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Abstract We prove a lower semicontinuity result for polyconvex functionals of the Calculus of Variations along sequences of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ in $W^{1,m}$, $2 \leq m \leq n$, bounded in $W^{1,m-1}$ and convergent in L^1 under mild technical conditions but without any extra coercivity assumption on the integrand.

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1 Introduction

Let n, m be positive integers, let Ω be an open set of $\mathbb{R}^n$ and let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a *map* in the *Sobolev space* $W^{1,p}(\Omega, \mathbb{R}^m)$ for some $p \geq 1$. We denote by ∇u the *gradient* of the

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map u, i.e., the $m \times n$ matrix of the first derivatives of u. The *energy functional* associated to the map u is an integral of the type

$$F(u) = \int_\Omega f(\mathcal{M}^\ell(\nabla u(x)))\, dx\,, \tag{1.1}$$

where $\ell = m \wedge n$ and $\mathcal{M}^\ell(\mathbb{A})$ denotes the vector whose components are all the *minors* of order up to ℓ of the gradient matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, i.e.,

$$\mathcal{M}^\ell(\mathbb{A}) = \left(\mathbb{A}, \mathrm{adj}_2\mathbb{A}, \ldots, \mathrm{adj}_i\mathbb{A}, \ldots, \mathrm{adj}_\ell\mathbb{A}\right).$$

For instance, $\mathcal{M}^1(\mathbb{A}) = \mathbb{A}$ if $\ell = 1$, while if $\ell = m = n$ the "*last" component* of $\mathcal{M}^\ell(\mathbb{A})$ is the *determinant* $\det \mathbb{A}$ of the matrix $\mathbb{A}$.

Energy functionals as in (1.1) are considered in *nonlinear elasticity*, when $\ell = m = n = 3$; in particular $\det \nabla u$ takes into account the contribution to the energy given by *changes of volume* of the deformation u. The integrand f in (1.1) is assumed to be a convex function; this makes the integral F consistent with the theory of *polyconvex* and *quasiconvex* integrals (see Morrey [24], Ball [5], see also the book by Dacorogna [7]). We assume that f is bounded below, say $f(\mathcal{M}^\ell(\mathbb{A})) \geq 0$ for all $\mathbb{A} \in \mathbb{R}^{m\times n}$.

To fix the ideas let us assume $m = n \geq 2$. Then well-known results by Morrey [24] (see also Acerbi-Fusco [2] and Marcellini [22]) imply that the functional in (1.1) is lower semicontinuous with respect to the weak convergence in $W^{1,n}(\Omega, \mathbb{R}^n)$.

The modelling of *cavitation phenomena* forces then naturally to consider maps in the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^m)$ for $p < n$. For instance, deformation maps of the type

$$u(x) = v(|x|)\frac{x}{|x|}\,, \tag{1.2}$$

with $v : [0, \infty) \to \mathbb{R}$ an increasing smooth function and $v(0) > 0$, belong to $W^{1,p}(\Omega, \mathbb{R}^n)$ for every $p < n$, but not to $W^{1,n}(\Omega, \mathbb{R}^n)$. An extension of the energy functional outside the space $W^{1,n}(\Omega, \mathbb{R}^n)$ is needed in this case. Different choices have been investigated. Referring to the prototype case of the determinant of ∇u in (1.2) (see Ball [5], Fonseca et al. [11,12], Giaquinta et al. [20] and Müller [25]), we recall the so called *distributional determinant* $\mathrm{Det}\nabla u$ as opposed to the *total variation* of the determinant, i.e.,

$$TV^p(u, \Omega) := \inf\left\{\liminf_j \int_\Omega |\det \nabla u_j|\, dx :\ (u_j)_j \subset W^{1,n},\ u_j \rightharpoonup u \text{ in } W^{1,p}\right\}.$$

TV^p is an extension of the orginal integral, i.e.,

$$TV^p(u, \Omega) = \int_\Omega |\det \nabla u|\, dx \quad \text{for} \quad u \in W^{1,n},$$

if and only if for every sequence $(u_j)_j \subset W^{1,n}$ converging weakly to u in $W^{1,p}$

$$\int_\Omega |\det \nabla u|\, dx \leq \liminf_j \int_\Omega |\det \nabla u_j|\, dx. \tag{1.3}$$

The lower semicontinuity inequality as in (1.3) for general integrands is the object of investigation in Theorem 1.1 below under the weak convergence in $W^{1,p}$ for $p < n$. Marcellini observed in [23] that the lower semicontinuity inequality still holds below the critical exponent n. Later on Dacorogna and Marcellini [8] proved the lower semicontinuity for $p > n-1$ (see also [15]), while Malý [21] exhibited a counterexample in the case $p < n-1$. Finally, the limit case $p = n-1$ was addressed by Acerbi and Dal Maso [1], Dal Maso and Sbordone [9], Celada and Dal Maso [6] and Fusco and Hutchinson [18]. In particular, in [6] the integrand f can be any nonnegative convex function with no coercivity assumptions.

The situation significantly changes when an explicit dependence either on x and/or on u is also allowed, since the presence of these variables cannot be treated as a simple perturbation. Results in this context are due to Gangbo [19], under a structure assumption, and to Fusco and Hutchinson [18] and Fonseca and Leoni [13], assuming the coercivity of the integrand. More recently, Amar et al. [3] studied the non-coercive case with $(x, u)-$dependence on the integrand, assuming the strict inequality $p > n-1$.

In this paper we deal with the limit case $p = n-1$ and consider integrals of the general form

$$F(u) = \int_{\Omega} f\left(x, u(x), \mathcal{M}^n(\nabla u(x))\right) dx.$$

Other new issues here are that we do not assume coerciveness of the integrand and we allow the dependence on lower order variables and minors.

Theorem 1.1 *Let $m = n \geq 2$ and $f = f(x, u, \xi) : \Omega \times \mathbb{R}^n \times \mathbb{R}^\sigma \to [0, \infty)$ be such that*

(i) $f \in C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^\sigma)$ and $f(x, u, \cdot)$ is convex for all (x, u);
(ii) denoting $\xi = (z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$, if $f(x_0, u_0, z_0, \cdot)$ is constant for some point $(x_0, u_0, z_0) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{\sigma-1}$, then for all $(z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$

$$f(x_0, u_0, z, t) = \inf\left\{f(y, v, z, s) : (y, v, s) \in \Omega \times \mathbb{R}^n \times \mathbb{R}\right\}. \tag{1.4}$$

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ satisfying

$$u_j \to u \text{ in } L^1 \text{ and } \sup_j \|u_j\|_{W^{1,n-1}} < \infty \tag{1.5}$$

we have

$$F(u) \leq \liminf_j F(u_j). \tag{1.6}$$

Note that (1.5) assures that $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ for $n \geq 3$, and $u \in BV(\Omega, \mathbb{R}^2)$ if $n = 2$; in the latter case ∇u denotes the *approximate gradient* of u (see [4, Theorem 3.83]). Let us point out that assumption (ii) is automatically satisfied both in the autonomous case $f = f(\xi)$ and in the following coercive case $f = f(x, u, \xi) \geq c\,|t|$, with $c > 0$ and $\xi = (z, t)$. This is the content of Corollary 2.3 in the former case, while in the second just note that no such point (x_0, u_0, z_0) exists.

To highlight the main ideas of our strategy we choose to study first autonomous functionals as in (1.1) for which the proof simplifies. Semicontinuity actually holds in a more general setting, i.e. under the assumption $m \leq n$ (see Theorem 3.1). On the other hand, we are not able to remove the coerciveness assumption when $m > n$ (see Propositions 3.3 and 3.5), a difficulty which was also present in the papers [6], [3] (details are in Remark 2.9). Further results for $m \leq n$ are discussed for some special integrands (see Sect. 3).

The main tools used in the proof of Theorem 1.1 are: (i) the De Giorgi's approximation theorem for convex integrands; (ii) a suitable generalization of a truncation lemma by Fusco and Hutchinson [18] (see Proposition 2.8); (iii) a measure-theoretic lemma by Celada and

Dal Maso [6] (see Lemma 2.5). All these tools are carefully combined in an argument which exploits assumption (1.4) in the statement above and the blow-up technique.

A resume of the contents of the paper is described next briefly. Section 2 is devoted to some technical results instrumental in the rest of the paper: we state a version of De Giorgi's celebrated approximation theorem, we prove some results concerning truncation for minors and a blow-up type lemma that will be repeatedly employed to reduce ourselves to target affine functions. In Sect. 3 we deal with the model case of autonomous integrands for dimensions $m \leq n$ and related results. Finally, in Sect. 4, we provide the proof of Theorem 1.1.

The research presented in this paper took origin by the work of two different groups, one in Firenze and the other one in Napoli. Before Summer 2012 the two groups independently reached quite similar results. Then our colleague Bernard Dacorogna, talking separately with some of us, pointed out the similarities. What to do: *cooperation or competition?* We decided to continue to study together the problem, and this is the reason why the paper has six authors.

