to nonlinear elliptic systems of the form $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha (x, Du) = 0$, $\alpha = 1, 2, \ldots, N$.

Keywords Elliptic systems · Local minimizers · Local Lipschitz continuity · p, q-growth · Variable exponents

Mathematics Subject Classification Primary 35J60 · 35B65 · 49N60; Secondary 35J70 · 35B45

1 Introduction

Aim of this paper is to prove the local Lipschitz continuity for solutions to elliptic systems of the form

$$\operatorname{div} A(x, Du) = 0, \tag{1.1}$$

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where A is the $N \times n$ matrix $A = (a_i^\alpha(x,\xi))_{N\times n}$, with $\alpha = 1, 2, \ldots, N$, $i = 1, 2, \ldots, n$ and $a_i^\alpha : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$. Here, Ω is a bounded open subset of $\mathbb{R}^n$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ ($N \geq 1$) is a map in a Sobolev space, and Du denotes the gradient of u. Our approach is of variational nature, i.e., the solution is achieved through a minimization property. In fact, we assume that there exists a function $f = f(x,\xi)$ such that $a_i^\alpha(x,\xi) = f_{\xi_i^\alpha}(x,\xi)$.

In contrast with the scalar case, solutions to elliptic systems with general structure may lack regularity, see [11,22]. On the contrary, everywhere regularity has been obtained for systems with Uhlenbeck structure [40]; therefore, it is natural to consider the case $f(x,\xi) = g(x,|\xi|)$. Under p, q-growth conditions, we obtain the local Lipschitz continuity of minimizers; this also gives the existence of weak solutions for the associated Dirichlet problem to the elliptic system (1.1).

Motivated by applications to the theory of elasticity for strongly anisotropic materials (see Zhikov [41] and also Zhikov et al. [42]), in recent years the integral of the calculus of variations

$$\int_\Omega \left(|Du|^p + |x|^\alpha |Du|^q\right) \,\mathrm{d}x \tag{1.2}$$

with $1 < p < q$ and $\alpha \in (0,1)$ has been investigated from the point of view of the regularity of local minimizers. The condition $q \neq p$ may produce not smooth minimizers. Recently Colombo and Mingione ([6,7], see also [2]) proved the regularity of minimizers for integrals of the type (1.2) when, more generally, $|x|^\alpha$ is replaced by a function $a = a(x)$ which is Hölder continuous with exponent α and p, q are related to α by the inequality

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{1.3}$$

The condition (1.3) is considered sharp in view of some examples given in [15,17].

A natural question now arises: is it necessary to assume the *precise structure condition* for the integrand as in (1.2)? That is, is it possible to investigate more general integrals of the calculus of variations of the type

$$F(u) = \int_\Omega g(x,|Du|)\,\mathrm{d}x \tag{1.4}$$

where

$$g(x,|Du|) = |Du|^p + a(x)\,|Du|^q \tag{1.5}$$

is a model example?

The structure $f(x, Du) = g(x,|Du|)$ is necessary to treat the general case $N \geq 1$, since in the vectorial framework minimizers can be unbounded even when $p = q$ [11,38]. In this paper, we study the general case (1.4), by assuming an higher Sobolev summability of g with respect to x instead of the precise structure as in (1.5). We also require uniform convexity and growth conditions on $g = g(x,t)$ only *for large values of* t.

Precisely, let $g : \Omega \times [0,+\infty) \to [0,+\infty)$ be a convex function with respect to the t-variable such that for some $t_0 > 0$, g_{tt} and g_{tx} are Carathéodory functions in $\Omega \times [t_0,+\infty)$ and, for $t \geq t_0$ and a.e. $x \in \Omega$,

$$\lambda\, t^{p-2} \leq g_{tt}(x,t) \leq \Lambda\, \left(1 + t^{q-2}\right), \tag{1.6}$$

$$|g_{tx}(x,t)| \leq h(x)\, \left(1 + t^{q-1}\right), \tag{1.7}$$

where $h \in L^r(\Omega)$ for some $r > n$. Then, $g_t(\cdot,t) \in W^{1,r}(\Omega)$ for every $t \geq t_0$. We also assume that $g(x,t_0)$ and $g_t(x,t_0)$ are bounded functions.

As we stated above, assumptions (1.6) and (1.7) hold only for large values of the gradient variable. This is a relevant point of view, which allows us to consider *uniform convexity only at infinity*. In the mathematical literature, this fact was first pointed out by the pioneristic paper by Chipot and Evans [4], who considered the general vector-valued case $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and integrands satisfying some convexity conditions as $|Du| \to +\infty$. We also refer to [9,12,16,18,21,35,37] for convexity and growth conditions at infinity. The Sobolev dependence on x recently has been considered in [33].

In the general vector-valued context $N \geq 1$, we prove that the local minimizers of the integral in (1.4) are locally Lipschitz continuous if the exponents p, q, with $q \geq p > 1$, are close to each other, precisely if (note that $r > n$)

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{1.8}$$

In the model case (1.5) with $g(x,t) = t^p + a(x)\, t^q$ and $a \in W^{1,r}(\Omega)$, the inequality (1.8) gives back (1.3): $q/p < 1 + \alpha/n$ by the Sobolev embedding theorem, since $a \in C^{0,\alpha}(\Omega)$ with exponent

$$\alpha = 1 - \frac{n}{r}.$$

Note that this framework $g(x,t) = t^p + a(x)\, t^q$ with $a \in W^{1,r}(\Omega)$ holds for the model functional (1.2). The precise statement of our regularity theorem is given in Theorem 2.1.

The local boundedness of the gradient Du is a fundamental property; in fact in this case, the behavior of $|Du|$ at infinity becomes irrelevant for further regularity. With conditions on $g(x,t)$ near $t = 0$ we get $C^{1,\beta}$ regularity for some $0 < \beta < 1$, see Corollary 2.4.

The mathematical literature on the regularity under p, q growth is very large; it is now well known that a restriction between p and q must be imposed since the counterexamples in [20,24–26]; we refer to [32] for a complete survey on the subject. For similar results, we refer to [26–31] and more recently [1,3,10]. A new impulse to the subject has been given by the recent articles already cited [6,7]. Everywhere Lipschitz continuity up to the boundary for either the Dirichlet or the Neumann problem has been recently considered by Cianchi and Maz'ya [5] under uniformly elliptic conditions.

As previously noted, the special structure $g = g(x, |Du|)$ is necessary to treat the general case $N \geq 1$ even if $p = q$, see [29,30,40]. Moreover, we stress that the non-autonomous case still contains many issues not fully solved and that the x-dependence increases significantly the difficulties in the proof.

We also consider here a more general context, in particular functionals with variable exponent of the form

$$\int_\Omega a(x)\, |Du|^{p(x)}\, \mathrm{d}x, \tag{1.9}$$

which are studied by Rajagopal and Růžička [34] and Růžička [36] (see also [31]), in the context of special fluids called *electrorheological*; the associated system takes the form

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(m(x)\, |Du|^{p(x)-2}\, u^\alpha_{x_i} \right) = 0, \quad \alpha = 1, 2, \ldots, N,$$

with $m(x) = a(x)\, p(x)$. More generally, our regularity results apply to the model example

$$\int_\Omega a(x)|Du|^{p(x)} + b(x)|Du|^{q(x)}\, \mathrm{d}x, \tag{1.10}$$

with $p(x)$ and $q(x)$ related by the inequality

$$\frac{q(x)}{p(x)} < 1 + \frac{1}{n} - \frac{1}{r},$$

with $p, q, a, b \in W^{1,r}(\Omega)$ for $r > n$ and $a(x) \geq C > 0$, $b(x) \geq 0$, $q(x) \geq p(x) \geq p > 1$ for $x \in \Omega$.

Let us discuss briefly the techniques to the regularity. First for smooth minimizers of (1.4), we prove an *a priori estimate* for the L^∞-norm of the gradient. Then, we construct a sequence of functions $g^{k\ell}$, and for u local minimizer of (1.4) and in a ball $B_R = B(x_0, R) \subset\subset \Omega$, we consider the sequence of variational problems

$$\inf \left\{ \int_{B_R} g^{k\ell}(x, |Dv|)\, \mathrm{d}x, \quad v \in u + W_0^{1,p}\left(B_R; \mathbb{R}^N\right) \right\}. \tag{1.11}$$

By applying the a priori estimate to the solutions $v^{k\ell}$ to (1.11), we obtain an L^∞ bound on $Dv^{k\ell}$ independent of k, ℓ. As $k, \ell \to +\infty$, the uniform convexity of g for $t \geq t_0$ allow us to transfer the Lipschitz continuity property to the minimizer u.

The plan of the paper is briefly described. In Sect. 2 we give the precise assumptions and the statement of the main results. Section 3 is devoted to the a priori estimate. In Sect. 4 we construct the suitable double approximation and in Sects. 5 and 6 we complete the proof of the main results. Finally in Sect. 6 we transfer the regularity results, obtained for minimizers, to weak solutions to systems.

2 Assumptions and statement of the main results

Let Ω be an open bounded subset of $\mathbb{R}^n$, for $n \geq 2$. Let $u : \Omega \to \mathbb{R}^N$ $(N \geq 1)$, $u \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and consider the following functional of the calculus of variations

$$F(u) = \int_\Omega f(x, Du)\, \mathrm{d}x. \tag{2.1}$$

We say that u is a *local minimizer* of F in (2.1) if $f(x, Du) \in L^1(\Omega)$ and

$$\int_{\operatorname{supp} w} f(x, Du)\, \mathrm{d}x \leq \int_{\operatorname{supp} w} f(x, Du + Dw)\, \mathrm{d}x, \tag{2.2}$$

for $w \in W^{1,1}(\Omega; \mathbb{R}^N)$ with $\operatorname{supp} w \subset\subset \Omega$. We assume that $f : \Omega \times \mathbb{R}^{Nn} \to [0, +\infty)$ and its derivatives $f_{\xi\xi}$, $f_{\xi x}$ are Carathéodory functions in $\Omega \times [t_0, +\infty)$ and f is represented in the form $f(x, \xi) = g(x, |\xi|)$ for a given function $g : \Omega \times [0, +\infty) \to [0, +\infty)$. Moreover, there exist positive constants t_0, λ, Λ such that for all $\mu, \xi \in \mathbb{R}^{Nn}$, $\mu = \mu_i^\alpha, \xi = \xi_i^\alpha$, $i = 1, 2, \ldots, n$, $\alpha = 1, 2, \ldots N$, for $|\xi| \geq t_0$ and a.e. $x \in \Omega$

$$\lambda\, |\xi|^{p-2} |\mu|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, \xi) \mu_i^\alpha \mu_j^\beta, \tag{2.3}$$

$$|f_{\xi_i^\alpha \xi_j^\beta}(x, \xi)| \leq \Lambda\, |\xi|^{q-2}, \tag{2.4}$$

$$|f_{\xi x}(x, \xi)| \leq h(x)\, |\xi|^{q-1}, \tag{2.5}$$

for some exponents $1 < p \leq q$ and $h \in L^r(\Omega)$ for some $r > n$. We also assume $g(x, 0) = 0$. Throughout the paper, we will denote by B_ρ and B_R balls of radii, respectively, ρ and R (with $\rho < R$) compactly contained in Ω and with the same center, let us say, $x_0 \in \Omega$.

