LOCAL LIPSCHITZ CONTINUITY OF MINIMIZERS WITH MILD ASSUMPTIONS ON THE x-DEPENDENCE

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Abstract. We are interested in the *regularity* of local minimizers of energy integrals of the Calculus of Variations. Precisely, let Ω be an open subset of $\mathbb{R}^n$. Let $f(x,\xi)$ be a real function defined in $\Omega \times \mathbb{R}^n$ satisfying the growth condition $|f_{\xi x}(x,\xi)| \leq h(x)|\xi|^{p-1}$, for$x \in \Omega$ and $\xi \in \mathbb{R}^n$ with $|\xi| \geq M_0$ for some $M_0 \geq 0$, with $h \in L^r_{\mathrm{loc}}(\Omega)$ for some $r > n$. This growth condition is more general than those considered in the mathematical literature and allows us to handle some cases recently studied in similar contexts. We associate to $f(x,\xi)$ the so-called *natural $p-$growth conditions* on the second derivatives $f_{\xi\xi}(x,\xi)$; i.e., $(p-2)-$growth for $|f_{\xi\xi}(x,\xi)|$ from above and $(p-2)-$growth from below for the quadratic form $(f_{\xi\xi}(x,\xi)\lambda,\lambda)$; for details see either (1.3) or (2.2) below. We prove that these conditions are sufficient for the *local Lipschitz continuity* of any minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the energy integral $\int_\Omega f(x,Du(x))\,dx$.

1. Introduction

In recent years many authors considered differential problems, in the context of the calculus of variations and of partial differential equations of elliptic and parabolic type, under general $p,q-$growth conditions and, as a special relevant case, related to $p(x)-$growth; i.e., variable exponents. Among them Vicenţiu D. Rădulescu who studied, in this framework of general growth, multiplicity of solutions for some nonlinear problems, qualitative analysis, anisotropic elliptic equations, eigenvalue problems and several other related questions; see for instance [20], [21], [5], [23], [2]. We like to explicitly dedicate this manuscript to Vicenţiu D. Rădulescu, with esteem and sympathy.

In order to introduce the problem we first consider the classical Dirichlet energy integral

$$\int_\Omega a(x)|Du(x)|^2\,dx\,; \tag{1.1}$$

then any minimizer u of this functional is locally Lipschitz continuous in the open set $\Omega \subset \mathbb{R}^n$ if also the coefficient $a(x)$ is assumed to be locally Lipschitz continuous and positive in Ω. In the nonlinear case too, more generally, a minimizer of an energy integral of the type

$$\int_\Omega f(x,Du(x))\,dx \tag{1.2}$$

comes out to be locally Lipschitz continuous in Ω if the integrand $f:\Omega\times\mathbb{R}^n \to \mathbb{R}$ satisfies the ellipticity and the growth conditions

$$\begin{cases} (f_{\xi\xi}(x,\xi)\lambda,\lambda) \geq M_1\left(1+|\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \\ |f_{\xi\xi}(x,\xi)| \leq M_2\left(1+|\xi|^2\right)^{\frac{p-2}{2}} \end{cases}, \tag{1.3}$$

for some positive constants M_1, M_2, for an exponent $p \in (1,+\infty)$ and for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$. The assumption usually considered in the mathematical literature for Lipschitz

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continuity of solutions is the Lipschitz continuity of $f(x,\xi)$ with respect to x; more precisely, similarly to the Dirichlet integral in (1.1), the condition often assumed on the $x-$dependence is

$$|f_{\xi x}(x,\xi)| \leq L\left(1+|\xi|^2\right)^{\frac{p-1}{2}}. \tag{1.4}$$

A classical reference is, for instance, the book by Ladyzhenskaya-Uraltseva [19]. The $x-$dependence cannot be only considered as a perturbation, but it is a relevant difference with the case $f=f(\xi)$ from several points of view. For instance recently several authors studied the $x-$dependence under Hölder continuity assumptions as well as under Sobolev summability assumptions, in the general context of $p,q-$growth conditions; see [7], [8], [13], [14]; see also [11], [3], [4], [15].

In this paper we show that we can obtain the local Lipschitz continuity in Ω of the local minimizers by assuming a mild condition on the $x-$dependence, weaker than (1.4). Precisely, instead of (1.4), we assume that

$$|f_{\xi x}(x,\xi)| \leq h(x)\left(1+|\xi|^2\right)^{\frac{p-1}{2}}, \tag{1.5}$$

where $h \in L^r_{\mathrm{loc}}(\Omega)$ for some exponent $r>n$. A precise statement (with assumptions only for $|\xi| \to +\infty$) is described in the next section. The following regularity theorem holds.

Theorem 1.1. *Let $p>1$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the energy integral* (1.2) *under the growth assumptions* (1.3), (1.5). *Then u is locally Lipschitz continuous in Ω.*

An explicit bound for the $L^\infty-$norm of u in any compactly contained subset of $\Omega \subset \mathbb{R}^n$ is given in terms of the L^r- norm of h and of the L^p- norm of the gradient Du. See the details in the next Section, precisely in the statement of Theorem 2.1.

A motivation for the previous result, other that its intrinsic interest in the framework of regularity theory, also relies in the approximation procedure to pass from a-priori estimates to existence and regularity under general $p,q-$growth conditions. A place where this procedure has been used is the author's paper [15]; we plan to go back to this problem in the next future to explain with more details the use of Theorem 1.1 to get local Lipschitz continuity and regularity of solutions in a general context.

2. A-priori estimates

Let Ω be an open set of $\mathbb{R}^n$. In the following, we say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *local minimizer* of the energy integral (1.2) if

$$\int_{\Omega'} f(x,Du)\,dx \leq \int_{\Omega'} f(x,D(u+\varphi))\,dx, \tag{2.1}$$

for every open set Ω' compactly contained in Ω and for every $\varphi \in W^{1,p}_0(\Omega')$.