2 Definitions and preliminary results

The aim of this section is to introduce some notations and to recall some basic definitions and results which will be used in the sequel.

We begin with some algebraic notation.

Let $n, m \geq 2$ and $\mathbb{M}^{m\times n}$ be the linear space of all $m \times n$ real matrices. For $\mathbb{A} \in \mathbb{M}^{m\times n}$, we denote $\mathbb{A} = (\mathbb{A}^i_j)$, $1 \leq i \leq m$, $1 \leq j \leq n$, where upper and lower indices correspond to rows and columns respectively.

The euclidean norm of $\mathbb{A}$ will be denoted by $|\mathbb{A}|$. The number of all minors of any matrix in $\mathbb{M}^{m\times n}$ is given by

$$\sigma := \sum_{i=1}^{m\wedge n} \binom{m}{i}\binom{n}{i}.$$

We shall also adopt the following notations. We set $I_{l,k} = \{\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l : 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_l \leq k\}$, where $1 \leq l \leq k$. If $\alpha \in I_{l,m}$ and $\beta \in I_{l,n}$, then $M_{\alpha,\beta}(\mathbb{A}) = \det(\mathbb{A}^{\alpha_i}_{\beta_j})$.

By $\mathcal{M}_l(\mathbb{A})$ we denote the vector whose components are all the minors of order l, and by $\mathcal{M}^l(\mathbb{A})$ the vector of all minors of order up to l.

As usual, $Q_r(x)$, $B_r(x)$ denote the open euclidean cube, ball in $\mathbb{R}^n$, $n \geq 2$, with side r, radius r and center the point x, respectively. The center shall not be indicated explicitly if it coincides with the origin.

We shall often deal in what follows with convergences of measures. As usual, we shall name local weak * convergence of Radon measures the one defined by duality with $C_c(\Omega)$, and weak * convergence the one defined by duality with $C_0(\Omega)$ on the subset of finite Radon measures.

2.1 Approximation of convex functions

We survey now on an approximation theorem for convex functions, due to De Giorgi, that plays an important role in the framework of lower semicontinuity problems (see [10]). Given a convex function $f : \mathbb{R}^\sigma \to \mathbb{R}$, $\sigma \geq 1$ a natural number, consider the affine functions $\xi \to a_i + \langle b_i, \xi\rangle$, with $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}^\sigma$, given by

$$a_i := \int_{\mathbb{R}^\sigma} f(\eta)\big((\sigma+1)\alpha_i(\eta) + \langle \nabla\alpha_i(\eta), \eta\rangle\big) d\eta \tag{2.1}$$

$$b_i := -\int_{\mathbb{R}^\sigma} f(\eta)\nabla\alpha_i(\eta) d\eta, \tag{2.2}$$

where, for all $i \in \mathbb{N}$, $\alpha_i(\xi) := i^\sigma \alpha(i(q_i - \xi))$, $(q_i)_i = \mathbb{Q}^\sigma$ and $\alpha \in C_0^1(\mathbb{R}^\sigma)$ is a non negative function such that $\int_{\mathbb{R}^\sigma} \alpha(\eta) d\eta = 1$.

Lemma 2.1 *Let $f : \mathbb{R}^\sigma \to \mathbb{R}$ be a convex function and a_i, b_i be defined as in (2.1)-(2.2). Then,*

$$f(\xi) = \sup_{i\in\mathbb{N}} (a_i + \langle b_i, \xi\rangle), \quad \textit{for all } \xi \in \mathbb{R}^\sigma.$$

The main feature of the approximation in Lemma 2.1 above is the explicit dependence of the coefficients a_i and b_i on f. In particular, if f depends on the lower order variables (x, u) regularity properties of the coefficients a_i and b_i with respect to (x, u) can be easily deduced from related hypotheses satisfied by f thanks to formulas (2.1) and (2.2) and Lemma 2.1.

In particular, the following approximation result holds.

Theorem 2.2 *Let $f = f(x, u, \xi) : \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma \to [0, \infty)$, be a continuous function, convex in the last variable ξ. Then, there exist two sequences of compactly supported continuous functions $a_i : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $b_i : \Omega \times \mathbb{R}^m \to \mathbb{R}^\sigma$ such that, setting for every $i \in \mathbb{N}$,*

$$f_i(x, u, \xi) := (a_i(x, u) + \langle b_i(x, u), \xi\rangle)^+,$$

then

$$f(x, u, \xi) = \sup_i f_i(x, u, \xi).$$

Moreover, for every $i \in \mathbb{N}$ there exists a positive constant C_i such that

(a) f_i is continuous, convex in ξ and

$$0 \le f_i(x, u, \xi) \le C_i(1 + |\xi|) \quad \textit{for all } (x, u, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma; \tag{2.3}$$

(b) if ω_i denotes a modulus of continuity of $a_i + |b_i|$ we have

$$|f_i(x, u, \xi) - f_i(y, v, \eta)| \le C_i|\xi - \eta| + \omega_i(|x - y| + |u - v|)(1 + |\xi| \wedge |\eta|) \tag{2.4}$$

for all (x, u, ξ) and $(y, v, \eta) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma$.

The compactness of the supports of a_i and b_i is obtained by first approximating f with a monotone sequence $f_j(x, u, \xi) := m_j(x, u) f(x, u, \xi)$, where $m_j \in C_c(\Omega \times \mathbb{R}^m)$, $m_j = 1$ on $\Omega_j \times \{|u| < j\}$ and $m_j = 0$ on $\Omega \times \mathbb{R}^m \setminus (\Omega_{j+1} \times \{|u| < j+1\})$, $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ a family of open sets exhausting Ω; and then applying to each f_j De Giorgi's approximation result in Lemma 2.1.

Finally, by virtue of Lemma 2.1 we can show that the technical assumption stated in (1.4) of Theorem 1.1 is satisfied by autonomous integrands.

Corollary 2.3 *Let $f = f(\xi) : \mathbb{R}^\sigma \to \mathbb{R}$ be a convex function, and denote $\xi = (z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$. If $\mathbb{R} \ni t \mapsto f(z_0, t)$ is bounded from above for some $z_0 \in \mathbb{R}^{\sigma-1}$, then $f(z, t) = f(z)$ for all $(z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$, i.e. f does not depend on t.*

Proof By Lemma 2.1 $f = \sup_{i\in\mathbb{N}} f_i$, where $f_i(\xi) := a_i + \langle\beta_i, z\rangle + \gamma_i\, t$, for constants a_i, $\gamma_i \in \mathbb{R}$ and a vector $\beta_i \in \mathbb{R}^{\sigma-1}$. Then, if $f(z_0, \cdot)$ is bounded from above, necessarily $\gamma_i = 0$ for all $i \in \mathbb{N}$. In turn, this implies that $f_i(z,t) = f_i(z)$. The conclusion then follows at once. □

Remark 2.4 More generally, the thesis of Corollary 2.3 holds for lower semicontinuous convex functions $f : X \times Y \to \mathbb{R}$, X and Y two real locally convex topological vector spaces. Indeed, such a function f is the supremum of all continuous affine functionals below f itself.

2.2 A truncation method for minors

We first recall a lemma proved in [6, Lemma 3.2].

Lemma 2.5 *Let $(\mu_k)_k$ be a sequence of signed Radon measures on Ω. Assume that*

(a) there exists $T \in \mathcal{D}'(\Omega)$ such that $\mu_k \to T$ in the sense of distributions on Ω;
(b) there exists a positive Radon measure ν such that $\mu_k^+ \rightharpoonup \nu$ (locally) weakly in the sense of measures on Ω.*

Then, there exists a Radon measure μ such that $T = \mu$ on Ω and $\mu_k \rightharpoonup T$ locally weakly in the sense of measures on Ω.*

An immediate consequence is the following corollary (see [18, Lemma 2.2], and also [6, Corollary 3.3] for $m = n = l$).