Theorem 2.1 *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the integral functional (2.1), whose integrand f satisfies (2.3), (2.4), (2.5) with exponents p, q fulfilling*

$$\frac{q}{p} < 1 + \frac{\alpha}{n} \quad with \quad \frac{\alpha}{n} = \frac{1}{n} - \frac{1}{r}. \tag{2.6}$$

Then, u is locally Lipschitz continuous, and for all $0 < \rho < R$, the following estimate holds

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le C\left(\left(1 + \|h\|^2_{L^r(\Omega)}\right)^{\frac{n}{2\alpha}} \int_{B_R} \{1 + f(x, Du)\}\, dx\right)^{\beta}, \tag{2.7}$$

with $C \equiv C(n, r, p, q, \lambda, \Lambda, R, \rho)$ and $\beta \equiv \beta(n, p, q, \lambda, \Lambda, R, \rho)$.

As a consequence of Theorem 2.1, under the stated assumptions, *the Lavrentiev phenomenon* for the integral functional (2.1) cannot occur.

A further relevant consequence is the following regularity result for weak solutions to elliptic systems. In order to state it, we consider a nonlinear elliptic system of PDEs of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha(x, Du) = 0, \qquad \alpha = 1, 2, \ldots, N, \tag{2.8}$$

where $a_i^\alpha(x,\xi) = f_{\xi_i^\alpha}(x,\xi)$ and $f(x,\xi) = g(x, |\xi|)$. Under the assumptions (2.3), (2.4), (2.5), a solution *in the sense of distributions* to the elliptic system (2.8) is a map $u \in W^{1,p}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^N\right)$ such that

$$\int_\Omega \sum_{i=1}^{n} a_i^\alpha(x, Du) \frac{\partial \varphi^\alpha}{\partial x_i} = 0, \qquad \alpha = 1, 2, \ldots, N, \tag{2.9}$$

for every $\varphi = (\varphi^\alpha)_{\alpha=1,2,\ldots,N} \in C_0^1\left(\Omega;\mathbb{R}^N\right)$. Note that, in general, for differential problems under *p, q-growth conditions* (if p, q are not close enough, precisely, if (2.6) is not satisfied), the notion of solution to the elliptic system (2.8) *in the sense of distributions* may differ from the notion of *weak solution*, the difference being in the class of the allowed test functions φ, which in this second case is $W^{1,q}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^N\right)$ (as necessary to treat variations). That is, a *weak solution* to the elliptic (2.8) is a map $u \in W^{1,q}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^N\right)$ which satisfies the integral condition (2.9) for every test function $\varphi = (\varphi^\alpha)_{\alpha=1,2,\ldots,N} \in W^{1,q}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^N\right)$. By Theorem 2.1, we have

Corollary 2.2 *Every weak solution to the system (2.8) is locally Lipschitz continuous in Ω.*

In general, the elliptic system (2.8) may even lack a weak solution. Nevertheless, under the assumptions (2.3), (2.4), (2.5), the associated Dirichlet problem can be solved and the two notions of weak solution and solution in the sense of distributions turn out to be equivalent. We have in fact the following regularity results for systems. We consider below a Dirichlet problem, but a similar result could be stated for Neumann conditions, or for more general variational boundary value problems.

Corollary 2.3 *Let $u_0 \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$ be a map such that $\int_\Omega f(x, Du_0)\, dx < +\infty$, with f satisfying the assumptions of Theorem 2.1. Then, the Dirichlet problem*

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha(x, Du) = 0 \quad \alpha = 1, 2, \ldots, N, & in\ \Omega, \\ u = u_0 & on\ \partial\Omega, \end{cases} \tag{2.10}$$

has a weak solution $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right) \cap W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *. Moreover,* $u \in W^{1,\infty}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^N\right)$*, that is u is locally Lipschitz continuous in* Ω*.*

Corollary 2.4 *Let* $f \in C^2\left(\Omega \times \mathbb{R}^{Nn}\right)$ *with* $f(x,\xi) = g(x,|\xi|)$ *satisfying the assumptions of Theorem 2.1. Assume that there exist two positive constants* m, M *such that for* $0 < t \le 1$ *and a.e.* $x \in \Omega$

$$m\left(\mu^2+t^2\right)^{\frac{p-2}{2}} \le \frac{g_t(x,t)}{t} \le M\left(\mu^2+t^2\right)^{\frac{p-2}{2}}, \tag{2.11}$$

$$m\left(\mu^2+t^2\right)^{\frac{p-2}{2}} \le g_{tt}(x,t) \le M\left(\mu^2+t^2\right)^{\frac{p-2}{2}}, \tag{2.12}$$

$$|g_{tx}(x,t)| \le M\left(\mu^2+t^2\right)^{\frac{p-1}{2}}, \tag{2.13}$$

for some $\mu \in [0,1]$*. Then, every weak solution* $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ *to (2.8) is of class* $C^{1,\beta}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$*, for some* $0 < \beta < 1$*.*

Further regularity of solutions to nonlinear elliptic systems with continuous coefficients applies when we know that the gradient Du is locally in $C^{0,\beta}$ for some $0 < \beta < 1$. Indeed we state the following result.

Corollary 2.5 *Assume that* $f \in C^{k-1,\beta}\left(\Omega \times \mathbb{R}^{Nn}\right)$ *with* $f(x,\xi) = g(x,|\xi|)$ *for some* $k \ge 2$ *and* $g_{tt}(x,t) \ge m > 0$ *for a.e.* $x \in \Omega$*, for all* $t > 0$*. Then, every weak solution to elliptic system (2.8) is of class* $C^{k,\beta}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$*.*

Finally, we would like to focus on the fact that our assumptions allow us to consider a class of integrals of the calculus of variations with variable exponent, which can be typified by the model integral

$$I(u) = \int_\Omega a(x)\,|Du|^{p(x)}\,\mathrm{d}x. \tag{2.14}$$

Theorem 2.6 *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *be a local minimizer of the integral functional (2.14) with* $a(x), p(x)$ *satisfying*

$$a(x) \ge a > 0, \qquad p(x) \ge p > 1, \qquad a, p \in W^{1,r}(\Omega), \quad \text{with } r > n. \tag{2.15}$$

Then, u is locally Lipschitz continuous in Ω*.*

The model integral in (2.14) has been already studied by Coscia and Mingione in [8], where the Hölder continuity of the exponent $p(x)$ is assumed. However, we emphasis that the integral in (2.14), in our context, is just a model example and our techniques permit to consider more general integrands as in (1.10).

The Lipschitz regularity for the case $f(x, Du) = a(x)h(|Du|)^{p(x)}$ is considered by the authors in [13].

3 A priori estimates

Let u be a local minimizer of functional (2.1) under the assumptions (2.3), (2.4), (2.5) for a given $t_0 > 0$. We can transform $f(x,\xi)$ into $f(x,t_0\xi)$, which satisfies the same assumptions (2.3), (2.4), (2.5) for $|\xi| \ge 1$ (with different constants depending on t_0). Then, it is sufficient to obtain the a priori bound and the regularity results for $v = \frac{1}{t_0}u$. Therefore, for clarity of exposition and without loss of generality, we can assume $t_0 = 1$.

In this section, we make some supplementary assumptions on f.

Assumption 3.1 Assume that $f \in C^2(\Omega \times \mathbb{R}^{Nn})$ and there exist two positive constants k and K such that for $\xi \in \mathbb{R}^{Nn}$ and a.e. $x \in \Omega$

$$k\left(1+|\xi|^2\right)^{\frac{p-2}{2}}|\mu|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\mu_i^\alpha \mu_j^\beta, \tag{3.1}$$

$$|f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)| \leq K\left(1+|\xi|^2\right)^{\frac{p-2}{2}}, \tag{3.2}$$

$$|f_{\xi x}(x,\xi)| \leq K\left(1+|\xi|^2\right)^{\frac{p-1}{2}}, \tag{3.3}$$

In the next proposition, we obtain an a priori estimate for the L^∞-norm of the gradient of u, which is independent of k and K.

Proposition 3.2 *Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a local minimizer of the integral functional* (2.1)*, whose integrand f satisfies Assumption* 3.1 *and* (2.3)*,* (2.4)*,* (2.5)*, with exponents p, q fulfilling* (2.6)*.*

Then, there exist constants $C \equiv C(n,r,p,q,\lambda,\Lambda)$ and $\beta \equiv \beta(n,r,p,q,\lambda,\Lambda)$ such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \leq C\left(\left[\frac{\left(1+\|h\|^2_{L^r(\Omega)}\right)^{\frac{1}{2}}}{(R-\rho)}\right]^{\frac{n}{\alpha}} \int_{B_R}\{1+f(x,Du)\}\,dx\right)^{\beta}. \tag{3.4}$$

Proof Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a local minimizer of (2.1). We observe that by Assumption 3.1, Df has $p-1$ growth; then, u satisfies the Euler's first variation

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\varphi^\alpha_{x_i}(x)\,\mathrm{d}x = 0 \qquad \forall \varphi = (\varphi^\alpha)_{\alpha=1,\ldots,N} \in W_0^{1,p}\left(\Omega;\mathbb{R}^N\right).$$

Since $D^2 f$ has $p-2$ growth, by using the technique of the difference quotients (see for example [14,19,23]), we have that

$$u \in W^{2,\min(2,p)}_{\mathrm{loc}}\left(\Omega;\mathbb{R}^N\right) \quad \text{and} \quad \left(1+|Du|^2\right)^{\frac{p-2}{2}}|D^2u|^2 \in L^1_{\mathrm{loc}}(\Omega). \tag{3.5}$$

and the second variation system holds

$$\int_\Omega \left\{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\varphi^\alpha_{x_i} u^\beta_{x_s x_j} + \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\varphi^\alpha_{x_i}\right\}\mathrm{d}x = 0$$
$$\forall s = 1,\ldots,n, \qquad \forall \varphi = (\varphi^\alpha)_{\alpha=1,\ldots,N} \in W_0^{1,\min(2,p)}\left(\Omega;\mathbb{R}^N\right). \tag{3.6}$$

Let $\eta \in C^1_0(\Omega)$. For any fixed $s \in \{1,\ldots,n\}$, we choose

$$\varphi^\alpha = \eta^2 u^\alpha_{x_s}\Phi\left((|Du|-1)_+\right)$$

for $\Phi : [0,+\infty) \to [0,+\infty)$ increasing, locally Lipschitz continuous function, with Φ and Φ' bounded on $[0,+\infty)$, such that $\Phi(0) = 0$ and

$$\Phi'(s)s \leq c_\Phi\,\Phi(s) \tag{3.7}$$

for a suitable constant $c_\Phi > 0$. Here, $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following, we denote $\Phi((|Du|-1)_+) = \Phi(|Du|-1)_+$. We compute then

$$\varphi^\alpha_{x_i} = 2\eta\eta_{x_i} u^\alpha_{x_s} \Phi\left(|Du|-1\right)_+ + \eta^2 u^\alpha_{x_s x_i} \Phi\left(|Du|-1\right)_+ \\ +\eta^2 u^\alpha_{x_s} \Phi'\left(|Du|-1\right)_+ [(|Du|-1)_+]_{x_i}.$$

Here, we used the fact that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, see Proposition 3.1 of [9] and [39]. Plugging this expression in (3.6) we obtain:

$$\begin{aligned} 0 &= \int_\Omega 2\eta\Phi\left(|Du|-1\right)_+ \sum_{i,j,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}\left(x, Du\right) \eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j}\, \mathrm{d}x \\ &\quad + \int_\Omega \eta^2 \Phi\left(|Du|-1\right)_+ \sum_{i,j,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}\left(x, Du\right) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, \mathrm{d}x \\ &\quad + \int_\Omega \eta^2 \Phi'\left(|Du|-1\right)_+ \sum_{i,j,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}\left(x, Du\right) u^\alpha_{x_s} u^\beta_{x_s x_j} [(|Du|-1)_+]_{x_i}\, \mathrm{d}x \\ &\quad + \int_\Omega 2\eta\Phi\left(|Du|-1\right)_+ \sum_{i,\alpha} f_{\xi^\alpha_i x_s}\left(x, Du\right) \eta_{x_i} u^\alpha_{x_s}\, \mathrm{d}x \\ &\quad + \int_\Omega \eta^2 \Phi\left(|Du|-1\right)_+ \sum_{i,\alpha} f_{\xi^\alpha_i x_s}\left(x, Du\right) u^\alpha_{x_s x_i}\, \mathrm{d}x \\ &\quad + \int_\Omega \eta^2 \Phi'\left(|Du|-1\right)_+ \sum_{i,\alpha} f_{\xi^\alpha_i x_s}\left(x, |Du|\right) u^\alpha_{x_s} [(|Du|-1)_+]_{x_i}\, \mathrm{d}x \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{3.8}$$

In the following, constants will be denoted by C, regardless of their actual value.