We assume that $f:\Omega\times\mathbb{R}^n \to [0,+\infty)$ is a convex function with respect to the gradient variable $\xi \in \mathbb{R}^n$ and it is strictly convex only at infinity. Precisely, the second derivatives of f are Carathéodory functions satisfying the growth conditions

$$\begin{cases} (f_{\xi\xi}(x,\xi)\lambda,\lambda) \geq M_1\,|\xi|^{p-2}\,|\lambda|^2, \\ |f_{\xi\xi}(x,\xi)| \leq M_2\,|\xi|^{p-2}, \\ |f_{\xi x}(x,\xi)| \leq h(x)\,|\xi|^{p-1}, \end{cases} \tag{2.2}$$

for a.e. $x\in\Omega$ and for all $\lambda,\xi \in \mathbb{R}^n$, with $|\xi| \geq M_0$, for some constants $M_0 \geq 0$, $M_1, M_2 > 0$. Here $h \in L^r_{\mathrm{loc}}(\Omega)$ for some $r>n$.

We observe that here the ellipticity and growth assumptions hold only for large values of the gradient variable, i.e., we consider functionals which are *uniformly convex only at infinity.*

In this context see [6], [17], [10] and recently [13], [14] and [9]. The Sobolev dependence on x recently has been considered in [22], [1] and for obstacle problems in [16].

We observe that we can transform $f(x,\xi)$ into $f(x,M_0\xi)$, which satisfies the same assumptions for $|\xi|\geq 1$ (with different constants depending on M_0). Therefore and without loss of generality, for clarity of exposition, we assume $M_0=1$.

Throughout the paper we will denote by B_ρ and B_R balls of radii ρ and R ($\rho<R$) compactly contained in Ω and with the same center.

In this section we assume the following supplementary assumptions on f which will be automatically satisfied in Section 3. Assume that $f\in\mathcal{C}^2(\Omega\times\mathbb{R}^n)$ and there exist two positive constants k,K such that $\forall\xi\in\mathbb{R}^n$, $\forall x\in\Omega$

$$\begin{cases} k\,(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2\leq\sum_{i,j}f_{\xi_i\xi_j}(x,\xi)\lambda_i\lambda_j,\\ |f_{\xi\xi}(x,\xi)|\leq\ K\,(1+|\xi|^2)^{\frac{p-2}{2}},\\ |f_{\xi x}(x,\xi)|\leq\ K\,(1+|\xi|^2)^{\frac{p-1}{2}}.\end{cases}\tag{2.3}$$

The following a-priori estimate holds.

Theorem 2.1. (A-PRIORI ESTIMATE) *Let the growth assumptions* (2.2) *and* (2.3) *hold and let $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the energy integral in* (1.2). *Then u is locally Lipschitz continuous in Ω and the following a-priori estimate holds*

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)}\leq\ C\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^\beta\left(\int_{B_R}\{1+|Du|^p\}\,dx\right)^{\frac{1}{p}},\tag{2.4}$$

for every ρ,R, with $\rho<R\leq\rho+1$ and $B_{\rho+1}\subset\Omega$, where C,β are positive constants depending on n,r,p,M_1,M_2 but independent of k and K in (2.3).

Proof. Let $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of (1.2). First of all we obtain an a-priori estimate for the $L^\infty-$norm of the gradient of u which is independent of k and K, i.e. for every $0<\rho<R\leq\rho+1$, we prove that there exists a positive constant C depending only on n,r,p,M_1,M_2 such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)}\leq\ C\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^{\tilde\beta}\left(\int_{B_R}\{1+|Du|^{pm}\}\,dx\right)^{\frac{1}{pm}},\tag{2.5}$$

where

$$m:=\frac{r}{r-2}\qquad\tilde\beta:=\frac{2}{p\left(1-\frac{2m}{2*}\right)}.\tag{2.6}$$

Let us observe that since $r>n$

$$1<m:=\frac{r}{r-2}<\frac{n}{n-2}=\frac{2^*}{2},\tag{2.7}$$

where $2^*:=\frac{2n}{n-2}$ when $n>3$ and 2^* is any fixed real number greater than $2m=\frac{2r}{r-2}$ when $n=2$. Therefore $\tilde\beta>0$ in (2.6).

The local minimizer u satisfies the following Euler first variation

$$\int_\Omega\sum_{i=1}^n f_{\xi_i}(x,Du)\varphi_{x_i}(x)\,dx=0\qquad\forall\varphi\in W^{1,p}_0(\Omega).$$

By (2.3), the technique of the difference quotients (see [19], [12], in particular [18], Chapter 8, Sections 8.1 and 8.2) gives

$$u\in W^{1,\infty}_{\mathrm{loc}}(\Omega)\cap W^{2,\min(2,p)}_{\mathrm{loc}}(\Omega)\ \text{ and }\ (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\in L^1_{\mathrm{loc}}(\Omega).\tag{2.8}$$

Let $\eta \in C_0^1(\Omega)$ and for any fixed $s \in \{1,\dots,n\}$ define $\varphi = \eta^2 u_{x_s} \Phi((|Du|-1)_+)$ for $\Phi : [0,+\infty) \to [0,+\infty)$ increasing, locally Lipschitz continuous function, with Φ and Φ' bounded on $[0,+\infty)$, such that $\Phi(0)=\Phi'(0)=0$ and

$$\Phi'(s)s \leq c_\Phi \, \Phi(s) \tag{2.9}$$

for a suitable constant $c_\Phi > 1$. Here $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following we denote $\Phi((|Du|-1)_+) = \Phi(|Du|-1)_+$. We have then

$$\varphi_{x_i} = 2\eta\eta_{x_i} u_{x_s} \Phi(|Du|-1)_+ + \eta^2 u_{x_s x_i} \Phi(|Du|-1)_+ + \eta^2 u_{x_s} \Phi'(|Du|-1)_+ [(|Du|-1)_+]_{x_i}. \tag{2.10}$$