Corollary 2.6 *Let $2 \le l \le m \wedge n$, $u_k, u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ be maps such that*

(a) $(u_k)_k$ converges to u in $L^\infty(\Omega, \mathbb{R}^m)$;
(b) $\sup_k \|\mathcal{M}^{l-1}(\nabla u_k)\|_{L^1} < \infty$;
(c) there exists $c \in \mathbb{R}^\tau$, $\tau = \binom{m}{l}\binom{n}{l}$, such that

$$\sup_k \int_\Omega \langle c, \mathcal{M}_l(\nabla u_k)\rangle^+\, dx < \infty.$$

Then, $\mathcal{M}^{l-1}(\nabla u_k) \rightharpoonup \mathcal{M}^{l-1}(\nabla u)$ weakly and $\langle c, \mathcal{M}_l(\nabla u_k)\rangle \rightharpoonup \langle c, \mathcal{M}_l(\nabla u)\rangle$ locally weakly* in the sense of measures on Ω.*

Proof Passing to a subsequence, not relabeled for convenience, we may suppose that for any $\alpha \in I_{h,m}$, $\beta \in I_{h,n}$, $h \le l-1$

$$M_{\alpha,\beta}(\nabla u_k) \rightharpoonup \mu_{\alpha,\beta} \quad \text{weakly}^*\text{in the sense of measures.}$$

We claim that for every $\alpha \in I_{h,m}$, $\beta \in I_{h,n}$, $h \le l-1$ then

$$\mu_{\alpha,\beta} = M_{\alpha,\beta}(\nabla u)\mathcal{L}^n \llcorner \Omega.$$

To establish the case $h = 1$ fix a test function $\varphi \in C_c^\infty(\Omega)$ and $\alpha \in I_{1,m}$, $\beta \in I_{1,n}$, then in view of (a) and being u a Sobolev function we have

$$\int_\Omega \varphi\, d\mu_{\alpha,\beta} = \lim_k \int_\Omega \varphi \frac{\partial u_k^\alpha}{\partial x^\beta}\, dx = -\lim_k \int_\Omega u_k^\alpha \frac{\partial\varphi}{\partial x^\beta}\, dx = -\int_\Omega u^\alpha \frac{\partial\varphi}{\partial x^\beta}\, dx = \int_\Omega \varphi \frac{\partial u^\alpha}{\partial x^\beta}\, dx.$$

By induction, suppose that the result is true for any $\alpha \in I_{h-1,m}, \beta \in I_{h-1,n}$ with $2 \le h \le l-1$. Fix now $\alpha \in I_{h,m}$, $\beta \in I_{h,n}$ and a test function $\varphi \in C_c^\infty(\Omega)$, then

$$\begin{aligned}\int_\Omega \varphi\, d\mu_{\alpha,\beta} &= \lim_k \int_\Omega \varphi \det\left(\frac{\partial u_k^{\alpha_i}}{\partial x_{\beta_j}}\right) dx \\ &= -\lim_k \int_\Omega u_k^{\alpha_1} \det \frac{\partial(\varphi, u_k^{\alpha_2}, \dots, u_k^{\alpha_h})}{\partial(x_{\beta_1}, \dots, x_{\beta_h})} \\ &= -\lim_k \sum_{j=1}^h (-1)^{j-1} \int_\Omega u_k^{\alpha_1} \frac{\partial \varphi}{\partial x_{\beta_j}} M_{\hat{\alpha_1},\hat{\beta}_j}(\nabla u_k)\, dx,\end{aligned}$$

where $\hat{\alpha_1} = (\alpha_2, \dots, \alpha_h)$, $\hat{\beta_j} = (\beta_1, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_h)$. Since $u_k \to u$ in L^∞ and by induction assumption $M_{\hat{\alpha_1},\hat{\beta}_j}(\nabla u_k) \rightharpoonup M_{\hat{\alpha_1},\hat{\beta}_j}(\nabla u)$ weakly* in the sense of measures, we have

$$\int_\Omega \varphi\, d\mu_{\alpha,\beta} = -\lim_k \sum_{j=1}^h (-1)^{j-1} \int_\Omega u_k^{\alpha_1} \frac{\partial \varphi}{\partial x_{\beta_j}} M_{\hat{\alpha_1},\hat{\beta}_j}(\nabla u_k)\, dx = \int_\Omega \varphi M_{\alpha,\beta}(\nabla u)\, dx,$$

that gives the claim.

To prove that $(\langle c, \mathcal{M}_l(\nabla u_k)\rangle)_k$ converges to $\langle c, \mathcal{M}_l(\nabla u)\rangle$ locally weakly* in the sense of measures, we first observe that arguing as before, the induction assumption and conditions in items (a) and (b) ensure that $(\mathcal{M}_l(\nabla u_k))_k$ converge in $\mathcal{D}'(\Omega, \mathbb{R}^\tau)$ to $\mathcal{M}_l(\nabla u)$. In particular, $(\langle c, \mathcal{M}_l(\nabla u_k)\rangle)_k$ converges to $\langle c, \mathcal{M}_l(\nabla u)\rangle$ in $\mathcal{D}'(\Omega)$.

Then, using assumption (c), Lemma 2.5 yields that up to a subsequence $(\langle c, \mathcal{M}_l(\nabla u_k)\rangle)_k$ converges locally weakly* in the sense of measures to some Radon measure μ with $\mu = \langle c, \mathcal{M}_l(\nabla u)\rangle$.

Since the limit is independent of the extracted subsequence the convergence for the whole sequence follows immediately. □

Remark 2.7 If condition (c) above is strengthened to $\sup_k \|\mathcal{M}^l(\nabla u_k)\|_{L^1} < \infty$, [18, Lemma 2.2] establishes the weak* convergence in the sense of measures of $(\mathcal{M}^l(\nabla u_k))_k$ to $\mathcal{M}^l(\nabla u)$.

The next result is inspired by [18, Proposition 2.5].

Proposition 2.8 *Let $n \ge m \ge 2$ and let u_j be $W^{1,m}(\Omega, \mathbb{R}^m)$ functions and u in $W^{1,\infty}(\Omega, \mathbb{R}^m)$. Suppose that $u_j \to u$ in $L^1(\Omega, \mathbb{R}^m)$ and that for some $c \in \mathbb{R}^\sigma$*

$$\sup_j \left(\|\mathcal{M}^{m-1}(\nabla u_j)\|_{L^1} + \|\langle c, \mathcal{M}_m(\nabla u_j)\rangle\|_{L^1}\right) < \infty.$$

Then, there exists a sequence $(v_j)_j \in W^{1,m}(\Omega, \mathbb{R}^m)$ converging to u in $L^\infty(\Omega, \mathbb{R}^m)$, such that

$$\mathcal{M}^{m-1}(\nabla v_j) \rightharpoonup \mathcal{M}^{m-1}(\nabla u), \quad \langle c, \mathcal{M}_m(\nabla v_j)\rangle \rightharpoonup \langle c, \mathcal{M}_m(\nabla u)\rangle \tag{2.5}$$

weakly* in the sense of measures. Moreover, an infinitesimal sequence of positive numbers s_j exists such that

$$\{x \in \Omega : u_j(x) \neq v_j(x)\} \subseteq A_j := \{x \in \Omega : |u_j(x) - u(x)| > s_j\} \tag{2.6}$$

and

$$\lim_j \int_{A_j} \left(1 + |\mathcal{M}^{m-1}(\nabla v_j)| + |\langle c, \mathcal{M}_m(\nabla v_j)\rangle|\right) dx = 0. \tag{2.7}$$

Proof Let us first fix some notations that shall be used throughout this proof. If $\mathbb{A}$, $\mathbb{B}$ are in $\mathbb{M}^{m\times n}$ and $1 \le k \le m$ the components of $\mathcal{M}_k(\mathbb{A}+\mathbb{B})$ can be written as certain linear combinations of products of components of the vectors $\mathcal{M}_i(\mathbb{A})$ and $\mathcal{M}_{k-i}(\mathbb{B})$. We denote these linear combinations writing

$$\mathcal{M}_k(\mathbb{A}+\mathbb{B}) = \sum_{i=0}^{k} \mathcal{M}_i(\mathbb{A}) \odot \mathcal{M}_{k-i}(\mathbb{B}), \tag{2.8}$$

with the convention that $\mathcal{M}_0(\mathbb{A}) = \mathcal{M}_0(\mathbb{B}) = 1$.

Let $0 < s < t$ and denote by

$$\varphi_{s,t}(r) := \begin{cases} 1 & r \le s \\ \dfrac{t-r}{t-s} & s \le r \le t \\ 0 & r \ge t, \end{cases}$$

and by $\Phi_{s,t}$ the function

$$\Phi_{s,t}(y) := y\,\varphi_{s,t}(|y|) \quad \text{for all } y \in \mathbb{R}^m.$$

We set

$$v^j_{s,t} := u + \Phi_{s,t}(u_j - u).$$

Note that

$$\begin{aligned} &\mathcal{L}^n(\{x \in \Omega :\ v^j_{s,t} \ne u_j\}) \le \mathcal{L}^n(\{x \in \Omega :\ |u_j - u| \ge s\}) \le s^{-1}\|u_j - u\|_{L^1}, \\ &\|v^j_{s,t} - u\|_{L^\infty} \le t, \end{aligned} \tag{2.9}$$

and in addition

$$\nabla v^j_{s,t} = \nabla u + D\Phi_{s,t}(u_j - u) \circ (\nabla u_j - \nabla u)$$

where $D\Phi_{s,t}(y) = \varphi_{s,t}(|y|) Id + \varphi'_{s,t}(|y|) \frac{y \otimes y}{|y|}$. Note that if $1 \le i \le m$, and $\alpha, \lambda \in I_{i,m}$, since $y \otimes y$ is a rank one matrix one easily infers that

$$|M_{\alpha,\lambda}(D\Phi_{s,t}(y))| \le (\varphi_{s,t}(|y|))^i + |y||\varphi'_{s,t}(|y|)|(\varphi_{s,t}(|y|))^{i-1}. \tag{2.10}$$

Finally, we define

$$C_j := \{x \in \Omega :\ \nabla(|u_j - u|)(x) = 0\},$$

and it is easy to check that

$$\nabla v^j_{s,t}(x) = \nabla u(x) + \varphi_{s,t}(|u_j(x) - u(x)|)(\nabla u_j(x) - \nabla u(x))$$

for $\mathcal{L}^n$ a.e. x in C_j.