We now sum the previous equation with respect to s from 1 to n, and we denote by $\tilde{I}_1 - \tilde{I}_6$ the corresponding integrals.

Let us start with the estimate of the integral $\tilde{I}_1$. By the Cauchy–Schwartz inequality, the Young inequality and (2.4), we have

$$\begin{aligned} |\tilde{I}_1| &= \left| \int_\Omega 2\eta\Phi\left(|Du|-1\right)_+ \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du)\eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j}\, \mathrm{d}x \right| \\ &\le \int_\Omega 2\eta\Phi\left(|Du|-1\right)_+ \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du)\eta_{x_i} u^\alpha_{x_s} \eta_{x_j} u^\beta_{x_s} \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j} \right\}^{\frac{1}{2}} \mathrm{d}x \\ &\le C \int_\Omega |D\eta|^2 \Phi\left(|Du|-1\right)_+ |Du|^q\, \mathrm{d}x \\ &\quad + \frac{1}{2}\int_\Omega \eta^2 \Phi\left(|Du|-1\right)_+ \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, \mathrm{d}x. \end{aligned} \tag{3.9}$$

Let us consider $\tilde{I}_3$. First of all, we have that

$$f_{\xi_i^\alpha \xi_j^\beta}(x,\xi) = \left(\frac{g_{tt}(x,|\xi|)}{|\xi|^2} - \frac{g_t(x,|\xi|)}{|\xi|^3}\right)\xi_i^\alpha \xi_j^\beta + \frac{g_t(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_j^\beta}.$$

At this point,

$$\begin{aligned}
&\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du) u_{x_s}^\alpha u_{x_s x_j}^\beta [(|Du|-1)_+]_{x_i} \\
&\quad = \left(\frac{g_{tt}(x,|Du|)}{|Du|^2} - \frac{g_t(x,|Du|)}{|Du|^3}\right) \sum_{i,j,s,\alpha,\beta} u_{x_s}^\alpha u_{x_s x_j}^\beta u_{x_j}^\beta u_{x_i}^\alpha [(|Du|-1)_+]_{x_i} \\
&\qquad + \frac{g_t(x,|Du|)}{|Du|} \sum_{s,i,\alpha} u_{x_s}^\alpha u_{x_s x_i}^\alpha [(|Du|-1)_+]_{x_i} \\
&\quad = \left(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2}\right) \sum_\alpha \left[\sum_i u_{x_i}^\alpha (|Du|)_{x_i}\right]^2 \\
&\qquad + g_t(x,|Du|)|D(|Du|-1)_+|^2,
\end{aligned} \tag{3.10}$$

where we used the fact that

$$\left[(|Du|-1)_+\right]_{x_i} = (|Du|)_{x_i} = \frac{1}{|Du|}\sum_{\alpha,s} u_{x_i x_s}^\alpha u_{x_s}^\alpha \qquad |Du| \geq 1.$$

Thus, coming back to the estimate of $\tilde{I}_3$ from (3.10), we deduce

$$\begin{aligned}
\tilde{I}_3 = \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \Bigg\{ &\left(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2}\right) \sum_\alpha \left[\sum_i u_{x_i}^\alpha (|Du|)_{x_i}\right]^2 \\
&+ g_t(x,|Du|)|D(|Du|-1)_+|^2 \Bigg\} \, \mathrm{d}x.
\end{aligned}$$

Now we argue as in the proof of Lemma 4.1 of [30]. Using the inequality

$$|D(|Du|-1)_+|^2 \leq |D^2 u|^2, \qquad |Du| \geq 1 \tag{3.11}$$

we conclude that

$$\tilde{I}_3 \geq \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \frac{g_{tt}(x,|Du|)}{|Du|} \sum_\alpha \left(\sum_i u_{x_i}^\alpha [(|Du|-1)_+]_{x_i}\right)^2 \mathrm{d}x \geq 0,$$

where we used the fact that $g_{tt}(x,|Du|) \geq 0$ and $\Phi'(|Du|-1)_+ \geq 0$. By (2.3), we have that

$$\begin{aligned}
\frac{1}{2}\int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2}|D^2u|^2 \, \mathrm{d}x &\leq \frac{1}{2}\tilde{I}_2 + \tilde{I}_3 \\
&\leq |\tilde{I}_4| + |\tilde{I}_5| + |\tilde{I}_6| + C\int_\Omega |D\eta|^2 \Phi(|Du|-1)_+ |Du|^q \, \mathrm{d}x.
\end{aligned}$$

We now deal with $|\tilde{I}_4|$. We have

$$\begin{aligned}
|\tilde{I}_4| &= \left| \int_\Omega 2\eta\Phi\left(|Du|-1\right)_+ \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\eta_{x_i} u^\alpha_{x_s}\,\mathrm{d}x \right| \\
&\overset{(2.5)}{\leq} \int_\Omega 2\eta\Phi\left(|Du|-1\right)_+ h(x)|Du|^{q-1} \sum_{i,s,\alpha} |\eta_{x_i} u^\alpha_{x_s}|\,\mathrm{d}x \\
&\leq \int_\Omega (\eta^2+|D\eta|^2) h(x)\Phi\left(|Du|-1\right)_+ |Du|^q\,\mathrm{d}x.
\end{aligned}$$

Consider $|\tilde{I}_5|$, we have

$$\begin{aligned}
|\tilde{I}_5| &= \left| \int_\Omega \eta^2\Phi\left(|Du|-1\right)_+ \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x,Du) u^\alpha_{x_s x_j}\,\mathrm{d}x \right| \\
&\overset{(2.5)}{\leq} \int_\Omega \eta^2\Phi\left(|Du|-1\right)_+ h(x)|Du|^{q-1}|D^2u|\,\mathrm{d}x \\
&\leq \int_\Omega \left[\eta^2\Phi\left(|Du|-1\right)_+ |Du|^{p-2}|D^2u|^2\right]^{1/2} \left[\eta^2\Phi\left(|Du|-1\right)_+ |h(x)|^2|Du|^{2q-p}\right]^{1/2}\,\mathrm{d}x \\
&\leq \varepsilon\int_\Omega \eta^2\Phi\left(|Du|-1\right)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x + C_\varepsilon \int_\Omega \eta^2\Phi\left(|Du|-1\right)_+ |h(x)|^2|Du|^{2q-p}\,\mathrm{d}x,
\end{aligned}$$

where in the last line we used the Young inequality. Finally, for any $0<\delta<1$

$$\begin{aligned}
|\tilde{I}_6| &= \left| \int_\Omega \eta^2 \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x,Du) u^\alpha_{x_s} \Phi'\left(|Du|-1\right)_+ \left[\left(|Du|-1\right)_+\right]_{x_i}\,\mathrm{d}x \right| \\
&\overset{(2.5)}{\leq} \int_\Omega \eta^2\Phi'\left(|Du|-1\right)_+ h(x)|Du|^{q-1}|Du||D\left(|Du|-1\right)_+|\,\mathrm{d}x \\
&\overset{(3.11)}{\leq} \int_\Omega \eta^2\Phi'\left(|Du|-1\right)_+ h(x)|Du|^q\,|D^2u|\,\mathrm{d}x \\
&= \int_\Omega \eta^2\Phi'\left(|Du|-1\right)_+ h(x)\left[\left(|Du|-1\right)_+ + \delta\right]\left[\left(|Du|-1\right)_+ + \delta\right]^{-1}|Du|^q\,|D^2u|\,\mathrm{d}x \\
&\leq \int_\Omega \eta^2 \left\{\frac{1}{c_\Phi}\Phi'\left(|Du|-1\right)_+\left[\left(|Du|-1\right)_+ + \delta\right]|Du|^{p-2}|D^2u|^2\right\}^{1/2} \\
&\quad \times \left\{c_\Phi\Phi'\left(|Du|-1\right)_+ |h(x)|^2|Du|^{2q-p+2}\left[\left(|Du|-1\right)_+ + \delta\right]^{-1}\right\}^{1/2}\,\mathrm{d}x \\
&\leq \frac{\varepsilon}{c_\Phi}\int_\Omega \eta^2\Phi'\left(|Du|-1\right)_+\left[\left(|Du|-1\right)_+ + \delta\right]|Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\quad + C_\varepsilon\, c_\Phi \int_\Omega \eta^2\Phi'\left(|Du|-1\right)_+ |h(x)|^2|Du|^{2q-p+2}\left[\left(|Du|-1\right)_+ + \delta\right]^{-1}\,\mathrm{d}x.
\end{aligned}$$

We concentrate our attention on the first term in the last inequality. We split the set Ω as $\Omega=\{x:|Du(x)|\geq 2\}\cup\{x:|Du(x)|<2\}$, and we observe that in the set $\{x:|Du(x)|\geq 2\}$, we also have $\left(|Du|-1\right)_+\geq 1$ which in turn implies

$$\left(|Du|-1\right)_+ + \delta \leq 2\left(|Du|-1\right)_+ \tag{3.12}$$

as long as we have chosen $\delta < 1$. Therefore, we have, using (3.7)

$$
\begin{aligned}
&\int_\Omega \eta^2 \Phi'\left(|Du|-1\right)_+ \left[(|Du|-1)_+ + \delta\right] |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&= \int_{|Du|\geq 2} \eta^2 \Phi'\left(|Du|-1\right)_+ \left[(|Du|-1)_+ + \delta\right] |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\quad + \int_{1<|Du|<2} \eta^2 \Phi'\left(|Du|-1\right)_+ \left[(|Du|-1)_+ + \delta\right] |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\overset{(3.12)}{\leq} 2 \int_{|Du|\geq 2} \eta^2 \Phi'\left(|Du|-1\right)_+ (|Du|-1)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\quad + \int_{1<|Du|<2} \eta^2 \Phi'\left(|Du|-1\right)_+ (|Du|-1)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\quad + \delta \int_{1<|Du|<2} \eta^2 \Phi'\left(|Du|-1\right)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\overset{(3.7)}{\leq} 2\,c_\Phi \int_\Omega \eta^2 \Phi\left(|Du|-1\right)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\quad + \delta \int_{1<|Du|<2} \eta^2 \Phi'\left(|Du|-1\right)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x.
\end{aligned}
$$

Now, choosing ε sufficiently small and putting together all the estimates obtained for $|\tilde{I}_4|, |\tilde{I}_5|, |\tilde{I}_6|$, we deduce

$$
\begin{aligned}
&\int_\Omega \eta^2 \Phi\left(|Du|-1\right)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x \\
&\leq C\,c_\Phi \int_\Omega \left(\eta^2 + |D\eta|^2\right)\left(1+h^2(x)\right)|Du|^{2q-p} \\
&\quad \times \left[\Phi\left(|Du|-1\right)_+ |Du|^{2q-p} + \Phi'\left(|Du|-1\right)_+ |Du|^2 \left[(|Du|-1)_+ + \delta\right]^{-1}\right]\mathrm{d}x \\
&\quad + \delta \int_{1<|Du|<2} \eta^2 \Phi'\left(|Du|-1\right)_+ |Du|^{p-2}|D^2u|^2\,\mathrm{d}x, \qquad (3.13)
\end{aligned}
$$

with a constant C depending on n, r, p, q.