Let $p \geq 2$, by (2.8) we have that $|D^2u|^2 \in L^1_{\mathrm{loc}}(\Omega)$. Otherwise if $1 < p < 2$, we use the fact that $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ to infer that there exists $M = M(\operatorname{supp}\varphi)$ such that $|Du(x)| \leq M$ for a.e. $x \in \operatorname{supp}\varphi$. Now since $p-2<0$ we have

$$(1+M^2)^{\frac{p-2}{2}} |D^2u|^2 \leq (1+|Du|^2)^{\frac{p-2}{2}} |D^2u|^2,$$

and by (2.8) we again get $|D^2u|^2 \in L^1(\operatorname{supp}\varphi)$. Therefore we can insert φ_{x_i} in the following second variation

$$\int_\Omega \left\{ \sum_{i,j=1}^n f_{\xi_i\xi_j}(x,Du) u_{x_j x_s} \varphi_{x_i} + \sum_{i=1}^n f_{\xi_i x_s}(x,Du)\varphi_{x_i} \right\} dx = 0 \qquad \forall s = 1,\dots,n,$$

and we obtain

$$\begin{aligned} 0 \;=\; & \sum_s \Bigg[\int_\Omega 2\eta \Phi(|Du|-1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x,Du)\eta_{x_i} u_{x_s} u_{x_s x_j}\, dx \\ & + \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x,Du) u_{x_s x_i} u_{x_s x_j}\, dx \\ & + \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x,Du) u_{x_s} u_{x_s x_j} [(|Du|-1)_+]_{x_i}\, dx \\ & + \int_\Omega 2\eta \Phi(|Du|-1)_+ \sum_i f_{\xi_i x_s}(x,Du)\eta_{x_i} u_{x_s}\, dx \\ & + \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_i f_{\xi_i x_s}(x,Du) u_{x_s x_i}\, dx \\ & + \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \sum_i f_{\xi_i x_s}(x,Du) u_{x_s} [(|Du|-1)_+]_{x_i}\, dx \Bigg] \\ =: \; & \sum_s (I_1^s + I_2^s + I_3^s + I_4^s + I_5^s + I_6^s). \end{aligned} \tag{2.11}$$

In the following, constants will be denoted by C, regardless of their actual value.

First of all, using the Cauchy-Schwarz inequality, the Young inequality and (2.2), we have

$$\begin{aligned} \left| \sum_s I_1^s \right| = & \left| \int_\Omega 2\eta\Phi(|Du|-1)_+ \sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)\eta_{x_i} u_{x_s} u_{x_s x_j}\, dx \right| \qquad (2.12) \\ \leq & \int_\Omega 2\eta\Phi(|Du|-1)_+ \left\{ \sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)\eta_{x_i} u_{x_s} \eta_{x_j} u_{x_s} \right\}^{\frac12} \left\{ \sum_{i,j,s} f_{\xi_i\xi_j}(x,Du) u_{x_s x_i} u_{x_s x_j} \right\}^{\frac12} dx \\ \leq & \; C \int_\Omega |D\eta|^2 \Phi(|Du|-1)_+ \, |Du|^p \, dx + \frac12 \int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{i,j,s} f_{\xi_i\xi_j}(x,Du) u_{x_s x_i} u_{x_s x_j}\, dx. \end{aligned}$$

Moreover, by the growth of $f_{\xi\xi}$ in (2.2), we obtain

$$\begin{aligned}\sum_s I_3^s &= \int_\Omega \eta^2\Phi'(|Du|-1)_+\sum_{i,j,s} f_{\xi_i\xi_j}(x,Du)u_{x_s}u_{x_sx_j}[(|Du-1|_+)]_{x_i}\,dx\\ &\geq M_1\int_\Omega \eta^2\Phi'(|Du|-1)_+|Du|^{p-1}|D(|Du|-1)_+|^2\,dx\geq 0\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left|\sum_s I_4^s\right| &= \left|\int_\Omega 2\eta\Phi(|Du|-1)_+\sum_{i,s} f_{\xi_ix_s}(x,Du)\eta_{x_i}u_{x_s}\,dx\right|\\ &\overset{(2.2)}{\leq} \int_\Omega 2\eta\Phi(|Du|-1)_+h(x)|Du|^{p-1}\sum_{i,s}|\eta_{x_i}u_{x_s}|\,dx\\ &\leq C\int_\Omega(\eta^2+|D\eta|^2)h(x)\Phi(|Du|-1)_+|Du|^p\,dx.\end{aligned}$$

Consider now the fifth term in (2.11):

$$\begin{aligned}\left|\sum_s I_5^s\right| &= \left|\int_\Omega \eta^2\Phi(|Du|-1)_+\sum_{i,s} f_{\xi_ix_s}(x,Du)u_{x_sx_j}\,dx\right|\\ &\overset{(2.2)}{\leq} \int_\Omega \eta^2\Phi(|Du|-1)_+h(x)|Du|^{p-1}|D^2u|\,dx\\ &\leq \int_\Omega\left[\eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\right]^{1/2}\left[\eta^2\Phi(|Du|-1)_+|h(x)|^2|Du|^p\right]^{1/2}\,dx\\ &\leq \varepsilon\int_\Omega \eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx+C_\varepsilon\int_\Omega\eta^2\Phi(|Du|-1)_+|h(x)|^2|Du|^p\,dx.\end{aligned}$$