We now estimate the linear combination of the m-th order minors on the set $E^j_{s,t} := \{x \in \Omega :\ s < |u_j(x) - u(x)| < t\}$ by taking into account (2.8)

$$
\begin{aligned}
\int_{E^j_{s,t}} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\,dx &= \int_{E^j_{s,t}\cap C_j} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\,dx + \int_{E^j_{s,t}\setminus C_j} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\,dx \\
&\le \sum_{i=0}^{m} \int_{E^j_{s,t}\cap C_j} |\langle c, \mathcal{M}_{m-i}(\nabla u)\odot \mathcal{M}_i(\varphi_{s,t}(|u_j-u|)(\nabla u_j-\nabla u))\rangle|\,dx \\
&\quad + \sum_{i=0}^{m} \int_{E^j_{s,t}\setminus C_j} |\langle c, \mathcal{M}_{m-i}(\nabla u)\odot \mathcal{M}_i(D\Phi_{s,t}(u_j-u)\circ(\nabla u_j-\nabla u))\rangle|\,dx \\
&\le C\int_{E^j_{s,t}} \left(1+|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\right)\,dx \\
&\quad + C\int_{E^j_{s,t}\setminus C_j} \big(|\mathcal{M}^{m-1}(D\Phi_{s,t}(u_j-u)\circ(\nabla u_j-\nabla u))| \\
&\quad + |\langle c, \mathcal{M}_m(D\Phi_{s,t}(u_j-u)\circ(\nabla u_j-\nabla u))\rangle|\big)\,dx,
\end{aligned}
\tag{2.11}
$$

where $C = C(m, n, \|\nabla u\|_{L^\infty})$.

An elementary but lengthy algebraic computation shows that if $\mathbb{A} \in \mathbb{M}^{m\times m}$ and $\mathbb{B} \in \mathbb{M}^{m\times n}$ then

$$
\begin{aligned}
&|\mathcal{M}_i(\mathbb{A}\circ\mathbb{B})| \le \sum_{\alpha,\lambda\in I_{i,m}} \sum_{\beta\in I_{i,n}} |M_{\alpha,\lambda}(\mathbb{A})||M_{\lambda,\beta}(\mathbb{B})| \quad \text{for } 1 \le i \le m-1 \\
&\mathcal{M}_m(\mathbb{A}\circ\mathbb{B}) = (\det \mathbb{A})\mathcal{M}_m(\mathbb{B}).
\end{aligned}
$$

By taking into account this estimate and (2.10) we conclude that

$$
\begin{aligned}
&\int_{E^j_{s,t}\setminus C_j} |\mathcal{M}^{m-1}(D\Phi_{s,t}(u_j-u)\circ(\nabla u_j-\nabla u))|\,dx \\
&\qquad \le \int_{E^j_{s,t}\setminus C_j} \left(1+\frac{C}{t-s}|u_j-u|\right)|\mathcal{M}^{m-1}(\nabla u_j-\nabla u))|\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{E^j_{s,t}\setminus C_j} |\langle c, \mathcal{M}_m(D\Phi_{s,t}(u_j-u)\circ(\nabla u_j-\nabla u))\rangle|\,dx \\
&\qquad = \int_{E^j_{s,t}\setminus C_j} |\det(D\Phi_{s,t}(u_j-u))||\langle c, \mathcal{M}_m(\nabla u_j-\nabla u)\rangle|\,dx.
\end{aligned}
$$

Therefore, recalling (2.11) we get

$$
\begin{aligned}
\int_{E^j_{s,t}} |\langle c, \mathcal{M}_m(\nabla v^j_{s,t})\rangle|\,dx &\le C\int_{E^j_{s,t}} \left(1+|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\right)\,dx \\
&\quad + \frac{C}{t-s}\int_{E^j_{s,t}\setminus C_j} \left(|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c, \mathcal{M}_m(\nabla u_j)\rangle|\right)|u_j-u|\,dx.
\end{aligned}
$$

Repeating for the lower order minors the argument used to infer the previous estimate we get

$$\int_{E^j_{s,t}} \Big(|\mathcal{M}^{m-1}(\nabla v^j_{s,t})|+|\langle c,\mathcal{M}_m(\nabla v^j_{s,t})\rangle|\Big) \le C\int_{E^j_{s,t}} \Big(1+|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|\Big)\,dx$$
$$+\frac{C}{t-s}\int_{E^j_{s,t}\setminus C_j} \Big(|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|\Big)|u_j-u|\,dx. \tag{2.12}$$

To deal with the last integral in (2.12) we recall that $\nabla(|u_j-u|)\neq 0$ on $E^j_{s,t}\setminus C_j$, then we use the coarea formula to get for $\mathcal{L}^1$ a.e. $t>0$

$$\lim_{s\uparrow t}\frac{1}{t-s}\int_{\{x\in\Omega\setminus C_j:\, s<|u_j-u|<t\}} \big(|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|\big)|u_j-u|\,dx$$
$$=\lim_{s\uparrow t}\frac{1}{t-s}\int_s^t dr\int_{\{x\in\Omega\setminus C_j:\,|u_j-u|=r\}} \big(|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|\big)$$
$$\times\frac{|u_j-u|}{|\nabla(|u_j-u|)|}\,d\mathcal{H}^{n-1}$$
$$=t\int_{\{x\in\Omega\setminus C_j:\,|u_j-u|=t\}} \frac{|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|}{|\nabla(|u_j-u|)|}\,d\mathcal{H}^{n-1} \tag{2.13}$$

Let us now denote by C_0 a constant such that for all $j\in\mathbb{N}$

$$\int_0^\infty dr\int_{\{x\in\Omega\setminus C_j:\,|u_j-u|=r\}} \frac{|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|}{|\nabla(|u_j-u|)|}\,d\mathcal{H}^{n-1}$$
$$=\int_{\Omega\setminus C_j}\big(|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|\big)\,dx<C_0.$$

We now recall the following elementary fact: If g is a nonnegative measurable function in $[0,\infty)$ with $\int_0^\infty g(r)\,dr\le C_0$, then for every $0<r_1<r_2$ there exists a set J of positive measure in (r_1,r_2) such that for all $r\in J$

$$rg(r)\le\frac{C_0}{\ln(r_2/r_1)}.$$

By applying for j sufficiently large this inequality with

$$0<r_1=\|u_j-u\|_{L^1}^{1/2}<r_2=\|u_j-u\|_{L^1}^{1/4}<1$$

we find $t_j\in(\|u_j-u\|_{L^1}^{1/2},\|u_j-u\|_{L^1}^{1/4})$ so that $\mathcal{L}^n(\{x\in\Omega:\,|u_j-u|=t_j\})=0$ and

$$t_j\int_{\{x\in\Omega\setminus C_j:\,|u_j-u|=t_j\}} \frac{|\mathcal{M}^{m-1}(\nabla u_j)|+|\langle c,\mathcal{M}_m(\nabla u_j)\rangle|}{|\nabla(|u_j-u|)|}\,d\mathcal{H}^{n-1}\le\frac{4C_0}{-\ln\|u_j-u\|_{L^1}}.$$

The latter estimate and (2.12) and (2.13) imply that for every $j \in \mathbb{N}$ sufficiently large there exists $s_j \in (\|u_j - u\|_{L^1}^{1/2}, t_j)$ such that

$$\int_{\{x\in\Omega:\, s_j<|u_j-u|<t_j\}} \left(|\mathcal{M}^{m-1}(\nabla v^j_{s_j,t_j})| + |\langle c, \mathcal{M}_m(\nabla v^j_{s_j,t_j})\rangle|\right) dx \le \frac{5C_0}{-\ln \|u_j - u\|_{L^1}}.$$

Therefore, we may conclude by choosing $v_j := v^j_{s_j,t_j}$. Indeed, from (2.9), setting

$$A_j := \{x \in \Omega : |u_j - u| > s_j\},$$

we have that

$$\mathcal{L}^n(\{x \in \Omega : v_j \ne u_j\}) \le \mathcal{L}^n(A_j) \le \frac{1}{s_j}\|u_j - u\|_{L^1} \le \|u_j - u\|_{L^1}^{1/2},$$
$$\|v_j - u\|_{L^\infty} \le t_j \le \|u_j - u\|_{L^1}^{1/4},$$

and

$$\begin{aligned}
&\int_{A_j} \left(|\mathcal{M}^{m-1}(\nabla v_j)| + |\langle c, \mathcal{M}_m(\nabla v_j)\rangle|\right) dx \\
&= \int_{\{x\in\Omega:\, s_j<|u_j-u|<t_j\}} \left(|\mathcal{M}^{m-1}(\nabla v_j)| + |\langle c, \mathcal{M}_m(\nabla v_j)\rangle|\right) dx \\
&+ \int_{\{x\in\Omega:\, |u_j-u|\ge t_j\}} \left(|\mathcal{M}^{m-1}(\nabla u)| + |\langle c, \mathcal{M}_m(\nabla u)\rangle|\right) dx \\
&\le \frac{5C_0}{-\ln \|u_j - u\|_{L^1}} + C\int_{A_j} |\mathcal{M}^m(\nabla u)|\, dx \to 0.
\end{aligned}$$