Now we define

$$\Phi(s) := (1+s)^{\gamma-2}s^2 \qquad \gamma \geq 0; \tag{3.14}$$

we have

$$\Phi'(s) = (\gamma s + 2)s(1+s)^{\gamma-3}. \tag{3.15}$$

This function satisfies (3.7) with $c_\Phi = 2(1+\gamma)$.
We now approximate this function Φ by a sequence of functions Φ_h, each of them being equal to Φ in the interval $[0, h]$, and then extended to $[h, +\infty)$ with the constant value $\Phi(h)$. Moreover, Φ_h and Φ'_h converge monotonically to Φ and Φ', respectively. The expression of Φ_h can be inserted in (3.13), and then, it is possible to pass to the limit as $h \to +\infty$ by the

Monotone Convergence Theorem. Therefore, we obtain for every $0 < \delta < 1$

$$\int_\Omega \eta^2 \left(1 + (|Du| - 1)_+\right)^{\gamma-2} (|Du| - 1)_+^2 \, |Du|^{p-2} |D^2 u|^2 \,\mathrm{d}x$$
$$\leq C \; (1+\gamma)^2 \int_\Omega \left(\eta^2 + |D\eta|^2\right) \left(1 + h(x)^2\right) \left(1 + (|Du| - 1)_+\right)^{\gamma+2q-p} \,\mathrm{d}x$$
$$+ \delta \, C(\gamma) \int_{1<|Du|<2} \eta^2 |Du|^{p-2} |D^2 u|^2 \,\mathrm{d}x,$$

where we used the fact that

$$\frac{(|Du| - 1)_+}{(|Du| - 1)_+ + \delta} \leq 1 \qquad \forall \, \delta > 0$$

and $\Phi'(t-1)_+ \leq C(\gamma)$ when $1 < t < 2$. Inequality (3.51) of the following Lemma 3.3 and (3.5) imply

$$\int_{1<|Du|<2} \eta^2 |Du|^{p-2} |D^2 u|^2 \,\mathrm{d}x \leq C \int_{1<|Du|<2} \eta^2 \left(1 + |Du|^2\right)^{\frac{p-2}{2}} |D^2 u|^2 \,\mathrm{d}x < +\infty$$

so we can pass to the limit for $\delta \to 0$ and the last term in the previous inequality vanishes.

Since $h \in L^r(\Omega)$, by the Hölder inequality and by denoting

$$m := \left(\frac{r}{2}\right)' = \frac{r}{r-2}, \tag{3.16}$$

we have, using (3.11)

$$\int_\Omega \eta^2 \left(1 + (|Du| - 1)_+\right)^{\gamma-2} (|Du| - 1)_+^2 \, |Du|^{p-2} |D\left((|Du| - 1)_+\right)|^2 \,\mathrm{d}x$$
$$\leq C \, (1+\gamma)^2 \, H \left[\int_\Omega \left(\eta^2 + |D\eta|^2\right)^m \left(1 + (|Du| - 1)_+\right)^{(\gamma+2q-p)m} \,\mathrm{d}x\right]^{\frac{1}{m}}, \tag{3.17}$$

by denoting, from now on

$$H := \left(1 + \|h\|^2_{L^r(\Omega)}\right) \tag{3.18}$$

and where C now depends also on r and $|\Omega|$ (and so on n).

Let us introduce

$$G(t) = 1 + \int_0^t \sqrt{\Phi(s)} (1+s)^{\frac{p-2}{2}} \,\mathrm{d}s = 1 + \int_0^t (1+s)^{\frac{\gamma}{2} + \frac{p}{2} - 2} \, s \,\mathrm{d}s \tag{3.19}$$

and we obtain the following upper bound for $[G(t)]^2$

$$[G(t)]^2 \leq 4 \, (1+t)^{\gamma+p} \leq 4 \, (1+t)^{\gamma+2q-p}, \tag{3.20}$$

where we used the fact that $p \leq q \leq 2q - p$. On the other hand

$$G_t(t) = \sqrt{\Phi(t)} \, (1+t)^{\frac{p-2}{2}} \overset{(3.14)}{=} (1+t)^{\frac{\gamma}{2} + \frac{p}{2} - 2} t \tag{3.21}$$

which in turn allows us to give the following estimate for the gradient of the function $w = \eta G((|Du|-1)_+)$

$$\begin{aligned}&\int_\Omega |D(\eta G((|Du|-1)_+))|^2\,\mathrm{d}x\\&\quad\leq 2\int_\Omega |D\eta|^2|G((|Du|-1)_+)|^2\,\mathrm{d}x\\&\qquad+2\int_\Omega \eta^2[G_t((|Du|-1)_+)]^2[D((|Du|-1)_+)]^2\,\mathrm{d}x\\&\overset{(3.17),(3.20),(3.21)}{\leq} C\,(1+\gamma)^2\,H\left[\int_\Omega \left(\eta^2+|D\eta|^2\right)^m[1+(|Du|-1)_+^{(\gamma+2q-p)m}]\,\mathrm{d}x\right]^{\frac{1}{m}}.\end{aligned} \tag{3.22}$$

Now, let $2^* = \frac{2n}{n-2}$ for $n > 2$, while 2^* equal to any fixed real number greater than 2, if $n = 2$. By Sobolev's inequality there exists a constant C such that

$$\left\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\,\mathrm{d}x\right\}^{\frac{2}{2^*}} \leq C\int_\Omega |D(\eta G((|Du|-1)_+))|^2\,\mathrm{d}x. \tag{3.23}$$

Moreover, since $r > n$, we have

$$1 \leq m \overset{(3.16)}{:=} \frac{r}{r-2} < \frac{n}{n-2} = \frac{2^*}{2}. \tag{3.24}$$

Observe that

$$(2q-p)m = 2(q-p)m + pm; \tag{3.25}$$

moreover, in view of the strict inequality in (2.6), we infer the existence of $0 < \epsilon < 1$ such that

$$(q-p) + \epsilon\left(\frac{1}{n}-\frac{1}{r}\right) \leq p\left(\frac{1}{n}-\frac{1}{r}\right). \tag{3.26}$$

We also set

$$\tilde{M} := 2(q-p)m + p(m-1) + \epsilon \qquad \tilde{N} := p - \epsilon. \tag{3.27}$$

We remark that $\tilde{M} > 0$ because $q \geq p$, $m \geq 1$, $\epsilon > 0$ and $\tilde{N} > 0$ since $\epsilon < 1 < p$; moreover, we observe that

$$\tilde{M} + \tilde{N} = (2q-p)m \tag{3.28}$$

and

$$\tilde{M} > (2q-p)m - p. \tag{3.29}$$

Now we prove that

$$\frac{1}{(\gamma+p)^2}\left[\int_\Omega \eta^{2^*}[1+(|Du|-1)_+]^{\left(\gamma+\frac{\tilde{M}}{m}\right)\frac{2^*}{2}+\tilde{N}}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \leq 4\left(\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\right)^{\frac{2}{2^*}}. \tag{3.30}$$

In view of (3.19), by setting $t := (|Du|-1)_+$, (3.30) is proved if

$$\frac{1}{\gamma+p}(1+t)^{\left(\frac{\gamma}{2}+\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}\right)} \leq 2\left(1+\int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}\,s\,\mathrm{d}s\right). \tag{3.31}$$

Now, if $t \leq 1$, then easily

$$\frac{1}{\gamma+p}(1+t)^{\left(\frac{\gamma}{2}+\frac{\tilde M}{2m}+\frac{\tilde N}{2^*}\right)} \leq \frac{2}{\gamma+p} \leq 2 \leq 2\left(1+\int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}\, s\, \mathrm{d}s\right)$$

so that (3.31) is achieved. Let now $t \geq 1$, then (3.31) becomes, after differentiation

$$\frac{\frac{\gamma}{2}+\frac{\tilde M}{2m}+\frac{\tilde N}{2^*}}{\gamma+p}(1+t)^{\frac{\gamma}{2}+\frac{\tilde M}{2m}+\frac{\tilde N}{2^*}-1} \leq 2(1+t)^{\frac{\gamma}{2}+\frac{p}{2}-2} t. \tag{3.32}$$

If we are able to show that

$$\frac{\tilde M}{2m}+\frac{\tilde N}{2^*} \leq \frac{p}{2}, \tag{3.33}$$

then we would have

$$\frac{\frac{\gamma}{2}+\frac{\tilde M}{2m}+\frac{\tilde N}{2^*}}{\gamma+p}(1+t)^{\frac{\tilde M}{2m}+\frac{\tilde N}{2^*}} \leq \frac{1}{2}(1+t)^{\frac{p}{2}} = \frac{1}{2}(1+t)^{\frac{p-2}{2}}(1+t) \overset{t\geq 1}{\leq} (1+t)^{\frac{p-2}{2}} t$$

and so also (3.32) is satisfied.

Thus, all is reduced to prove (3.33), which is equivalent to

$$(q-p)+\frac{p}{2}-\frac{p}{2m}+\frac{\epsilon}{2m}+\frac{p}{2^*}-\frac{\epsilon}{2^*} \leq \frac{p}{2}.$$

Since

$$\frac{1}{2m}-\frac{1}{2^*} = \frac{r-2}{2r}-\frac{n-2}{2n} = \frac{n(r-2)-r(n-2)}{2nr} = \frac{r-n}{nr} = \frac{1}{n}-\frac{1}{r} = \frac{\alpha}{n},$$

then, by (3.26), the claim (3.30) is satisfied.

By collecting (3.22), (3.23) and (3.30), we obtain

$$\begin{aligned}
&\left[\int_\Omega \eta^{2^*}[1+(|Du|-1)_+]^{\left(\gamma+\frac{\tilde M}{m}\right)\frac{2^*}{2}}\left[1+(|Du|-1)_+\right]^{\tilde N}\,\mathrm{d}x\right]^{\frac{2}{2^*}}\\
&\quad\leq C\,H\,(\gamma+2q-p)^4\left[\int_\Omega \left(\eta^2+|D\eta|^2\right)^m\left[1+(|Du|-1)_+\right]^{(\gamma+2q-p)m}\,\mathrm{d}x\right]^{\frac{1}{m}},
\end{aligned} \tag{3.34}$$

where the constant C only depends on $n, r, p, q, \lambda, \Lambda$ but is independent of γ.

Now, let η to be equal to 1 in B_ρ, with $\operatorname{supp}\eta \subset B_R$ and such that $|D\eta| \leq \frac{1}{(R-\rho)}$. Let us denote by

$$\kappa := \gamma m + \tilde M \overset{(3.28)}{=} (\gamma+2q-p)m - \tilde N.$$

We notice that $\kappa \geq \tilde M$ since $\gamma \geq 0$; moreover, $\frac{2^*}{2m} > 1$ due to (3.24). Therefore, from (3.34) we now have

$$\begin{aligned}
&\left\{\int_{B_\rho}\left[1+(|Du|-1)_+\right]^{\kappa\frac{2^*}{2m}}\left[1+(|Du|-1)_+\right]^{\tilde N}\,\mathrm{d}x\right\}^{\frac{2m}{2^*}}\\
&\quad\leq C\,H^m\left(\frac{(\kappa+\tilde N)^2}{R-\rho}\right)^{2m}\int_{B_R}\left[1+(|Du|-1)_+\right]^{\kappa}\left[1+(|Du|-1)_+\right]^{\tilde N}\,\mathrm{d}x,
\end{aligned} \tag{3.35}$$

where the constant C only depends on $n, r, p, q, \lambda, \Lambda$.