Finally we need to estimate the sixth integral in (2.11). Let us observe that we want to consider growth conditions only at infinity, therefore we need to overcome the difficulty due to the presence of the term Φ' in this sixth integral. The idea is to use the same argument exploited in [13]. For any $0<\delta<1$ we have

$$\begin{aligned}\left|\sum_s I_6^s\right| &= \left|\int_\Omega \eta^2\sum_{i,s} f_{\xi_ix_s}(x,Du)u_{x_s}\Phi'(|Du|-1)_+[(|Du|-1)_+]_{x_i}\,dx\right|\\ &\leq \int_\Omega \eta^2\Phi'(|Du|-1)_+h(x)[(|Du|-1)_++\delta]\,[(|Du|-1)_++\delta]^{-1}|Du|^p\,|D^2u|\,dx\\ &\leq \int_\Omega\eta^2\left\{\frac{1}{c_\Phi}\Phi'(|Du|-1)_+[(|Du|-1)_++\delta]|Du|^{p-2}|D^2u|^2\right\}^{1/2}\\ &\quad\times\left\{c_\Phi\Phi'(|Du|-1)_+|h(x)|^2|Du|^{p+2}[(|Du|-1)_++\delta]^{-1}\right\}^{1/2}\,dx\\ &\leq C_\varepsilon\,c_\Phi\int_\Omega\eta^2\Phi'(|Du|-1)_+|h(x)|^2|Du|^{p+2}[(|Du|-1)_++\delta]^{-1}\,dx\\ &\quad+\frac{\varepsilon}{c_\Phi}\int_\Omega\eta^2\Phi'(|Du|-1)_+[(|Du|-1)_++\delta]|Du|^{p-2}|D^2u|^2\,dx.\end{aligned}$$

In $\{x:|Du(x)|\geq 2\}$ we get $(|Du|-1)_++\delta\leq 2(|Du|-1)_+$ and we estimate the last integral using the properties of Φ in (2.9)

$$\int_\Omega \eta^2\Phi'(|Du|-1)_+[(|Du|-1)_++\delta]|Du|^{p-2}|D^2u|^2\,dx$$

$$
\begin{aligned}
\le\ & 2 \int_{|Du|\ge 2} \eta^2 \Phi'(|Du|-1)_+ (|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx \\
& + \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+ (|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx \\
& + \delta \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx \\
\overset{(2.9)}{\le}\ & 2\, c_\Phi \int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx + \delta \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx.
\end{aligned}
$$

Therefore we finally obtain

$$
\begin{aligned}
\left| \sum_s I_6^s \right| \le\ & C_\varepsilon\, c_\Phi \int_\Omega \eta^2 \Phi'(|Du|-1)_+ |h(x)|^2 |Du|^{p+2} [(|Du|-1)_+ + \delta]^{-1}\, dx \\
& + 2\,\varepsilon \int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx \\
& + \delta\, \varepsilon \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx.
\end{aligned}
$$

Now putting together all the previous estimates, for ε sufficiently small, we deduce that there exists a constant C depending on n, p, M_1, M_2 such that

$$
\begin{aligned}
& \int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx \qquad (2.13) \\
\le\ & C\, c_\Phi \int_\Omega (\eta^2 + |D\eta|^2)(1+h(x))^2 |Du|^p \\
& \times \left[\Phi(|Du|-1)_+ + \Phi'(|Du|-1)_+ |Du|^2 [(|Du|-1)_+ + \delta]^{-1} \right] dx \\
& + \delta \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-2} |D^2u|^2\, dx.
\end{aligned}
$$

At this point we set

$$
\Phi(s) := (1+s)^{\gamma-2} s^2 \qquad \gamma \ge 0; \tag{2.14}
$$

with

$$
\Phi'(s) = (\gamma s + 2) s (1+s)^{\gamma-3}. \tag{2.15}
$$

It is easy to check that Φ satisfies (2.9) with $c_\Phi = 2(1+\gamma)$.
Let Φ_h be a sequence of functions, with Φ_h equal to Φ in $[0,h]$ and extended to $[h,+\infty)$ with the constant value $\Phi(h)$. Then (2.13) holds for each Φ_h and since Φ_h and Φ_h' converge monotonically to Φ and Φ', by passing to the limit we have (2.5) with Φ defined in (2.14).

Therefore, for every $0 < \delta < 1$, since

$$
\frac{(|Du|-1)_+}{(|Du|-1)_+ + \delta} \le 1 \qquad \forall\, \delta > 0
$$

and $\Phi'(t-1)_+ \le C(\gamma)$ when $1 < t < 2$, we obtain

$$
\begin{aligned}
& \int_\Omega \eta^2 (1 + (|Du|-1)_+)^{\gamma-2} (|Du|-1)_+^2 |Du|^{p-2} |D^2u|^2\, dx \\
\le\ & C\,(1+\gamma)^2 \int_\Omega (\eta^2 + |D\eta|^2)(1+h(x))^2 (1+(|Du|-1)_+)^{\gamma+p}\, dx \\
& + \delta\, C(\gamma) \int_{1<|Du|<2} \eta^2 |Du|^{p-2} |D^2u|^2\, dx. \qquad (2.16)
\end{aligned}
$$

Since $|Du|^{p-2} \leq C(p)\,(1+|Du|^2)^{\frac{p-2}{2}}$ when $|Du|>1$ (see Lemma 3.3 in [13]), we get

$$\int_{1<|Du|<2} \eta^2 |Du|^{p-2}|D^2u|^2\,dx \leq C\int_{1<|Du|<2} \eta^2 (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx < +\infty$$

by (2.8), and the last term in (2.16) vanishes as $\delta \to 0$. Using the Hölder inequality, since $h \in L^r(\Omega)$ and $\frac{1}{m}+\frac{2}{r}=1$, by (2.16) we have

$$\begin{aligned}
&\int_\Omega \eta^2(1+(|Du|-1)_+)^{\gamma-2}(|Du|-1)_+^2|Du|^{p-2}|D((|Du|-1)_+)|^2\,dx \\
\leq\ & C\,(1+\gamma)^2\,(\|1+h\|^2_{L^r(\Omega)})\left[\int_\Omega (\eta^2+|D\eta|^2)^m(1+(|Du|-1)_+)^{(\gamma+p)m}\,dx\right]^{\frac{1}{m}}. \qquad (2.17)
\end{aligned}$$