Finally, the weak* convergence stated in (2.5) follows from Corollary 2.6 and the boundedness of $(\|\langle c, \mathcal{M}_m(\nabla v_j)\rangle\|_{L^1})_j$, in turn implied by (2.6). □

Remark 2.9 Let us point out that the assumption $m \le n$ plays a crucial role in the proof of Proposition 2.8. The reason is essentially of algebraic nature. In fact if $m \le n$ and $\mathbb{A} \in \mathbb{M}^{m\times m}$ and $\mathbb{B} \in \mathbb{M}^{m\times n}$ from the algebraic equality

$$\mathcal{M}_m(\mathbb{A} \circ \mathbb{B}) = (\det \mathbb{A})\mathcal{M}_m(\mathbb{B}) \tag{2.14}$$

we can trivially estimate $\langle c, \mathcal{M}_m(\mathbb{A} \circ \mathbb{B})\rangle$ with $\langle c, \mathcal{M}_m(\mathbb{B})\rangle$.

Instead, when $m > n$ equality (2.14) is replaced by a more complicate expression that involves suitable linear combinations of higher order minors and it is no longer true that $|\langle c, \mathcal{M}_m(\mathbb{A} \circ \mathbb{B})\rangle| \le C|\langle c, \mathcal{M}_m(\mathbb{B})\rangle|$, for some positive constant C depending on $\mathbb{A}$.

We do not know if Proposition 2.8 fails to be true if $m > n$.

Remark 2.10 For every m and $n \in \mathbb{N}$, if the second condition in the statement of Proposition 2.8 is replaced with $\sup_j \|\mathcal{M}^\ell(\nabla u_j)\|_{L^1} < \infty$, $\ell = m \wedge n$, then [18, Proposition 2.5] establishes a stronger result. More precisely, if $\sup_j \|\mathcal{M}^\ell(\nabla u_j)\|_{L^1} < \infty$, then the sequence $(v_j)_j$ defined accordingly, converges to u in L^∞ and satisfies $\sup_j \|\mathcal{M}^\ell(\nabla v_j)\|_{L^1} < \infty$, (2.6), and

$$\lim_j \int_{A_j} (1 + |\mathcal{M}^\ell(\nabla v_j)|)\, dx = 0. \tag{2.15}$$

Furthermore, $(\mathcal{M}^\ell(\nabla v_j))_j$ weakly* converges in the sense of measures to $\mathcal{M}^\ell(\nabla u)$.

2.3 A blow-up type lemma

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma \to [0, \infty)$ be a continuous function, and consider

$$F(v, U) := \int_U f(x, v(x), \mathcal{M}^\ell(\nabla v(x)))dx,$$

where U is any open subset in Ω, $v \in W^{1,\ell-1}(\Omega, \mathbb{R}^m)$, if $\ell = m \wedge n \geq 3$, $v \in BV(\Omega, \mathbb{R}^m)$ for $\ell = 2$. In the latter case ∇v is the density of the absolutely continuous part of the distributional gradient of v (see, for instance, [4, Theorem 3.83]).

We shall show that to infer the lower semicontinuity inequality

$$F(u) \leq \liminf_j F(u_j), \tag{2.16}$$

along sequences $(u_j)_j \subset W^{1,\ell}(\Omega, \mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1(\Omega, \mathbb{R}^m), \quad \text{and } \sup_j \|u_j\|_{W^{1,\ell-1}} < \infty,$$

we can always reduce ourselves to affine target maps thanks to the next lemma.

Lemma 2.11 *Suppose that for $\mathcal{L}^n$ a.e. $x_0 \in \Omega$, and for all sequences $\varepsilon_k \downarrow 0$ and $(v_k)_k \subset W^{1,\ell}(Q_1, \mathbb{R}^m)$ such that*

$$v_k \to v_0 := \nabla u(x_0) \cdot y \text{ in } L^1(Q_1, \mathbb{R}^m), \quad \text{and } \sup_k \|v_k\|_{W^{1,\ell-1}} < \infty,$$

we have

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k, \mathcal{M}^\ell(\nabla v_k))\, dy \geq f(x_0, u(x_0), \mathcal{M}^\ell(\nabla u(x_0))), \tag{2.17}$$

then the lower semicontinuity inequality (2.16) *holds.*

Proof We employ the blow-up technique introduced by Fonseca & Müller [17]. Without loss of generality we may assume that

$$\liminf_j F(u_j) = \lim_j F(u_j) < \infty,$$

and define the (traces of the) non negative Radon measures $\mu_j(U) := F(u_j, U)$, $U \subseteq \Omega$ open. Then, as $\sup_j \mu_j(\Omega) < \infty$, by passing to a subsequence if necessary, there exists a non negative Radon measure μ such that $\mu_j \rightharpoonup \mu$ weakly* in the sense of measures.

In what follows, by using (2.17) we shall prove that

$$\frac{d\mu}{d\mathcal{L}^n}(x) = \lim_{\varepsilon \downarrow 0} \frac{\mu(Q_\varepsilon(x))}{\mathcal{L}^n(Q_\varepsilon(x))} \geq f(x, u(x), \mathcal{M}^\ell(\nabla u(x))), \tag{2.18}$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$. Clearly, given (2.18) for granted, the conclusion easily follows as

$$\liminf_j F(u_j) = \liminf_j \mu_j(\Omega) \geq \mu(\Omega) \geq F(u).$$

To this aim we consider the Radon measures $\nu_j(U) := \|u_j\|^{\ell-1}_{W^{1,\ell-1}(U,\mathbb{R}^m)}$ and suppose that $(\nu_j)_j$ converges weakly* in the sense of measures to a Radon measure ν.

The ensuing properties are satisfied for all x in a set Ω_0 of full measure in Ω

$$\frac{d\mu}{d\mathcal{L}^n}(x),\ \frac{d\nu}{d\mathcal{L}^n}(x) \quad \text{exist finite},$$

and

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^{n+1}} \int_{Q_\varepsilon(x)} |u(y) - u(x) - \nabla u(x)\cdot(y-x)|\, dy = 0. \tag{2.19}$$

We shall establish (2.18) for all points in Ω_0. Thus, with fixed $x_0 \in \Omega_0$, let $\varepsilon_k \downarrow 0$ be any sequence such that for every $k \in \mathbb{N}$ we have

$$\mu(\partial Q_{\varepsilon_k}(x_0)) = \nu(\partial Q_{\varepsilon_k}(x_0)) = 0. \tag{2.20}$$

By changing variables, (2.19) rewrites for $x = x_0$ as

$$\lim_k \int_{Q_1} \left| \frac{u(x_0+\varepsilon_k y) - u(x_0)}{\varepsilon_k} - \nabla u(x_0)\cdot y \right| dy = 0.$$

The choice of $(\varepsilon_k)_k$ in (2.20) yields

$$\begin{aligned} \frac{d\mu}{d\mathcal{L}^n}(x_0) &= \lim_k \frac{\mu(Q_{\varepsilon_k}(x_0))}{\varepsilon_k^n} = \lim_k \lim_j \frac{1}{\varepsilon_k^n} \int_{Q_{\varepsilon_k}(x_0)} f(x, u_j, \mathcal{M}^\ell(\nabla u_j))dx \\ &= \lim_k \lim_j \int_{Q_1} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_{j,k}, \mathcal{M}^\ell(\nabla v_{j,k}))dy, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \frac{d\nu}{d\mathcal{L}^n}(x_0) &= \lim_k \frac{\nu(Q_{\varepsilon_k}(x_0))}{\varepsilon_k^n} = \lim_k \lim_j \frac{1}{\varepsilon_k^n} \|u_j\|^{\ell-1}_{W^{1,\ell-1}(Q_{\varepsilon_k}(x_0),\mathbb{R}^m)} \\ &= \lim_k \lim_j \|v_{j,k}\|^{\ell-1}_{W^{1,\ell-1}(Q_1,\mathbb{R}^m)}, \end{aligned} \tag{2.22}$$

where we have set

$$v_{j,k}(y) := \frac{u_j(x_0+\varepsilon_k y) - u(x_0)}{\varepsilon_k}.$$

Clearly, $v_{j,k} \in W^{1,\ell}$, and by denoting

$$v_0(y) := \nabla u(x_0)\cdot y$$

we have that

$$\lim_k \lim_j \|v_{j,k} - v_0\|_{L^1} = 0. \tag{2.23}$$

A standard diagonalization argument provides a subsequence $j_k \uparrow \infty$ for which (2.21), (2.22) and (2.23) become

$$v_k := v_{j_k,k} \to v_0 \quad L^1, \quad \sup_k \|v_k\|_{W^{1,\ell-1}} < \infty, \text{ and}$$

$$\infty > \frac{d\mu}{d\mathcal{L}^n}(x_0) = \lim_k \int_{Q_1} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k, \mathcal{M}^\ell(\nabla v_k))\, dy,$$

and thus inequality (2.18) is implied by (2.17). □

3 The model case

In this section we discuss, for $m \leq n$, the model case of autonomous functionals of the form

$$F(v) := \int_\Omega f\left(\mathcal{M}^m(\nabla v(x))\right) dx.$$

where $v \in W^{1,m-1}(\Omega, \mathbb{R}^m)$, if $n \geq m \geq 3$, $v \in BV(\Omega, \mathbb{R}^2)$ for $n \geq m = 2$.