Fixed $\bar{R}$ and $\bar{\rho}$, with $\bar{R} > \bar{\rho}$, we define the decreasing sequence of radii $\{\rho_i\}_{i\geq 0}$

$$\rho_i = \bar{\rho} + \frac{\bar{R}-\bar{\rho}}{2^i} \qquad \forall i \geq 0.$$

We observe that $\rho_0 = \bar{R} > \rho_i > \rho_{i+1} > \bar{\rho}$. We also define the increasing sequence of exponents $\{\kappa_i\}_{i\geq 0}$ such that

$$\kappa_0 := \tilde{M} \qquad \kappa_{i+1} = \kappa_i \frac{2^*}{2m} \qquad i \geq 0.$$

We notice that $\kappa_0 > 0$ because $\tilde{M} > 0$. We rewrite (3.35) with $R = \rho_i$, $\rho = \rho_{i+1}$, $\kappa = \kappa_i$; then, after observing that

$$R - \rho := \rho_i - \rho_{i+1} = \frac{\bar{R}-\bar{\rho}}{2^{i+1}}$$

we obtain for every $i \geq 0$

$$\begin{aligned}
&\left\{\int_{B_{\rho_{i+1}}} [1+(|Du|-1)_+]^{\kappa_{i+1}} [1+(|Du|-1)_+]^{\tilde{N}} \, \mathrm{d}x\right\}^{\frac{1}{\kappa_{i+1}}} \\
&\quad \leq \left[C\, H^m \left(\frac{(\kappa_i+\tilde{N})^{\frac{3}{2}} 2^{i+1}}{\bar{R}-\bar{\rho}}\right)^{2m}\right]^{\frac{1}{\kappa_i}} \left(\int_{B_{\rho_i}} [1+(|Du|-1)_+]^{\kappa_i} [1+(|Du|-1)_+]^{\tilde{N}} \, \mathrm{d}x\right)^{\frac{1}{\kappa_i}}.
\end{aligned}$$

The last inequality can be rewritten as

$$A_{i+1} \leq C_i\, A_i \tag{3.36}$$

having set

$$\begin{aligned}
A_i &:= \left(\int_{B_{\rho_i}} [1+(|Du|-1)_+]^{\kappa_i} [1+(|Du|-1)_+]^{\tilde{N}} \, \mathrm{d}x\right)^{\frac{1}{\kappa_i}} \\
C_i &:= \left[C\, H^m \left(\frac{(\kappa_i+\tilde{N})^{\frac{3}{2}} 2^{i+1}}{\bar{R}-\bar{\rho}}\right)^{2m}\right]^{\frac{1}{\kappa_i}}.
\end{aligned}$$

By iteration of (3.36), we deduce

$$\begin{aligned}
&\left\{\int_{B_{\bar{\rho}}} [1+(|Du|-1)_+]^{\kappa_0\left(\frac{2^*}{2m}\right)^{i+1}} [1+(|Du|-1)_+]^{\tilde{N}} \, \mathrm{d}x\right\}^{\left(\frac{2m}{2^*}\right)^{i+1}} \\
&\quad \leq \tilde{C} \int_{B_{\bar{R}}} [1+(|Du|-1)_+]^{(2q-p)m} \, \mathrm{d}x,
\end{aligned} \tag{3.37}$$

where by taking into account that $m > 1$, we have

$$\tilde{C} \le \prod_{k=0}^{\infty} \left[C\, H^m \left(\frac{(\kappa_k + \tilde{N})^{\frac{3}{2}} 2^{k+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\left(\frac{2m}{2^*}\right)^k}$$

$$= \prod_{k=0}^{\infty} \left[C\, H^m \left(\frac{\left[\tilde{M} \left(\frac{2^*}{2m}\right)^k + \tilde{N} \right]^{\frac{3}{2}} 2^{k+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\left(\frac{2m}{2^*}\right)^k}$$

$$\le \prod_{k=0}^{\infty} \left[\frac{2\, C\, H^m\, [(2q-p)m]^{3m}}{(\bar{R} - \bar{\rho})^{2m}} \right]^{\left(\frac{2m}{2^*}\right)^k}$$

$$\le \frac{C\, H^{\frac{1}{2\left(\frac{1}{n} - \frac{1}{r}\right)}}}{(\bar{R} - \bar{\rho})^{\frac{2\, 2^*\, m}{2^* - 2m}}},$$

with a constant $C = C(n, r, p, q)$. Let us denote

$$\tau := 2\, \frac{2^*\, m}{2^* - 2m} = \frac{1}{\frac{1}{n} - \frac{1}{r}}; \tag{3.38}$$

thus (3.37) implies

$$\left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\kappa_0 \left(\frac{2^*}{2m}\right)^{i+1}} \, \mathrm{d}x \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}}$$
$$\le C \left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\tau} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, \mathrm{d}x. \tag{3.39}$$

At this point, we pass to the limit as $i \to +\infty$, obtaining

$$\sup \left\{ [1 + (|Du| - 1)_+]^{\tilde{M}} : x \in B_{\bar{\rho}} \right\} = \lim_{i \to +\infty} \left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\tilde{M} \left(\frac{2^*}{2m}\right)^{i+1}} \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}}$$
$$\le C \left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\tau} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, \mathrm{d}x. \tag{3.40}$$

Let us now set

$$V(x) := 1 + (|Du|(x) - 1)_+ \quad \text{and} \quad s := (2q - p)m; \tag{3.41}$$

then, estimate (3.40) becomes

$$\sup_{x \in B_\rho} |V(x)| \le C \left(\left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\frac{\tau}{s}} \|V\|_{L^s(B_R)} \right)^{\frac{s}{\tilde{M}}} \tag{3.42}$$

for every ρ, R such that $0 < \rho < R \le \rho + 1$ and where $C = C(n, r, p, q)$.

We now use the classical interpolation inequality

$$\|V\|_{L^s(B_\rho)} \leq \|V\|_{L^p(B_\rho)}^{\frac{p}{s}} \|V\|_{L^\infty(B_\rho)}^{1-\frac{p}{s}}, \tag{3.43}$$

which permits to estimate the essential supremum of $|Du|$ in terms of its L^p-norm. In fact, (3.42) and (3.43) give

$$\|V\|_{L^s(B_\rho)} \leq C^{1-\frac{p}{s}} \|V\|_{L^p(B_\rho)}^{\frac{p}{s}} \left(\left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})} \right]^{\frac{\tau}{s}} \|V\|_{L^s(B_R)} \right)^{\theta} \tag{3.44}$$

where

$$\theta := \frac{s}{\tilde{M}} \left(1 - \frac{p}{s}\right) = \frac{1}{\tilde{M}}(s-p) \overset{(3.41)}{=} \frac{1}{\tilde{M}}[(2q-p)m - p] \overset{(3.29)}{<} 1. \tag{3.45}$$

For $0 < \bar{\rho} < \bar{R}$ and for every $k \geq 0$, let us define

$$\rho_k := \bar{R} - (\bar{R} - \bar{\rho})2^{-k} \qquad B_k := \|V\|_{L^s(B_{\rho_k})}.$$

By inserting in (3.44) $\rho = \rho_k$ and $R = \rho_{k+1}$ (so that $R - \rho = (\bar{R} - \bar{\rho})2^{-(k+1)}$), we have for every $k \geq 0$

$$B_k \leq C^{1-\frac{p}{s}} \|V\|_{L^p(B_{\bar{R}})}^{\frac{p}{s}} \left(2^{\frac{\tau}{s}(k+1)} \left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})} \right]^{\frac{\tau}{s}} B_{k+1} \right)^{\theta}. \tag{3.46}$$

By iteration of (3.46), we deduce for $k \geq 0$

$$B_0 \leq \left(C^{1-\frac{p}{s}} \left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})} \right]^{\frac{\tau}{s}\theta} \|V\|_{L^p(B_{\bar{R}})}^{\frac{p}{s}} \right)^{\sum_{i=0}^{k} \theta^i} 2^{\frac{\tau}{s} \sum_{i=0}^{k+1} i\theta^i} (B_{k+1})^{\theta^{k+1}}. \tag{3.47}$$

By (3.45), the series appearing in (3.47) are convergent.

Since B_k is bounded independently of k, i.e.,

$$B_{k+1} \leq \|V\|_{L^s(B_{\bar{R}})},$$

we can pass to the limit as $k \to +\infty$ and we obtain for every $0 < \rho < R$ with a constant $C = C(n, r, p, q)$ independent of k

$$\|V\|_{L^s(B_\rho)} \leq C \left(\left[\frac{\sqrt{H}}{(R-\rho)} \right]^{\frac{\tau}{s}\theta} \|V\|_{L^p(B_R)}^{\frac{p}{s}} \right)^{\frac{1}{1-\theta}}. \tag{3.48}$$

Combining (3.42) and (3.48), by setting $\rho' = \frac{(R+\rho)}{2}$, we have

$$\begin{aligned}
\|V\|_{L^\infty(B_\rho)} &\leq C \left(\left[\frac{\sqrt{H}}{(\rho'-\rho)} \right]^{\frac{\tau}{s}} \|V\|_{L^s(B_{\rho'})} \right)^{\frac{s}{\tilde{M}}} \\
&\leq C \left(\left[\frac{\sqrt{H}}{(\rho'-\rho)} \right]^{\frac{\tau}{s}(1-\theta)} \left[\frac{\sqrt{H}}{(R-\rho')} \right]^{\frac{\tau}{s}\theta} \|V\|_{L^p(B_R)}^{\frac{p}{s}} \right)^{\frac{s}{\tilde{M}} \frac{1}{1-\theta}};
\end{aligned}$$

now, since

$$\big(\rho' - \rho\big) = \big(R - \rho'\big) = \frac{R-\rho}{2},$$

this implies

$$\|Du\|_{L^\infty(B_\rho)} \le C \left[\frac{\sqrt{H}}{(R-\rho)} \right]^{\tilde\beta} \left(\int_{B_R} (1 + |Du|^p)\, \mathrm{d}x \right)^{\beta},$$

with

$$\beta := \frac{1}{\tilde M (1-\theta)} \overset{(3.45)}{=} \frac{1}{\tilde M \left(1 - \frac{s}{\tilde M} + \frac{p}{\tilde M}\right)} = \frac{1}{\tilde M - s + p} \overset{(3.25),(3.27)}{=} \frac{1}{\epsilon} > 1 \tag{3.49}$$

$$\tilde\beta := \frac{\tau}{s} \frac{s}{\tilde M} \frac{1}{1-\theta} \overset{(3.38),(3.49)}{=} \frac{1}{\frac{1}{n} - \frac{1}{r}} \frac{1}{\epsilon} = \frac{n}{\alpha} \frac{1}{\epsilon} \tag{3.50}$$

Since $f(x,\xi) \ge C|\xi|^p$ for every $|\xi| \ge 1$, (3.4) follows. □

We needed in the proof above the following elementary result.