Let us define

$$G(t) = 1+\int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}\,s\,ds \qquad (2.18)$$

so that

$$[G(t)]^2 \leq 4\,(1+t)^{\gamma+p} \qquad G_t(t) = (1+t)^{\frac{\gamma}{2}+\frac{p}{2}-2}t, \qquad (2.19)$$

which gives the following estimate for the gradient of the function $w=\eta G((|Du|-1)_+)$

$$\begin{aligned}
&\int_\Omega |D(\eta G((|Du|-1)_+))|^2\,dx \qquad (2.20)\\
\overset{(2.17),(2.19)}{\leq}\ & C\,(1+\gamma)^2\,(\|1+h\|^2_{L^r(\Omega)})\left[\int_\Omega (\eta^2+|D\eta|^2)^m[1+(|Du|-1)_+]^{(\gamma+p)m}\,dx\right]^{\frac{1}{m}}.
\end{aligned}$$

By Sobolev's inequality there exists a constant C, depending also on $|\Omega|$ when $n=2$, such that

$$\left\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\,dx\right\}^{\frac{2}{2^*}} \leq C\int_\Omega |D(\eta G((|Du|-1)_+))|^2\,dx.$$

By the previous inequality we get

$$\begin{aligned}
&\left\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\,dx\right\}^{\frac{2}{2^*}} \qquad (2.21)\\
\leq\ & C\,(1+\gamma)^2\,(\|1+h\|^2_{L^r(\Omega)})\left[\int_\Omega (\eta^2+|D\eta|^2)^m[1+(|Du|-1)_+]^{(\gamma+p)m}\,dx\right]^{\frac{1}{m}}.
\end{aligned}$$

We take into account the definition of $G\,(t)$ in (2.18) and we use Lemma 2.2 below, and in particular formula (2.36) with $\mu=\frac{\gamma+p}{2}$. Being $\mu \geq 0$, we have $\mu \geq \mu_0 := p/2 > 0$ and

$$(1+t)^{\frac{\gamma+p}{2}} \leq c''\left(\frac{\gamma+p}{2}\right)^2\left(1+\int_0^t (1+s)^{\frac{\gamma+p}{2}-2}\,s\,ds\right), \qquad (2.22)$$

for every $\gamma \geq 0$ and every $t \in [0,+\infty)$. In terms of $G(t)$ equivalently

$$(1+t)^{\frac{\gamma+p}{2}} \leq c''\left(\frac{\gamma+p}{2}\right)^2 G\,(t)\,, \qquad \forall\,\gamma \geq 0,\ \ \forall\,t \geq 0.$$

Therefore, if $t := (|Du|-1)_+$,

$$\left(1+(|Du|-1)_+\right)^{\frac{\gamma+p}{2}2^*} \leq (c'')^{2^*}\left(\frac{\gamma+p}{2}\right)^{2\cdot 2^*}\left[G\left((|Du|-1)_+\right)\right]^{2^*}, \quad \forall\,\gamma \geq 0,$$

and by (2.21) finally for every $\gamma \geq 0$ we obtain

$$\left\{\int_\Omega \eta^{2^*}\left[1+(|Du|-1)_+\right]^{\frac{\gamma+p}{2}2^*}\,dx\right\}^{\frac{2}{2^*}}$$

$$
\begin{aligned}
&\leq \; (c'')^2 \left(\frac{\gamma+p}{2}\right)^4 \left\{ \int_\Omega [\eta G((|Du|-1)_+)]^{2^*} \, dx \right\}^{\frac{2}{2^*}} \\
&\leq \; C\, (\gamma+1)^6 \, (1+\|h\|^2_{L^r(\Omega)}) \left[\int_\Omega (\eta^2+|D\eta|^2)^m [1+(|Du|-1)_+]^{(\gamma+p)m} \, dx \right]^{\frac{1}{m}} .
\end{aligned}
$$

For $\rho < R < \rho+1$ consider a function η with $\eta = 1$ in B_ρ, $\operatorname{supp}\eta \subset B_R$ and such that $|D\eta| \leq \frac{2}{(R-\rho)}$; we obtain

$$
\begin{aligned}
&\left[\int_{B_\rho} [1+(|Du|-1)_+]^{[(\gamma+p)m]\frac{2^*}{2m}} \, dx \right]^{\frac{2m}{2^*}} \\
&\leq \; C_0 \, (\|1+h\|_{L^r(\Omega)})^{2m} \frac{(\gamma+p)^{6m}}{(R-\rho)^{2m}} \int_{B_R} [1+(|Du|-1)_+]^{(\gamma+p)m} \, dx \,, \qquad (2.23)
\end{aligned}
$$

where the constant C_0 only depends on n, r, p, M_1, M_2 but is independent of γ.

Fixed $0 < \rho_0 < R_0 \leq \rho_0 + 1$, we define the decreasing sequence of radii $\{\rho_k\}_{k\geq 1}$

$$
\rho_k = \rho_0 + \frac{R_0-\rho_0}{2^k} \qquad \forall k \geq 1.
$$

We define recursively a sequence α_k in the following way

$$
\alpha_1 := 0 \qquad \alpha_{k+1} := (\alpha_k + pm)\frac{2^*}{2m} - pm = \alpha_k \frac{2^*}{2m} + p\left(\frac{2^*}{2} - m\right). \qquad (2.24)
$$