Our result improves upon [6, Theorems 3.1 and 4.1] (see also [16, Theorem 4.1] and [14, Theorem 10]).

Theorem 3.1 *Let $2 \leq m \leq n$ and $f : \mathbb{R}^\sigma \to [0, \infty)$ be a convex function.*

Then, for every sequence $(u_j)_j \subset W^{1,m}(\Omega, \mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1, \quad \text{and } \sup_j \|u_j\|_{W^{1,m-1}} < \infty \tag{3.1}$$

we have

$$F(u) \leq \liminf_j F(u_j).$$

Proof By Lemma 2.11 it is sufficient to show that

$$\liminf_k \int_{Q_1} f\left(\mathcal{M}^m(\nabla v_k)\right) dy \geq f\left(\mathcal{M}^m(\nabla u(x_0))\right), \tag{3.2}$$

for all points x_0 of approximate differentiability of u and for all sequences

$$v_k \to v_0 := \nabla u(x_0) \cdot y \quad \text{in } L^1, \text{ and } \sup_k \|v_k\|_{W^{1,m-1}} < \infty.$$

Without loss of generality we may assume that the inferior limit in (3.2) is finite.

Furthermore, to infer (3.2) we are left with proving for all $i \in \mathbb{N}$

$$\liminf_k \int_{Q_1} f_i\left(\mathcal{M}^m(\nabla v_k)\right) dy \geq f_i\left(\mathcal{M}^m(\nabla u(x_0))\right), \tag{3.3}$$

where $f_i(\xi) = (a_i + \langle b_i, \xi\rangle)^+$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}^\sigma$, are the functions in Lemma 2.1.

As $0 \leq f_i \leq f$ and being the inferior limit in (3.2) finite, we infer that

$$\sup_k \int_{Q_1} \left(\langle b_i, \mathcal{M}^m(\nabla v_k)\rangle\right)^+ dy < \infty. \tag{3.4}$$

Fix now $M \geq \|v_0\|_{L^\infty} + 1$ and set

$$v_{k,M}(x) := \begin{cases} v_k(x) & \text{if } |v_k(x)| \leq M \\ M\dfrac{v_k(x)}{|v_k(x)|} & \text{otherwise.} \end{cases}$$

Note that the sequence $(v_{k,M})_k$ is bounded in $W^{1,m-1} \cap L^\infty$, then $(\langle b_i, \mathcal{M}^m(\nabla v_{k,M})\rangle)_k$ has a limit in the sense of distributions. Moreover, (3.1) and (3.4) yield for every M as above

$$\sup_k \int_{Q_1} \left(\langle b_i, \mathcal{M}^m(\nabla v_{k,M})\rangle\right)^+ dy < \infty.$$

Hence, with fixed $\rho \in (0, 1)$, by the latter estimate and by Lemma 2.5, Proposition 2.8 applied on Q_ρ provides a new sequence $(w_k)_k$ satisfying all the conclusions there. We do not highlight the dependence of the various quantities on ρ for the sake of simplicity. In particular, we have

$$\begin{aligned} \int_{Q_\rho} f_i\left(\mathcal{M}^m(\nabla w_k)\right) dy &\leq \int_{Q_\rho \setminus A_k} f_i\left(\mathcal{M}^m(\nabla v_{k,M})\right) dy + \int_{A_k} (|a_i| + |\langle b_i, \mathcal{M}^m(\nabla w_k)\rangle|) dy \\ &\leq \int_{Q_1} f_i\left(\mathcal{M}^m(\nabla v_k)\right) dy + \int_{A_k} (|a_i| + |\langle b_i, \mathcal{M}^m(\nabla w_k)\rangle|) dy. \end{aligned} \tag{3.5}$$

The latter estimates follow from the positivity of f_i and recalling that, by the choice of M, for k sufficiently large we have

$$\{y \in Q_\rho : |v_k(y)| > M\} \subseteq A_k = \{y \in Q_\rho : |v_k(y) - v_0(y)| > s_k\}.$$

Therefore, thanks to (2.7) and (3.5), we get

$$\liminf_k \int_{Q_1} f_i\left(\mathcal{M}^m(\nabla v_k)\right) dy \geq \liminf_k \int_{Q_\rho} f_i\left(\mathcal{M}^m(\nabla w_k)\right) dy. \tag{3.6}$$

From this inequality, (3.3) follows at once recalling the weak* convergence of $(\mathcal{M}^{m-1}(\nabla w_k))_k$ to $\mathcal{M}^{m-1}(\nabla u(x_0))$ and of $(\langle b_i, \mathcal{M}_m(\nabla w_k)\rangle)_k$ to $\langle b_i, \mathcal{M}_m(\nabla u(x_0))\rangle$ on Q_ρ, and finally by letting $\rho \uparrow 1$. □

A simple variant of the proof of Theorem 3.1 allows us to treat also some special cases when a dependence on x and u appears (cp. with [19] for $m = n$ and $p > n - 1$, and with [16, Remark 4.3]). To be precise, let us consider the functional

$$F(u) = \int_\Omega h(x, u) f(\mathcal{M}^m(\nabla u(x)))\, dx. \tag{3.7}$$

Proposition 3.2 *Let m, n, f be as in Theorem 3.1 and F defined by (3.7). If h is a nonnegative continuous function in $\Omega \times \mathbb{R}^m$ such that either $h \geq c_0 > 0$ or $h \equiv h(x)$, then*

$$F(u) \leq \liminf_j F(u_j)$$

for every sequence $(u_j)_j \subset W^{1,m}(\Omega, \mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1, \quad \text{and } \sup_j \|u_j\|_{W^{1,m-1}} < \infty.$$

Proof Let us first assume that $h \equiv h(x)$ is nonnegative and continuous. In this case the result follows from Theorem 3.1 by a simple continuity and localization argument.

If instead $h \equiv h(x, u) \geq c_0 > 0$ we use Lemma 2.11 and reduce ourselves to show that

$$\liminf_k \int_{Q_1} h(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k) f(\mathcal{M}^m(\nabla v_k))\, dy \geq h(x_0, u(x_0)) f(\mathcal{M}^m(\nabla u(x_0))),$$

where $\varepsilon_k \downarrow 0$ and $v_k \to v_0 = \nabla u(x_0) \cdot y$ in $L^1(Q_1)$, and the liminf on the left hand side can be taken finite.

Then the inequality above is proved if we show that for all $i \in \mathbb{N}$ we have

$$\liminf_k \int_{Q_1} h(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_k) f_i(\mathcal{M}^m(\nabla v_k))\, dy \geq h(x_0, u(x_0)) f_i(\mathcal{M}^m(\nabla u(x_0))), \tag{3.8}$$

where $f_i(\xi) = (a_i + \langle b_i, \xi\rangle)^+$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}^\sigma$, are the functions in Lemma 2.1. Note that, since the above liminf is finite and $h \geq c_0$, we infer

$$\sup_k \int_{Q_1} \left(\langle b_i, \mathcal{M}^m(\nabla v_k)\rangle\right)^+ dy < \infty. \tag{3.9}$$

For $M \geq \|v_0\|_{L^\infty} + 1$ consider the truncated functions

$$v_{k,M}(x) = \begin{cases} v_k(x) & \text{if } |v_k(x)| \leq M \\ M \dfrac{v_k(x)}{|v_k(x)|} & \text{otherwise,} \end{cases}$$

the sequence $(v_{k,M})_k$ turns out to be bounded in $W^{1,m-1} \cap L^\infty$. Therefore, arguing as in Theorem 3.1 and taking into account (3.9) and Lemma 2.5, Proposition 2.8 gives a new sequence $(w_k)_k$ satisfying all the conclusions there on Q_ρ. Thus, using (2.6) and (2.7) as in the proof of (3.6), we get

$$\begin{aligned} &\liminf_k \int_{Q_1} h(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k) f_i\left(\mathcal{M}^m(\nabla v_k)\right) dy \\ &\qquad \geq \liminf_k \int_{Q_\rho} h(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k) f_i\left(\mathcal{M}^m(\nabla w_k)\right) dy. \end{aligned}$$