Lemma 3.3 *Let $t_0 > 0$ and $p \ge 1$. For every $t \ge t_0$, we have*

$$\min \left\{ \left(\frac{t_0^2}{t_0^2 + 1} \right)^{\frac{p-2}{2}}, 1 \right\} \left(1 + t^2\right)^{\frac{p-2}{2}} \le t^{p-2} \le \max \left\{ \left(\frac{t_0^2}{t_0^2 + 1} \right)^{\frac{p-2}{2}}, 1 \right\} \left(1 + t^2\right)^{\frac{p-2}{2}}. \tag{3.51}$$

Proof Since $1 \le t^2/t_0^2$ then

$$1 + t^2 \le \left(\frac{1}{t_0^2} + 1 \right) t^2$$

and thus, for every $t \ge t_0$,

$$t \le \left(1 + t^2\right)^{1/2} \le \left(\frac{t_0^2 + 1}{t_0^2} \right)^{1/2} t.$$

If $p \ge 2$, then

$$t^{p-2} \le \left(1 + t^2\right)^{\frac{p-2}{2}} \le \left(\frac{t_0^2 + 1}{t_0^2} \right)^{\frac{p-2}{2}} t^{p-2}.$$

On the contrary, if $p < 2$,

$$\left(\frac{t_0^2 + 1}{t_0^2} \right)^{\frac{p-2}{2}} t^{p-2} \le \left(1 + t^2\right)^{\frac{p-2}{2}} \le t^{p-2}.$$

In both cases, we obtain the conclusion (3.51). □

4 Approximation

In this section, we give two approximation lemmas, which are the main tools to prove Theorem 2.1. First, we need the following preliminary lemma. Note that the conditions (4.1), (4.2), (4.3) below are consequence of (1.6) and (1.7) since $g(x, 1)$ and $g_t(x, 1)$ are bounded functions.

Lemma 4.1 *Let f be as in Sect. 2 satisfying (2.3)–(2.5) and let $g = g(x,t)$ be such that $f(x,\xi) = g(x,|\xi|)$ for a.e. $x \in \Omega$. Then, (2.3) and (2.4) are equivalent to the following:*

- *There exist two positive constants λ, Λ such that for $|\xi| \geq 1$, for a.e. $x \in \Omega$*

$$\lambda|\xi|^{p-2} \leq \frac{g_t\,(x,|\xi|)}{|\xi|} \leq \Lambda|\xi|^{q-2} \tag{4.1}$$

$$\lambda|\xi|^{p-2} \leq g_{tt}\,(x,|\xi|) \leq \Lambda\,|\xi|^{q-2} \tag{4.2}$$

$$|g_{tx}\,(x,|\xi|)\,| \leq h(x)\,|\xi|^{q-1}. \tag{4.3}$$

Moreover, from the assumptions on f, we have that also $g(x,t)$ is strictly convex in the second variable for $t \geq 1$.

Proof First, we prove the equivalence (2.3)–(2.4) $\Leftrightarrow$ (4.1)–(4.2).
On the one hand, following [3], we first notice that

$$f_{\xi\xi}\,(x,\xi)(\mu,\mu) = g_{tt}\,(x,|\xi|)\,\frac{|\xi:\mu|^2}{|\xi|^2} + \frac{g_t\,(x,|\xi|)}{|\xi|}\left[|\mu|^2 - \frac{|\xi:\mu|}{|\xi|^2}\right]. \tag{4.4}$$

At this point, the choices $\mu = \xi$ and $\mu \perp \xi$ in (4.4) imply, recalling (2.3), (2.4), exactly (4.1) and (4.2), respectively.

On the other hand (see for instance [30]), we have that $f(x,\xi) = g(x,|\xi|)$ implies

$$f_{\xi_i^\alpha}(x,\xi) = g_t(x,|\xi|)\frac{\xi_i^\alpha}{|\xi|} \qquad f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)$$
$$= \left(\frac{g_{tt}\,(x,|\xi|)}{|\xi|^2} - \frac{g_t\,(x,|\xi|)}{|\xi|^3}\right)\xi_i^\alpha \xi_j^\beta + \frac{g_t\,(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_j^\beta}.$$

Since

$$\sum_{i,j,\alpha,\beta} \xi_i^\alpha \xi_j^\beta \mu_i^\alpha \mu_j^\beta = \left(\sum_{i,\alpha} \xi_i^\alpha \mu_i^\alpha\right)^2 \leq (|\xi||\mu|)^2, \qquad \forall \mu,\xi \in \mathbb{R}^{Nn}$$

with the equality holding when μ is proportional to ξ, we easily obtain the following ellipticity estimate

$$\min\left\{g_{tt}\,(x,|\xi|)\,, \frac{g_t\,(x,|\xi|)}{|\xi|}\right\} \leq \frac{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}\mu_i^\alpha \mu_j^\beta}{|\mu|^2} \leq \max\left\{g_{tt}\,(x,|\xi|)\,, \frac{g_t\,(x,|\xi|)}{|\xi|}\right\}$$

from which (2.3) and (2.4) follows from (4.1) and (4.2). Finally (2.5) implies (4.3). $\square$

We present now a first approximation lemma: we approximate g by a sequence of functions $g^k(x,t)$ monotonically converging to $g(x,t)$ and satisfying p-growth conditions with constants depending on k ((4.8)–(4.11) below) and for $t \geq 1$ the p,q-growth conditions as in (4.1)–(4.3) with constants independent of k.

Lemma 4.2 *Let $g : \Omega \times [0, +\infty) \to [0, +\infty)$ convex with respect to the second variable, g of class $C^2(\Omega \times [1, +\infty))$ satisfying*

$$g(x, 0) = g_t(x, 0) = 0 \quad \text{for a.e. } x \in \Omega \tag{4.5}$$

and (4.1)–(4.3). Then, there exists a sequence of functions $g^k \in W^{2,\infty}_{\text{loc}}(\Omega \times (1, +\infty))$ such that for all $t > 0$, we have

$$g^k(x, t) \le g(x, t) \tag{4.6}$$
$$g^k(x, t) \le g^{k+1}(x, t) \quad \forall k; \tag{4.7}$$

moreover, for a.e $x \in \Omega$ and for all $t \ge 1$, we have

$$g^k_t(x, t) \le C(k) t^{p-1} \tag{4.8}$$
$$\lambda \min\left\{\frac{1}{p-1}, 1\right\} t^{p-2} \le \frac{g^k_t(x, t)}{t} \le C(k) t^{p-2} \tag{4.9}$$
$$\lambda t^{p-2} \le g^k_{tt}(x, t) \le C(k) t^{p-2} \tag{4.10}$$
$$|g^k_{tx}(x, t)| \le C(k)\, h(x)\, t^{p-1} \tag{4.11}$$

and finally, for a.e. $x \in \Omega$ and for all $t \ge 1$, the following inequalities hold

$$\lambda \min\left\{\frac{1}{p-1}, 1\right\} t^{p-2} \le \frac{g^k_t(x, t)}{t} \le \Lambda t^{q-2} \tag{4.12}$$
$$\lambda t^{p-2} \le g^k_{tt}(x, t) \le \Lambda t^{q-2} \tag{4.13}$$
$$|g^k_{tx}(x, t)| \le h(x)\, t^{q-1}, \tag{4.14}$$

with λ, Λ as in (4.1)–(4.2).

Proof Let us define, for a.e. $x \in \Omega$

$$g^k(x, t) := \int_0^t g^k_t(x, s)\, \mathrm{d}s, \tag{4.15}$$

where

$$g^k_t(x, t) := \begin{cases} g_t(x, t) & 0 \le t < k \\ g_t(x, k) + \dfrac{\lambda}{p-1}[t^{p-1} - k^{p-1}] & t \ge k. \end{cases} \tag{4.16}$$

Direct computations show that

$$g^k_{tt}(x, t) := \begin{cases} g_{tt}(x, t) & 0 < t < k \\ \lambda t^{p-2} & t \ge k. \end{cases} \tag{4.17}$$

We claim that the sequence of functions defined by (4.16) satisfies the conditions in the statement of the Lemma. Before proceeding, we first observe that

$$\begin{cases} t^{q-1} \le C(k)\, t^{p-1} & t < k \\ k^{q-1} \le C(k)\, t^{p-1} & t \ge k. \end{cases} \tag{4.18}$$

It is not restrictive to assume $k > 1$. We also notice that for all $t \ge 0$ and a.e. $x \in \Omega$

$$g^k_t(x, t) \le g_t(x, t). \tag{4.19}$$

Indeed, (4.19) is trivial for $0 \le t < k$. Assume $t \ge k$. Then, $t \ge 1$. We need to prove that

$$g^k_t(x, t) := g_t(x, k) + \frac{\lambda}{p-1}\left[t^{p-1} - k^{p-1}\right] \le g_t(x, t). \tag{4.20}$$

By setting

$$\Psi(s) := g_t(x,s) - \frac{\lambda}{p-1}s^{p-1}$$

then Ψ is an increasing function as long as, due to (4.2), $\Psi'(s) \geq 0$. Then, (4.20) holds.

Now let us deal with the proof of (4.6)–(4.7).

- *Proof of* (4.6): it simply follows from (4.19) by integration (keeping into account (4.5) and the fact that by definition, also $g^k(x,0) = 0$.
- *Proof of* (4.7): we prove that $g_t^k(x,t) \leq g_t^{k+1}(x,t)$ for all $t \geq 0$ and a.e. $x \in \Omega$. When $t \leq k+1$, (4.7) follows easily from (4.19). Let $t \geq k+1$, since the function $\Psi(s)$ defined by

$$\Psi(s) = g_t(x,s) + \frac{\lambda}{p-1}\left[t^{p-1} - s^{p-1}\right]$$

  is increasing by (4.2), (4.7) is achieved.

  Before proceeding with the remaining inequalities, we notice that (4.8)–(4.14) are valid only for $t \geq 1$; therefore, we need to distinguish between the two cases $1 \leq t < k$ and $t \geq k$. Actually, when $1 \leq t < k$, the inequalities follow from the assumptions (4.1)–(4.3) possibly combined with (4.16) and (4.18). Thus, in the sequel we will focus just on the case $t \geq k$.
- *Proof of* (4.8): we have

$$g_t^k(x,t) \overset{(4.16)}{=} g_t(x,k) + \frac{\lambda}{p-1}[t^{p-1} - k^{p-1}] \overset{(4.1)}{\leq} \Lambda\, k^{q-1} + C\, t^{p-1} \overset{(4.18)}{\leq} C(k)\, t^{p-1}.$$

- *Proof of* (4.9): to prove the left inequality of (4.9) we distinguish two cases:

$$\frac{1}{p-1} \leq 1 \Leftrightarrow p \geq 2 \qquad \text{or} \qquad \frac{1}{p-1} > 1 \Leftrightarrow p < 2.$$

  In the first case

$$\begin{aligned} g_t^k(x,t) &\overset{(4.16)}{=} g_t(x,k) + \frac{\lambda}{p-1}t^{p-1} - \frac{\lambda}{p-1}k^{p-1} \\ &\geq g_t(x,k) + \frac{\lambda}{p-1}t^{p-1} - \lambda k^{p-1} \overset{(4.1)}{\geq} \frac{\lambda}{p-1}t^{p-1}, \end{aligned}$$

  while in the second case, using the fact that $t \geq k$

$$\begin{aligned} g_t^k(x,t) &\overset{(4.16)}{=} g_t(x,k) + \frac{\lambda}{p-1}[t^{p-1} - k^{p-1}] \\ &\overset{(4.1)}{\geq} \lambda\left(1 - \frac{1}{p-1}\right)k^{p-1} + \frac{\lambda}{p-1}t^{p-1} \\ &\geq \lambda\left(1 - \frac{1}{p-1}\right)t^{p-1} + \frac{\lambda}{p-1}t^{p-1} = \lambda\, t^{p-1}. \end{aligned}$$

  The right inequality instead follows as in the proof of (4.8).
- *Proof of* (4.10): in view of (4.17), it easily follows from (4.2) and (4.18).
- *Proof of* (4.11): observing that for $t \geq k$, $g_{tx}^k(x,t) = g_{tx}(x,k)$, the thesis follows combining (4.3) and (4.18).