The following representation formula for α_k can be easily proved by induction

$$
\alpha_k = pm \left[\left(\frac{2^*}{2m}\right)^{k-1} - 1 \right], \qquad (2.25)
$$

so that

$$
(\alpha_k + pm)\frac{2^*}{2m} = pm \left(\frac{2^*}{2m}\right)^k .
$$

We rewrite (2.23) with $R = \rho_k$, $\rho = \rho_{k+1}$, $\gamma = \frac{\alpha_k}{m}$ and we observe that

$$
R - \rho := \rho_k - \rho_{k+1} = \frac{R_0-\rho_0}{2^{k+1}}.
$$

For all $k \geq 1$, denote by

$$
A_k := \left(\int_{B_{\rho_k}} [1+(|Du|-1)_+]^{\alpha_k+pm} \, dx \right)^{1/(\alpha_k+pm)}
$$

$$
C_k := C_0 \, (\|1+h\|_{L^r(\Omega)})^{2m} \left(\frac{(\alpha_k+pm)^3 2^{k+1}}{R_0-\rho_0} \right)^{2m} ;
$$

for every $k \geq 1$, (2.23) implies $A_{k+1} \leq C_k^{1/(\alpha_k+pm)} A_k$. By iteration, we deduce that there exist $\tilde\beta$ and $\tilde C$ such that

$$
A_{k+1} \leq \tilde C \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R_0-\rho_0)} \right]^{\tilde\beta} A_1 \qquad k \geq 1, \qquad (2.26)
$$

such that

$$
\tilde C := C \exp\left[\sum_{i=1}^{\infty} \frac{\log[(\alpha_i+pm)^{6m} 2^{2m(i+1)}]}{\alpha_i + pm} \right] < +\infty
$$

which is finite because the series is convergent (α_i from the representation formula (2.25) grows exponentially) and

$$\tilde{\beta} := \sum_{i=1}^{\infty} \frac{2m}{\alpha_i + pm} = \frac{2}{p} \sum_{i=0}^{\infty} \left(\frac{2m}{2^*} \right)^i = \frac{2}{p} \frac{1}{\left(1 - \frac{2m}{2^*}\right)}.$$

By letting $k \to +\infty$ in (2.26), we have (2.5).

The a-priori estimate (2.4) follows by the classical interpolation inequality

$$\|v\|_{L^s(B_\rho)} \leq \|v\|_{L^p(B_\rho)}^{\frac{p}{s}} \|v\|_{L^\infty(B_\rho)}^{1-\frac{p}{s}}, \tag{2.27}$$

for any $s \geq p$, which permits to estimate the essential supremum of the gradient of the local minimizer in terms of its L^p-norm.

Let us denote

$$V(x) := 1 + (|Du|(x) - 1)_+ \tag{2.28}$$

then estimate (2.5) becomes

$$\|V\|_{L^\infty(B_\rho)} \leq C \left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \right]^{\tilde{\beta}} \|V\|_{L^{pm}(B_R)} \tag{2.29}$$

for every ρ, R such that $0 < \rho < R \leq \rho + 1$ and where $C = C(n, r, p, M_1, M_2)$. By applying at this point, (2.29) and (2.27) give

$$\|V\|_{L^{pm}(B_\rho)} \leq C^{1-\frac{1}{m}} \|V\|_{L^p(B_\rho)}^{\frac{1}{m}} \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \right]^{\tilde{\beta}} \|V\|_{L^{pm}(B_R)} \right)^{\left(1-\frac{1}{m}\right)}. \tag{2.30}$$

We observe that, since $m > 1$

$$\tau := \left(1 - \frac{1}{m}\right) < 1. \tag{2.31}$$

For $0 < \rho < R$ and for every $k \geq 0$, let us define $\rho_k := R - (R-\rho)2^{-k}$. By inserting in (2.30) $\rho = \rho_k$ and $R = \rho_{k+1}$, (so that $R - \rho = (R-\rho)2^{-(k+1)}$) we have, for every $k \geq 0$

$$\|V\|_{L^{pm}(B_{\rho_k})} \leq C^{1-\frac{1}{m}} \|V\|_{L^p(B_{\rho_k})}^{\frac{1}{m}} \left(2^{\tilde{\beta}(k+1)} \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \right]^{\tilde{\beta}} \|V\|_{L^{pm}(B_{\rho_{k+1}})} \right)^{\tau}. \tag{2.32}$$

By iteration of (2.32), we deduce for $k \geq 0$

$$\begin{aligned} \|V\|_{L^{pm}(B_{\rho_0})} &\leq \left(C^{1-\frac{1}{m}} \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \right]^{\tilde{\beta}\tau} \|V\|_{L^p(B_{\rho_k})}^{\frac{1}{m}} \right)^{\sum_{i=0}^{k} \tau^i} \\ &\quad \times 2^{\tilde{\beta} \sum_{i=0}^{k+1} i\tau^i} \left(\|V\|_{L^{pm}(B_{\rho_{k+1}})}\right)^{\tau^{k+1}}. \end{aligned} \tag{2.33}$$

By (2.31), the series in (2.33) are convergent. Since

$$\|V\|_{L^{pm}(B_{\rho_k})} \leq \|V\|_{L^{pm}(B_R)},$$

we can pass to the limit as $k \to +\infty$ and we obtain for every $0 < \rho < R$ with a constant $C = C(n, r, p, M_1, M_2)$ independent of k

$$\|V\|_{L^{pm}(B_\rho)} \leq C \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \right]^{\tilde{\beta}\tau} \|V\|_{L^p(B_R)}^{\frac{1}{m}} \right)^m, \tag{2.34}$$

as $1 - \tau = \frac{1}{m}$. Combining (2.29) and (2.34), by setting $\rho' = \frac{(R+\rho)}{2}$ we have

$$\|V\|_{L^\infty(B_\rho)} \leq C \left[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)} \right]^{\tilde{\beta}} \|V\|_{L^{pm}(B_{\rho'})}$$

$$\leq \; C \left(\left[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)} \right]^{\frac{\tilde\beta}{m}} \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho')} \right]^{\tilde\beta\left(1-\frac{1}{m}\right)} \|V\|_{L^p(B_R)}^{\frac{1}{m}} \right)^m ;$$

now, since $(\rho'-\rho) = (R-\rho') = (R-\rho)/2$, we get

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \right]^{\beta} \left(\int_{B_R} \{1+|Du|^p\}\, dx \right)^{1/p},$$

where

$$\beta := \tilde\beta m = \frac{2^*}{p\left(\frac{2^*}{2m}-1\right)}.$$

so (2.4) follows. □

Above we applied the following technical lemma whose proof is in [15].