From this inequality we easily get (3.8), since $(h(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k))_k$ converges uniformly in Q_ρ to $h(x_0, u(x_0))$, and since for any open subset $U \subseteq Q_\rho$

$$\mathcal{L}^n(U) f_i(\mathcal{M}^m(\nabla u(x_0))) \leq \liminf_k \int_U f_i\left(\mathcal{M}^m(\nabla w_k)\right) dy$$

thanks to the weak* convergence of $(\mathcal{M}^{m-1}(\nabla w_k))_k$ to $\mathcal{M}^{m-1}(\nabla u(x_0))$ and of $(\langle b_i, \mathcal{M}_m(\nabla w_k)\rangle)_k$ to $\langle b_i, \mathcal{M}_m(\nabla u(x_0))\rangle$ on Q_ρ, and finally letting $\rho \uparrow 1$. □

When trying to extend Theorem 3.1 to general integrands depending on lower order variables some difficulties arise. We have not been able neither to prove such a statement nor to find counterexamples in such a generality. Instead, we have established lower semicontinuity either strengthening condition (3.1) (cp. with Proposition 3.3) or adding a further condition on the integrand (cp. with Theorem 1.1).

In Sect. 4 we shall establish the lower semicontinuity property under condition (3.1) adding a mild technical assumption on the integrand f provided the domain and the target space have equal dimension, that is $m = n$.

Instead, for all values of m and n, we shall prove below lower semicontinuity without any further technical assumption on the integrand along sequences that have all the set of minors bounded in L^1. Actually, the latter condition holding true, we need only to suppose convergence in L^1, no bound on any Sobolev norm for the relevant sequence is needed (for a comparison with classical statements in literature see [1, Theorem 3.5], [6, Theorem 2.2, Corollary 2.3], [9, Theorem 3.1], [18, Theorem 3.3], and [13, Theorem 1.4, Corollary 1.5] for related results).

Proposition 3.3 *Let m and $n \geq 2$, $\ell = m \wedge n$, and $f = f(x, u, \xi) : \Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma \to [0, \infty)$, $\Omega \subset \mathbb{R}^n$ open, be in $C^0(\Omega \times \mathbb{R}^m \times \mathbb{R}^\sigma)$, and such that $f(x, u, \cdot)$ is convex for all $(x, u) \in \Omega \times \mathbb{R}^m$.*

Then, for every sequence $(u_j)_j \subset W^{1,\ell}(\Omega, \mathbb{R}^m)$ satisfying

$$u_j \to u \text{ in } L^1, \quad \text{and} \quad \sup_j \|\mathcal{M}^\ell(\nabla u_j)\|_{L^1} < \infty, \tag{3.10}$$

we have

$$F(u) \leq \liminf_j F(u_j).$$

Proof We can argue analogously to Lemma 2.11 and infer that to conclude we need to show

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k, \mathcal{M}^\ell(\nabla v_k))\, dy \geq f(x_0, u(x_0), \mathcal{M}^\ell(\nabla u(x_0))), \tag{3.11}$$

along sequences $(v_k)_k \subset W^{1,\ell}$ satisfying

$$v_k \to v_0 = \nabla u(x_0) \cdot y \quad \text{in } L^1, \text{ and } \sup_k \|\mathcal{M}^\ell(\nabla v_k)\|_{L^1} < \infty.$$

Thus, we can apply [18, Proposition 2.5] (cp. with Remarks 2.7 and 2.10) that provides a sequence $(w_k)_k \subset W^{1,\infty}$ converging to v_0 uniformly in Q_1, such that $\mathcal{M}^\ell(\nabla w_k) \rightharpoonup \mathcal{M}^\ell(\nabla u(x_0))$ weakly* in the sense of measures and satisfying (2.15).

In particular, by assuming that the left hand side in (3.11) is finite, for all functions f_i in Theorem 2.2, estimate (2.3) yields

$$\int_{Q_1} f_i(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k, \mathcal{M}^\ell(\nabla w_k))\, dy \leq \int_{Q_1} f_i(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_k, \mathcal{M}^\ell(\nabla v_k))\, dy + C_i \int_{\{v_k \neq w_k\}} (1+|\mathcal{M}^\ell(\nabla w_k)|)\, dy,$$

and moreover (2.4) gives

$$\int_{Q_1} f_i(x_0, u(x_0), \mathcal{M}^\ell(\nabla w_k))\,dy \le \int_{Q_1} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, w_k, \mathcal{M}^\ell(\nabla w_k))\,dy$$
$$+\omega_i(\varepsilon_k(1 + \|w_k\|_{L^\infty})) \int_{Q_1} (1 + |\mathcal{M}^\ell(\nabla w_k)|)\,dy.$$

Thus, the convexity of $f_i(x_0, u(x_0), \cdot)$, the weak* convergence of the minors and (2.15) imply that

$$\liminf_k \int_{Q_1} f_i(x_0 + \varepsilon_k\, y, u(x_0) + \varepsilon_k\, v_k, \mathcal{M}^\ell(\nabla v_k))\,dy$$
$$\ge \liminf_k \int_{Q_1} f_i(x_0, u(x_0), \mathcal{M}^\ell(\nabla w_k))\,dy \ge f_i(x_0, u(x_0), \mathcal{M}^\ell(\nabla u(x_0))).$$

Hence, inequality (3.11) follows. □

Remark 3.4 Note that the counterexample constructed in [19, Theorem 3.1] shows that a continuous, or better a lower-semicontinuous, dependence of the integrand f on the lower order variables is needed.

We point out that if we restrict to maps from $\mathbb{R}^n$ to itself with positive determinants, assumption (3.1) implies the local boundedness in L^1 of all the minors up to order n thanks to Müller's isoperimetric inequality (cp. with (3.10)). Indeed, [26, Lemma 1.3] states that for every $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $x_0 \in \Omega$ and for a.e. $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$ one has

$$\left| \int_{B_r(x_0)} \det \nabla u\,dx \right|^{\frac{n-1}{n}} \le C(n) \int_{\partial B_r(x_0)} |\mathcal{M}_{n-1}(\nabla u)|\,d\mathcal{H}^{n-1}.$$

Thus, if $\det \nabla u \ge 0$ $\mathcal{L}^n$ a.e. on Ω, by integrating this inequality on $(0, R)$, with $R < \mathrm{dist}(x_0, \partial\Omega)$, we get

$$\left| \int_{B_{\frac{R}{2}}(x_0)} \det \nabla u\,dx \right|^{\frac{n-1}{n}} \le \frac{C(n)}{R} \int_{B_R(x_0)} |\mathcal{M}_{n-1}(\nabla u)|\,dx.$$

Therefore, given a sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ with positive determinants and bounded in $W^{1,n-1}$ we infer that for every $\Omega' \subset\subset \Omega$

$$\sup_j \|\mathcal{M}^n(\nabla u_j)\|_{L^1(\Omega', \mathbb{R}^\sigma)} < \infty.$$

This observation leads to the following lower semicontinuity result for which we introduce some further notation: if $\xi \in \mathbb{R}^\sigma$ we write $\xi = (z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$.

Proposition 3.5 *Let $m = n \ge 2$, $\Omega \subset \mathbb{R}^n$ open, and $f = f(x, u, z, t) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{\sigma-1} \times [0, \infty) \to [0, \infty)$ be in $C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^{\sigma-1} \times [0, \infty))$, and such that $f(x, u, \cdot)$ is convex for all $(x, u) \in \Omega \times \mathbb{R}^n$.*

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ *satisfying*

$$u_j \to u \text{ in } L^1, \quad \det \nabla u_j \geq 0 \quad \mathcal{L}^n \text{ a.e. in } \Omega, \text{ and } \quad \sup_j \|\nabla u_j\|_{W^{1,n-1}} < \infty,$$

we have

$$F(u) \leq \liminf_j F(u_j).$$

Note that the lower semicontinuity property established in Proposition 3.5 above is also enjoyed by functionals with densities relevant in applications to elasticity, i.e. satisfying in addition $f(x, u, z, t) \uparrow \infty$ as $t \downarrow 0$.

4 The general case for $m = n$

In this section we address the case of integrands depending on the full set of variables in case $m = n$. More precisely, we consider the functional

$$F(v) := \int_\Omega f\left(x, v(x), \mathcal{M}^n(\nabla v(x))\right) dx,$$

where $v \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, if $n = m \geq 3$, $v \in BV(\Omega, \mathbb{R}^2)$ for $n = m = 2$. Below we give the proof of Theorem 1.1, a result that improves upon [3, Theorem 4.2] and [6, Theorems 3.1 and 4.1].