- *Proof of* (4.12): the left inequality follows as in the proof of (4.9), while concerning the right inequality, we have

$$\frac{g^k_t(x,t)}{t} \overset{(4.19)}{\leq} \frac{g_t(x,t)}{t} \overset{(4.1)}{\leq} \Lambda t^{q-2}.$$

- *Proof of* (4.13): as long as $t \geq k$, then $g^k_{tt}(x,t) = \lambda t^{p-2} \leq \Lambda\, t^{q-2}$.
- *Proof of* (4.14): as $k \leq t$, then

$$|g_{tx}(x,t)| \overset{(4.16)}{=} |g_{tx}(x,k)| \overset{(4.3)}{\leq} h(x)\, k^{q-1} \leq h(x)\, t^{q-1}.$$

□

We now construct a smooth approximation for each $g^k(x,t)$. In the following, we will use also the following condition

$$0 \leq g(x,1) \leq C \qquad \text{for a.e. } x \in \Omega. \tag{4.21}$$

Lemma 4.3 *Let g be as in Lemma 4.2. Then, there exists a sequence of functions $g^{k\ell} = g^{k\ell}(x,t)$ such that $g^{k\ell} \in C^2(\Omega \times \mathbb{R})$ and the following inequalities are satisfied for a.e. $x \in \Omega$ and for all $t > 0$*

$$g^{k\ell}(x,t) \leq C(k)\left(1+t^2\right)^{\frac{p}{2}} \tag{4.22}$$

$$\varepsilon_\ell \left(1+t^2\right)^{\frac{p-2}{2}} \leq \frac{g^{k\ell}_t(x,t)}{t} \leq C(k)\left(1+t^2\right)^{\frac{p-2}{2}} \tag{4.23}$$

$$\min\{p-1,1\}\,\varepsilon_\ell \left(1+t^2\right)^{\frac{p-2}{2}} \leq g^{k\ell}_{tt}(x,t) \leq C(k)\left(1+t^2\right)^{\frac{p-2}{2}} \tag{4.24}$$

$$|g^{k\ell}_{tx}(x,t)| \leq C(k,\ell,\Omega_0)\,\left(1+t^2\right)^{\frac{p-1}{2}} \quad \forall\, \Omega_0 \subset\subset \Omega, \tag{4.25}$$

where ε_ℓ is an infinitesimal sequence of positive numbers. Moreover, the functions $g^{k\ell}$ fulfill for a.e. $x \in \Omega$ and $t \geq 1$

$$\lambda \min\left\{\frac{1}{p-1}, 1\right\} t^{p-2} \leq \frac{g^{k\ell}_t(x,t)}{t} \leq 2\Lambda\, t^{q-2} \tag{4.26}$$

$$\lambda t^{p-2} \leq g^{k\ell}_{tt}(x,t) \leq \Lambda\, t^{q-2} \tag{4.27}$$

$$|g^{k\ell}_{tx}(x,t)| \leq h_{\varepsilon_\ell}(x)\left(1+t^2\right)^{\frac{q-1}{2}}, \tag{4.28}$$

where λ, Λ are as in (4.1)–(4.2)*, ℓ and $h_{\varepsilon_\ell} \in C^\infty(\Omega)$ is the regularized function of h in* (4.3) *defined by* (4.31) *below.*

Proof Let g^k as in (4.15). Fixed an open set $A \subset\subset \Omega$, an infinitesimal sequence ε_ℓ of positive numbers and two positive mollifiers ρ and ϕ. For ℓ large enough, we define for all $(x,t) \in A \times \mathbb{R}$ the sequence of functions $g^{k\ell} : \Omega \times \mathbb{R} \to [0,+\infty)$ such that

$$g^{k\ell}(x,t) := \tilde{g}^{k\ell}(x,t) + \varepsilon_\ell \left(1+t^2\right)^{\frac{p}{2}}, \qquad \text{for a.e. } x \in \Omega,\ \forall t > 0, \tag{4.29}$$

with

$$\tilde{g}^{k\ell}(x,t) := \int_{B\times B} \rho(y)\phi(\eta) g^k(x+\varepsilon_\ell y, t+\varepsilon_\ell \eta)\mathrm{d}y\,\mathrm{d}\eta, \tag{4.30}$$

where B is the unit open ball in $\mathbb{R}^n$.

It is not difficult to see that the sequence of functions $g_t^{k\ell} \in C^2(\Omega \times \mathbb{R})$ satisfy the inequalities (4.22)–(4.28). Indeed, for $0 < t < 1$, by (4.15), (4.16) and (4.5), we have $g^k(x,t) = g(x,t)$; using the fact that g is increasing in the second variable, this entails that

$$0 \leq \tilde{g}^{k\ell}(x,t) \overset{(4.21)}{\leq} C, \qquad \text{for a.e. } x \in \Omega,\ 0 < t < 1,$$

with a constant C independent of k, ℓ. On the other hand, for $t \geq 1$, to deduce the desired inequalities it is enough to take into account (4.15), (4.16) and the estimates obtained in Lemma 4.2 for g^k. In particular

$$\begin{aligned} |g_{tx}^{k\ell}(x,t)| &= |\tilde{g}_{tx}^{k\ell}(x,t)| \leq \int_{B\times B} \rho(y)\phi(\eta)|g_{tx}^k(x+\varepsilon_\ell y, t+\varepsilon_\ell \eta)|\,\mathrm{d}y\,\mathrm{d}\eta \\ &\overset{(4.11)}{\leq} C(k) \int_{B\times B} \rho(y)\phi(\eta)h(x+\varepsilon_\ell y)(t+\varepsilon_\ell\eta)^{p-1}\,\mathrm{d}y\,\mathrm{d}\eta \\ &= C(k)\int_B \rho(y)h(x+\varepsilon_\ell y)\,\mathrm{d}y \int_B \phi(\eta)(t+\varepsilon_\ell\eta)^{p-1}\mathrm{d}\eta \end{aligned}$$

so that

$$|g_{tx}^{k\ell}(x,t)| \leq C(k)h_{\varepsilon_\ell}(x)\left(1+t^2\right)^{\frac{p-1}{2}} \qquad \forall\, t > 1$$

and also

$$|g_{tx}^{k\ell}(x,t)| \leq C(k)\,\|h_{\varepsilon_\ell}\|_{L^\infty(\Omega)}\,\left(1+t^2\right)^{\frac{p-1}{2}} \qquad \forall\, t > 1,$$

where h_{ε_ℓ} denotes the regularization of the function h

$$h_{\varepsilon_\ell}(x,t) = \int_B \rho(y)h(x+\varepsilon_\ell y)\,\mathrm{d}y. \tag{4.31}$$

On the other hand, since $g^{k\ell} \in C^2(\Omega \times [0,+\infty))$, we have, for every $\Omega_0 \subset\subset \Omega$

$$|g_{tx}^{k\ell}(x,t)| \leq C(k,\ell,\Omega_0)\,\left(1+t^2\right)^{\frac{p-1}{2}} \qquad \forall\, 0 < t < 1,\ x \in \Omega_0$$

so that (4.25) is achieved

$$|g_{tx}^{k\ell}(x,t)| \leq C(k,\ell,\Omega_0)\,[1+\|h_\varepsilon\|_{L^\infty(\Omega_0)}]\,\left(1+t^2\right)^{\frac{p-1}{2}} \qquad \forall\, t > 0.$$

Arguing in a similar way but using (4.14) instead of (4.11) we obtain also (4.28). □

5 Proof of Theorem 2.1

For $k, \ell \in \mathbb{N}$, let us consider the following functional

$$F^{k\ell}(w) = \int_\Omega g^{k\ell}(x, |Dw|)\,\mathrm{d}x. \tag{5.1}$$

Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional (2.1); moreover, let us take $x_0 \in \Omega$ and $B_R = B_R(x_0) \subset\subset \Omega$ to be a ball of center x_0 and radius R compactly contained in Ω.

We consider the following variational problem

$$\inf\left\{F^{k\ell}(w) : w \in u + W_0^{1,p}\left(B_R;\mathbb{R}^N\right)\right\}. \tag{5.2}$$

It is not difficult to verify that $F^{k\ell}$ is lower semicontinuous; therefore, there exists $v^{k\ell} \in u + W_0^{1,p}(\Omega;\mathbb{R}^N)$ solution to Problem (5.2).

We observe that since

$$t^p \leq C(1+g^{k\ell}(x,t)) \qquad \forall t>0, \quad \text{for a.e. } x \in \Omega$$

by the minimality of $v^{k\ell}$, we have

$$\int_{B_R} |Dv^{k\ell}|^p \,\mathrm{d}x \leq C \int_{B_R} [1+g^{k\ell}(x,|Dv^{k\ell}|)]\,\mathrm{d}x \leq C \int_{B_R} [1+g^{k\ell}(x,|Du|)]\,\mathrm{d}x. \quad (5.3)$$

Moreover, by the convolution properties, as $\ell \to +\infty$

$$g^{k\ell}(x,|Du|) \to g^k(x,|Du|) \qquad \text{for a.e. } x \in \Omega,$$

since

$$g^{k\ell}(x,|Du|) \leq C(k)\left(1+|Du|^2\right)^{\frac{p}{2}} \in L^1(\Omega).$$

The Lebesgue Dominated Convergence Theorem and (4.6) then imply

$$\lim_{\ell} \int_{B_R} g^{k\ell}(x,|Du|)\,\mathrm{d}x = \int_{B_R} g^k(x,|Du|)\,\mathrm{d}x \overset{(4.6)}{\leq} \int_{B_R} g(x,|Du|)\,\mathrm{d}x. \qquad (5.4)$$

By collecting (5.3) and (5.4)

$$\sup_{\ell} \int_{B_R} |Dv^{k\ell}|^p \,\mathrm{d}x \leq C \int_{B_R} [1+g(x,|Du|)]\,\mathrm{d}x. \qquad (5.5)$$

and there exists $v^k \in u + W_0^{1,p}(B_R;\mathbb{R}^N)$ such that

$$v^{k\ell} \rightharpoonup v^k \qquad \text{weakly in } W^{1,p}\left(B_R;\mathbb{R}^N\right).$$

On the other hand, by Lemma 4.2 and Lemma 4.3, we have that $f^{k\ell}(x,\xi) = g^{k\ell}(x,|\xi|)$ satisfy Assumption 3.1 and (2.3), (2.4), (2.5) for $|\xi| \geq 1$ with constants independent of k,ℓ; thus, we can apply Proposition 3.2 to $v^{k\ell}$ and we obtain for $0<\rho<R$

$$\|Dv^{k\ell}\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \leq C \left[\frac{\left(1+\|h_{\varepsilon_\ell}\|^2_{L^r(\Omega)}\right)^{\frac{1}{2}}}{(R-\rho)}\right]^{\frac{\beta}{\frac{1}{n}-\frac{1}{r}}} \left[\int_{B_R} (1+g^{k\ell}(x,|Dv^{k\ell}|))\,\mathrm{d}x\right]^{\beta}, \qquad (5.6)$$

with constant C, β independent of k,ℓ. By taking into account that by convolution properties

$$\|h_{\varepsilon_\ell}\|_{L^r(\Omega)} \leq \|h\|_{L^r(\Omega)}$$

we also have

$$\|Dv^{k\ell}\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \leq C \left[\frac{\left(1+\|h\|^2_{L^r(\Omega)}\right)^{\frac{1}{2}}}{(R-\rho)}\right]^{\frac{\beta}{\frac{1}{n}-\frac{1}{r}}} \left[\int_{B_R} (1+g^{k\ell}(x,|Dv^{k\ell}|))\,\mathrm{d}x\right]^{\beta}. \qquad (5.7)$$