Lemma 2.2. *Let $\mu_0 > 0$. There exist constants c' and c'', depending on μ_0 but independent of $\mu \geq \mu_0$ and of $t \geq 0$, such that*

$$(1+t)^\mu \leq c' \frac{\mu^2}{\log(1+\mu)} \left(1 + \int_0^t (1+s)^{\mu-2}\, s\, ds \right), \tag{2.35}$$

$$(1+t)^\mu \leq c'' \mu^2 \left(1 + \int_0^t (1+s)^{\mu-2}\, s\, ds \right), \tag{2.36}$$

for every $\mu \in [\mu_0, +\infty)$ and every $t \in [0, +\infty)$.

3. Regularity

First of all we state an approximation theorem for f through a suitable sequence of regular functions. Let B be the unit ball of $\mathbb{R}^n$ centered in the origin and consider a positive decreasing sequence $\varepsilon_\ell \to 0$. Define

$$f^\ell(x,\xi) = \int_{B\times B} \rho(y)\rho(\eta) f(x+\varepsilon_\ell y, \xi+\varepsilon_\ell \eta)\, d\eta\, dy,$$

where ρ is a suitable symmetric mollifier, and set

$$f^{\ell k}(x,\xi) = f^\ell(x,\xi) + \frac{1}{k}(1+|\xi|^2)^{\frac{p}{2}}. \tag{3.1}$$

Proposition 3.1. *Let f be satisfying the growth conditions* (2.2) *with $M_0 = 1$ and f strictly convex at infinity. Then the sequence of $\mathcal{C}^2-$functions $f^{\ell k} : \Omega \times \mathbb{R}^n \to [0, +\infty)$ defined in* (3.1)*, convex in the last variable and strictly convex at infinity, is such that $f^{\ell k}$ converges to f as $\ell \to \infty$ and $k \to \infty$ for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and uniformly in $\Omega_0 \times K$ where $\Omega_0 \subset\subset \Omega$ and K being a compact set of $\mathbb{R}^n$. Moreover:*

- *there exists $\tilde C$, independently of k, ℓ such that*

$$|\xi|^p \leq f^{\ell k}(x,\xi) \leq \tilde C(1+|\xi|^p) \qquad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n, \tag{3.2}$$

- *there exists $\tilde M_1 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$*

$$\tilde M_1 |\xi|^{p-2} |\lambda|^2 \leq \sum_{i,j} f^{\ell k}_{\xi_i \xi_j}(x,\xi) \lambda_i \lambda_j \qquad \lambda \in \mathbb{R}^n, \tag{3.3}$$

- *there exists $c(k) > 0$ such that for all $(x,\xi) \in \Omega \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$*

$$c(k)(1+|\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \sum_{i,j} f^{\ell k}_{\xi_i \xi_j}(x,\xi) \lambda_i \lambda_j, \tag{3.4}$$

- *there exists $\tilde M_2 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$*

$$|f^{\ell k}_{\xi\xi}(x,\xi)| \leq \tilde M |\xi|^{p-2} \tag{3.5}$$

- *there exists $C(k)$ such that for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^n$*

$$|f^{\ell k}_{\xi\xi}(x,\xi)| \le C(k)(1+|\xi|^2)^{\frac{p-2}{2}} \tag{3.6}$$

- *there exists a constant $C > 0$ such that for a.e. $x \in \Omega$ and $|\xi| > 2$*

$$|f^{\ell k}_{\xi x}(x,\xi)| \le C\, h_\ell(x)|\xi|^{p-1} \tag{3.7}$$

where $h_\ell \in \mathcal{C}^\infty(\Omega)$ is the regularized function of h which converges to h in $L^r(\Omega)$

- *for $\Omega_0 \subset\subset \Omega$, there exists a constant $C(h,\Omega_0)$ such that for a.e. $x \in \Omega_0$ and $\xi \in \mathbb{R}^n$*

$$|f^{\ell k}_{\xi x}(x,\xi)| \le C(k,\Omega_0,\ell)(1+|\xi|^2)^{\frac{p-1}{2}}. \tag{3.8}$$

The proof follows with a similar argument as in [13] (see also [15]).

We are ready to prove the main result of the paper.

Theorem 3.2. *Let $p > 1$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the energy integral* (1.2) *under the growth assumptions* (2.2)*. Then u is locally Lipschitz continuous in Ω and we have*

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C\, \left[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\right]^\beta \left(\int_{B_R} \{1+|Du|^p\}\, dx\right)^{1/p}, \tag{3.9}$$

for some positive constants C, β (depending on n, r, p M_0, M_1, M_2) and for every ρ, R, with $\rho < R \le \rho+1$ and $B_{\rho+1} \subset \Omega$.