In the following, we denote by

$$\tilde{C} := C \left[\frac{\left(1 + \|h\|^2_{L^r(\Omega)}\right)^{\frac{1}{2}}}{(R-\rho)} \right]^{\frac{\beta}{\frac{1}{n}-\frac{1}{r}}}.$$

Therefore, by (5.3) and (5.4), we get, for all $B_\rho \subset\subset B_R$

$$v^{k\ell} \overset{*}{\rightharpoonup} v^k \qquad \text{weakly star in } W^{1,\infty}(B_\rho; \mathbb{R}^N).$$

At this point, by the semicontinuity of the norm and (5.5), we obtain

$$\int_{B_R} |Dv^k|^p \, dx \le \liminf_{\ell} \int_{B_R} |Dv^{k\ell}|^p \, dx \le \tilde{C} \int_{B_R} (1 + g(x, |Du|)) \, dx. \tag{5.8}$$

On the other hand, (5.4) and (5.6) imply

$$\|Dv^k\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le \liminf_{\ell} \|Dv^{k\ell}\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le \tilde{C} \left[\int_{B_R} (1 + g(x, |Du|)) \, dx\right]^\beta =: M. \tag{5.9}$$

Thus, we can deduce that up to subsequences, there exist $v \in u + W^{1,p}_0(B_R; \mathbb{R}^N)$ such that

$$v^k \rightharpoonup v \qquad \text{weakly in } W^{1,p}\left(B_R; \mathbb{R}^N\right)$$
$$v^k \overset{*}{\rightharpoonup} v \qquad \text{weakly star in } W^{1,\infty}\left(B_\rho; \mathbb{R}^N\right) \qquad \text{for all } B_\rho \subset\subset B_R.$$

Now we proceed in a similar way as in [9, 16]. First, we show that v is a solution to the problem

$$\inf\left\{\int_{B_R} g(x, Dw) \, dx : \; w \in u + W^{1,p}_0(B_R, \mathbb{R}^n)\right\}. \tag{5.10}$$

To this end, using the semicontinuity of the functional $\int_{B_\rho} g^{k_0}(x, |Du|) \, dx$ and (see (4.7))

$$g^{k_0}(x,t) \le g^k(x,t) \qquad \forall k \ge k_0,$$

we get

$$\int_{B_\rho} g^{k_0}(x, |Dv^k|) \, dx \le \liminf_{\ell} \int_{B_\rho} g^{k_0}(x, |Dv^{k\ell}|) \, dx \le \liminf_{\ell} \int_{B_\rho} g^k(x, |Dv^{k\ell}|) \, dx. \tag{5.11}$$

Since, up to subsequences, $g^{k\ell}(x,t)$ converges as $k \to +\infty$, a.e. in $\Omega \times [0, +\infty)$ to $g^k(x,t)$, by Egorov theorem, fixed $K = \{\xi \in \mathbb{R}^{Nn} : |\xi| \le M + 1\}$, for every $\delta > 0$ there exists A_δ with $|A_\delta| < \delta$ such that $g^{k\ell}$ converges to g^k uniformly in $(B_\rho \setminus A_\delta) \times K$. Thus,

$$\limsup_{\ell} \int_{B_\rho \setminus A_\delta} g^k(x, |Dv^{k\ell}|) \, dx = \limsup_{\ell} \int_{B_\rho \setminus A_\delta} g^{k\ell}(x, |Dv^{k\ell}|) \, dx$$

and due to (5.9)

$$\limsup_{\ell} \int_{B_\rho \cap A_\delta} g^k(x, |Dv^{hk}|) \, dx \le C(k) \, |A_\delta| \, (1 + M),$$

with $C(k)$ independent of δ. Thus, putting together the previous inequalities, (5.11) gives

$$\begin{aligned}\int_{B_\rho} g^{k_0}(x,|Dv^k|)\,dx &\le \liminf_{\ell} \int_{B_\rho} g^k(x,|Dv^{k\ell}|)\,dx\\ &\le \limsup_{\ell} \int_{B_\rho} g^k(x,|Dv^{k\ell}|)\,dx\\ &\le \limsup_{\ell} \int_{B_R} g^{k\ell}(x,|Dv^{k\ell}|)\,dx + C(k)\,|A_\delta|\,(1+M)\end{aligned}$$

so that letting $\delta \to 0$, by (5.4)

$$\int_{B_\rho} g^{k_0}(x,|Dv^k|)\,dx \le \limsup_{\ell} \int_{B_R} g^{k\ell}(x,|Du|)\,dx \le \int_{B_R} g^k(x,|Du|)\,dx.$$

At this point, exploiting the lower semicontinuity of the functional $\int_{B_\rho} g^{k_0}(x,|Du|)\,dx$, we obtain

$$\begin{aligned}\int_{B_\rho} g^{k_0}(x,|Dv|)\,dx &\le \liminf_{k} \int_{B_\rho} g^{k_0}(x,|Dv^k|)\,dx\\ &\le \liminf_{k} \int_{B_R} g^k(x,|Du|)\,dx = \int_{B_R} g(x,|Du|)\,dx,\end{aligned}$$

where in the last line, we applied once more the Lebesgue Dominated Convergence Theorem to the sequence of functions g^k. This has been possible because of (4.6) and due to the fact that $g^k(x,|Du|) \to g(x,|Du|)$ pointwise. Finally, letting $k_0 \to +\infty$ and $\rho \to R$

$$\int_{B_R} g(x,|Dv|)\,dx \le \int_{B_R} g(x,|Du|)\,dx, \tag{5.12}$$

and passing to the limit in (5.9), we get

$$\|Dv\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le \tilde{C}\left[\int_{B_R} (1+g(x,|Du|)\,dx\right]^\beta. \tag{5.13}$$

Therefore, u and v are two solutions to Problem (5.10), but since g is not strictly convex for all $t > 0$, we may not conclude that $u = v$ in B_R. Set

$$E_0 := \left\{x \in B_R : \left|\frac{Du(x)+Dv(x)}{2}\right| > 1\right\} \quad \text{and} \quad \bar{u} := \frac{u+v}{2}.$$

If E_0 has positive measure, then from the convexity of $g(x,.)$, we have:

$$\int_{B_R\setminus E_0} g(x,|D\bar{u}|)\,dx \le \frac{1}{2}\int_{B_R\setminus E_0} g(x,|Du|)\,dx + \frac{1}{2}\int_{B_R\setminus E_0} g(x,|Dv|)\,dx. \tag{5.14}$$

Now, by the strictly convexity of $g(x,t)$ for $t \ge 1$ and applying two times the following inequality

$$g(x,t) > g(x,s_0) + g_t(x,s_0)(t-s_0), \qquad s_0 \ge 1$$

first with $s_0 = D\bar{u}$ and $t = Du$, then for $s_0 = D\bar{u}$ and $t = Dv$, finally by adding up the two inequalities obtained, we have

$$\int_{B_R\cap E_0} g(x,|D\bar{u}|)\,dx < \frac{1}{2}\int_{B_R\cap E_0} g(x,|Du|)\,dx + \frac{1}{2}\int_{B_R\cap E_0} g(x,|Dv|)\,dx. \tag{5.15}$$

Adding (5.14) and (5.15), we get a contradiction with the minimality of u and v. Therefore, the set E_0 has zero measure, which implies that

$$\sup_{B_\rho} |Du(x)| \le \sup_{B_\rho} |Du(x) + Dv(x)| + \sup_{B_\rho} |Dv(x)| \le 2 + \sup_{B_\rho} |Dv(x)|$$

and estimate (2.7) follows by (5.13). □

Proof of Theorem 2.6. For every $x_0 \in \Omega$, there exists $R > 0$ such that the ball $B_R(x_0)$ contained in Ω and

$$\frac{\sup\{p(x) : x \in B_R(x_0)\} + \delta}{\inf\{p(x) : x \in B_R(x_0)\}} < 1 + \frac{\alpha}{n} \tag{5.16}$$

for some $0 < \delta < 1$. Observe that for the functional (2.14), with the notation $g(x,t) = a(x)\, t^{p(x)}$, we get

$$g_{tx_i}(x,t) = (a\, p)_{x_i} t^{p(x)-1} + a\, p\, p_{x_i}\, t^{p(x)-1} \log t, \qquad \forall i = 1, \dots, n. \tag{5.17}$$

Set

$$p := \inf\{p(x) : x \in B_R(x_0)\} \qquad q := \sup\{p(x) : x \in B_R(x_0)\} + \delta$$

where δ permits to absorb the logarithmic term in (5.17). Then, the following estimate

$$|g_{tx}(x,t)| \le \Lambda\, h(x)\, t^{q+\delta-1}, \qquad t \ge 1, \ \text{ a.e. in } B_R$$

holds for some $h \in L^r(B_R)$. This is due to the fact that the product $(a\, p)_{x_i} \in L^r(B_R)$, since $a, p \in W^{1,r}(B_R)$, $r > n$, are continuous functions.

We conclude that with this choice of p, q, $f(x,\xi) = a(x)\, |\xi|^{p(x)}$ satisfies (2.3), (2.4) and (2.5). Since a minimizer u of (2.14) in Ω is also a minimizer in $B_R(x_0)$, by proceeding as in Theorem 2.1, we obtain the u is locally Lipschitz continuous in $B_R(x_0)$. □

6 Back to systems

In this section, we give the proof of the previous statements about systems. The proof of Corollary 2.2 is similar to the next one, and we omit it.

Proof of Corollary 2.3. We consider the variational problem

$$\inf\left\{\int_\Omega f(x, Dv)\, dx \ : \ v \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)\right\}. \tag{6.1}$$

Assumption (2.3) guarantees the convexity of $f(x,\xi) = g(x, |\xi|)$ with respect to the second variable and its coercivity with p-growth. The lower semicontinuity of the integral in (6.1) gives the existence of (at least) a minimizing map $u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$. By the regularity Theorem 2.1, we have that $u \in W^{1,\infty}_{\mathrm{loc}}\left(\Omega; \mathbb{R}^N\right)$. Then, the q-growth from above in (2.3) allows us to apply the well-known variational technique to show that u in fact satisfies the weak form in (2.9), with test function $\varphi = (\varphi^\alpha)_{\alpha=1,2,\dots,N} \in W^{1,q}_{\mathrm{loc}}\left(\Omega; \mathbb{R}^N\right)$, and therefore, u is a weak solution to the Dirichlet system (2.10) too. This completes the proof of Corollary 2.3. □

Proof of Corollary 2.4. Once we get an estimate for the norm in L^∞ of the gradient of the solution to problem (2.10), with assumptions on the behavior of g, and then on f, as $t = |\xi| \to 0^+$, the $C^{1,\beta}$ regularity of the solution follows by well-known results for the systems considered in (2.8) (see [39] and [40]). □

Proof of Corollary 2.5. Since $g_{tt}(x, t) > 0$ for all $t > 0$, we are in the framework of uniformly elliptic systems. Since $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, each component of the gradient Du is a weak solution to a system with Hölder continuous coefficients. Then, further regularity follows from the regularity theory for linear elliptic systems with smooth coefficients (see for instance Section 3 of [19]). □

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