Proof. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional (1.2). Let $B_R \subset\subset \Omega$ and consider the following variational problem

$$\inf\left\{\int_{B_R} f^{\ell k}(x,Dv)\, dx,\ \ v \in W^{1,p}_0(B_R)+u\right\}, \tag{3.10}$$

where $f^{\ell k}$ are defined in (3.1). By semicontinuity arguments, there exists $v^{\ell k} \in u + W^{1,p}_0(\Omega)$ solution to (3.10). By the growth conditions and the minimality of $v^{\ell k}$, we get

$$\begin{aligned}\int_{B_R} |Dv^{\ell k}|^p\, dx &\le \int_{B_R} f^{\ell k}(x,Dv^{\ell k})\, dx \le \int_{B_R} f^{\ell k}(x,Du)\, dx\\ &= \int_{B_R} f^\ell(x,Du)\, dx + \frac{1}{k}\int_{B_R}(1+|Du|^2)^{\frac{p}{2}}\, dx.\end{aligned}$$

Moreover the properties of the convolutions imply that $f^\ell(x,Du) \stackrel{\ell\to\infty}{\to} f(x,Du)$ a.e. in B_R and since

$$\int_{B_R} f^\ell(x,Du)\, dx \le C \int_{B_R}(1+|Du|^2)^{\frac{p}{2}}\, dx,$$

by the Lebesgue Dominated Convergence Theorem we deduce therefore

$$\lim_{\ell\to\infty}\int_{B_R}|Dv^{\ell k}|^p\, dx = \int_{B_R} f(x,Du)\, dx + \frac{1}{k}\int_{B_R}(1+|Du|^2)^{\frac{p}{2}}\, dx.$$

By Proposition 3.1, $f^{\ell k}$ satisfy the growth conditions (2.2) and (2.3), so we can apply the a-priori estimate (2.4) to $v^{\ell k}$ and obtain

$$\|Dv^{\ell k}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C\, \left[\|1+h_\ell\|_{L^r(B_R)}\right]^\beta \left[\int_{B_R}(1+f^{\ell k}(x,Dv^{\ell k}))\, dx\right]^{\frac{1}{p}},$$

where C depends on $p, r, n, M_1, M_2, \rho, R$ but is independent of ℓ, k.

Since $\|1+h_\ell\|_{L^r(B_R)} = \|(1+h)_\ell\|_{L^r(B_R)} \le \|1+h\|_{L^r(B_R)}$, we obtain

$$\|Dv^{\ell k}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C\, \left[\|1+h\|_{L^r(B_R)}\right]^\beta \left[\int_{B_R} 1+f^\ell(x,Du)+\frac{1}{k}(1+|Du|^2)^{\frac{p}{2}} dx\right]^{\frac{1}{p}},$$

where C depends on $n, r, p, M_1, M_2, \rho, R$ but it is independent of ℓ, k. Therefore we conclude that

$$v^{\ell k} \stackrel{\ell\to\infty}{\to} v^k \text{ weakly in } W_0^{1,p}(B_R)+u, \qquad v^{\ell k} \stackrel{\ell\to\infty}{\to} v^k \text{ weakly star in } W^{1,\infty}_{\mathrm{loc}}(B_R).$$

Moreover by the previous estimates

$$\|Dv^k\|_{L^p(B_R;\mathbb{R}^n)} \leq \liminf_{\ell\to\infty} \|Dv^{\ell k}\|_{L^p(B_R;\mathbb{R}^n)} \leq \int_{B_R} f(x,Du)\,dx + \int_{B_R} (1+|Du|^2)^{\frac{p}{2}}\,dx$$

and, for $0<\rho<R$

$$\begin{aligned} \|Dv^k\|_{L^\infty(B_\rho;\mathbb{R}^n)} &\leq \liminf_{\ell\to\infty} \|Dv^{\ell k}\|_{L^\infty(B_\rho;\mathbb{R}^n)} \\ &\leq C\,\left[\|1+h\|_{L^r(B_R)}\right]^\beta \left[\int_{B_R} \{1+f(x,Du)\}\,dx\right]. \end{aligned}$$

Thus we can deduce that there exists $\bar v \in u + W_0^{1,p}(B_R)$ such that, up to subsequences

$$v^k \to \bar v \text{ weakly in } W_0^{1,p}(B_R)+u, \qquad v^k \to \bar v \text{ weakly star in } W^{1,\infty}_{\mathrm{loc}}(B_R)$$

and, for $0<\rho<R$

$$\|D\bar v\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C\,\left[\|1+h\|_{L^r(B_R)}\right]^\beta \left[\int_{B_R} \{1+f(x,Du)\}\,dx\right]. \tag{3.11}$$

Now, for any fixed $k\in\mathbb{N}$, using the uniform convergence of f^ℓ to f in $B_\rho \times K$ (for any K compact subset of $\mathbb{R}^n$) and the minimality of $v^{\ell k}$, we get

$$\begin{aligned} \int_{B_\rho} f(x,Dv^k)\,dx &\leq \liminf_{\ell\to\infty} \int_{B_\rho} f(x,Dv^{\ell k})\,dx = \liminf_{\ell\to\infty} \int_{B_\rho} f^\ell(x,Dv^{\ell k})\,dx \\ &\leq \liminf_{\ell\to\infty} \int_{B_\rho} f^\ell(x,Dv^{\ell k})\,dx + \frac{1}{k}\int_{B_R} (1+|Dv^{\ell k}|^2)^{\frac{p}{2}}\,dx \\ &\leq \liminf_{\ell\to\infty} \int_{B_R} f^\ell(x,Du)\,dx + \frac{1}{k}\int_{B_R} (1+|Du|^2)^{\frac{p}{2}}\,dx. \end{aligned}$$

Then, for $\rho\to R$

$$\int_{B_R} f(x,Dv^k)\,dx \leq \int_{B_R} f(x,Du)\,dx + \frac{1}{k}\int_{B_R} (1+|Du|^2)^{\frac{p}{2}}\,dx.$$

By the semicontinuity we get

$$\int_{B_R} f(x,D\bar v)\,dx \leq \liminf_{k\to\infty} \int_{B_R} f(x,Dv^k)\,dx \leq \int_{B_R} f(x,Du)\,dx. \tag{3.12}$$

Then u and $\bar v$ are two solutions to the problem

$$\inf\left\{\int_{B_R} f(x,Dv)\,dx, \;\; v\in W_0^{1,p}(B_R)+u\right\}.$$

By the strict convexity of f at infinity and by proceeding in an similar way as in [13], (see also [17]), we can conclude that also the gradient of Du is locally bounded and the estimate (3.9) follows by (3.11). $\square$

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