To begin with, we comment on assumption (1.4) by considering a simple functional with integrand a positive piecewise affine function

$$I(u) = \int_\Omega (a(x, u) + b(x, u) \det \nabla u)^+ \, dx.$$

In this particular case condition (1.4) becomes

$$b(x, u) = 0 \Longrightarrow a(x, u) \leq 0.$$

Moreover, we recall that assumption (1.4) is satisfied if f is coercive in t, or if f is autonomous, i.e. $f = f(\xi)$ (see Corollary 2.3).

Before proving Theorem 1.1 we recall the notation $\xi = (z, t) \in \mathbb{R}^{\sigma-1} \times \mathbb{R}$ if $\xi \in \mathbb{R}^\sigma$. Note also that condition (1.5) in Theorem 1.1 assures that the limit function $u \in BV(\Omega, \mathbb{R}^2)$ if $n = 2$, and $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ for $n \geq 3$.

Proof of Theorem 1.1 We split the proof into several intermediate steps.

Step 1. *Reduction to affine target maps*

By Lemma 2.11 to infer (1.6) we are left with proving

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k, \mathcal{M}^n(\nabla v_k)) \, dy \geq f(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))). \tag{4.1}$$

along sequences satisfying

$$v_k \to v_0 := \nabla u(x_0) \cdot y \quad L^1, \text{ and } \sup_k \|v_k\|_{W^{1,n-1}} < \infty,$$

for all points x_0 of approximate differentiability of u. As usual we can assume that the left hand side in (4.1) is finite.

Let us now distinguish two cases:

(a) there exists $z_0 \in \mathbb{R}^{\sigma-1}$ such that $t \mapsto f(x_0, u(x_0), z_0, t)$ is constant;
(b) no such a point exists.

Step 2. Proof in case (a)

In this case we apply assumption (ii) in the statement and use at once Theorem 3.1 to get

$$\lim_k \int_{Q_1} f(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy$$
$$\geq \liminf_k \int_{Q_1} g(\mathcal{M}^{n-1}(\nabla v_k))\, dy \geq g(\mathcal{M}^{n-1}(\nabla u(x_0))) = f(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))),$$

where we have denoted by g the convex function $g(z) := f(x_0, u(x_0), z, t)$ (cp. with (1.4)).

Step 3. Proof in case (b)

Let us now recall that by the approximation Theorem 2.2 there exist three sequences of continuous functions with compact support, $a_i, \gamma_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $\beta_i : \Omega \times \mathbb{R}^n \to \mathbb{R}^{\sigma-1}$ such that, setting for all $i \in \mathbb{N}$ and $(x, u, z, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{\sigma-1} \times \mathbb{R}$,

$$f_i(x, u, z, t) = (a_i(x, u) + \langle \beta_i(x, u), z\rangle + \gamma_i(x, u)\, t)^+,$$

we have

$$f(x, u, z, t) = \sup_{i\in\mathbb{N}} f_i(x, u, z, t).$$

Therefore, to prove (4.1) it is enough to show that for all $i \in \mathbb{N}$

$$\lim_k \int_{Q_1} f(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy \geq f_i(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))). \tag{4.2}$$

To this aim note that there exists $j \in \mathbb{N}$ such that $\gamma_j(x_0, u(x_0)) \neq 0$ since otherwise we would fall in case (a).

Without loss of generality we may assume $\gamma_j(x_0, u(x_0)) > 0$. Otherwise, we replace the functions $v_k = (v_k^1, \dots, v_k^n)$ with $(-v_k^1, v_k^2, \dots, v_k^n)$, the coefficient $\gamma_j(x, u)$ with $-\gamma_j(x, -u^1, \dots, u^n)$ and the remaining coefficients a_j and β_j accordingly.

Fix now $M > \|v_0\|_{L^\infty} + 1$ and set

$$v_{k,M}(x) := \begin{cases} v_k(x) & \text{if } |v_k(x)| \leq M \\ M\dfrac{v_k(x)}{|v_k(x)|} & \text{otherwise.} \end{cases} \tag{4.3}$$

Then, as $0 \leq f_j \leq f$, for all k we have

$$\int_{\{y\in Q_1: |v_k|\leq M\}} f_j(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_{k,M}, \mathcal{M}^n(\nabla v_{k,M}))\, dy$$
$$\leq \int_{Q_1} f(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy. \tag{4.4}$$

Therefore, since the sequence $(v_{k,M})_k$ is bounded in $W^{1,n-1}(Q_1,\mathbb{R}^n)$ we deduce that

$$\sup_k \int_{\{y\in Q_1:\,|v_k|\le M\}} \big(\gamma_j(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k) \det\nabla v_{k,M}\big)^+\, dy < \infty.$$

Recalling the choice $\gamma_j(x_0,u(x_0))>0$, the continuity of γ_j yields for k sufficiently large

$$\sup_k \int_{\{y\in Q_1:\,|v_k|\le M\}} \big(\det\nabla v_{k,M}\big)^+\, dy < \infty,$$

in turn implying

$$\sup_k \int_{Q_1} \big(\det\nabla v_{k,M}\big)^+\, dy < \infty.$$

An application of Lemma 2.5 gives that, up to a subsequence not relabeled for convenience, the sequence $(\det\nabla v_{k,M})_k$ converges locally weakly* in the sense of measures in Q_1. In particular, $(\det\nabla v_{k,M})_k$ is bounded in $L^1_{loc}(Q_1)$. Hence, with fixed $\rho\in(0,1)$, Proposition 2.8 provides sequences $s_k\downarrow 0$ and $(w_k)_k$ in $W^{1,n}(Q_\rho,\mathbb{R}^n)$ satisfying conclusions (2.5), (2.6) and (2.7) there. Note that, for k sufficiently large, recalling the choice of M, we have

$$\{y\in Q_\rho:\ |v_k(y)|>M\}\subseteq A_k=\{y\in Q_\rho:\ |v_k(y)-v_0(y)|>s_k\}.$$

Therefore, estimate (2.3) and equation (4.4) imply for all $i\in\mathbb{N}$

$$\begin{aligned}
&\int_{Q_\rho} f_i(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, w_k, \mathcal{M}^n(\nabla w_k))\, dy\\
&\le \int_{Q_\rho\setminus A_k} f_i(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_{k,M}, \mathcal{M}^n(\nabla v_{k,M}))\, dy + C_i\int_{A_k}\big(1+|\mathcal{M}^n(\nabla w_k)|\big)\, dy\\
&\le \int_{\{y\in Q_\rho:\,|v_k|\le M\}} f_i(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_{k,M}, \mathcal{M}^n(\nabla v_{k,M}))\, dy + C_i\int_{A_k}\big(1+|\mathcal{M}^n(\nabla w_k)|\big)\, dy\\
&\le \int_{Q_1} f(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy + C_i\int_{A_k}\big(1+|\mathcal{M}^n(\nabla w_k)|\big)\, dy.
\end{aligned}$$

The convergence of $(w_k)_k$ to v_0 in L^∞, the latter inequality, (2.7) and (2.4) imply

$$\liminf_k \int_{Q_1} f(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy \ge \liminf_k \int_{Q_\rho} f_i(x_0, u(x_0), \mathcal{M}^n(\nabla w_k))\, dy.$$

In turn, from this and by taking into account that $(\mathcal{M}^n(\nabla w_k))_k$ converges to $\mathcal{M}^n(\nabla u(x_0))$ weakly* in the sense of measures on Q_ρ, by the convexity of $f_i(x_0,u(x_0),\cdot)$ we get

$$\lim_k \int_{Q_1} f(x_0+\varepsilon_k\, y, u(x_0)+\varepsilon_k\, v_k, \mathcal{M}^n(\nabla v_k))\, dy \ge \rho^n\ f_i(x_0, u(x_0), \mathcal{M}^n(\nabla u(x_0))),$$

from which (4.2) follows straightforwardly as $\rho\uparrow 1$. □

In case the integrand depends only on the determinant, the same conclusion holds under a local version of the condition in (1.4).

Proposition 4.1 *Let* $f = f(x, u, t) : \Omega \times \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ *be such that*

(i) $f \in C^0(\Omega \times \mathbb{R}^n \times \mathbb{R})$, *and* $f(x, u, \cdot)$ *is convex for all* (x, u),
(ii) *if* $f(x_0, u_0, \cdot)$ *is constant with respect to* $t \in \mathbb{R}$ *for some point* (x_0, u_0), *then*

$$f(x_0, u_0, t) = h(x_0, u_0)$$

where

$$h(x, u) := \sup_{\delta > 0} h_\delta(x, u)$$

and for all $\delta > 0$

$$h_\delta(x, u) := \inf \{f(y, v, t) : (y, v, t) \in B_\delta(x, u) \times \mathbb{R}\}.$$

Then, for every sequence $(u_j)_j \subset W^{1,n}(\Omega, \mathbb{R}^n)$ *satisfying*

$$u_j \to u \text{ in } L^1, \quad \text{and } \sup_j \|u_j\|_{W^{1,n-1}} < \infty$$

we have

$$F(u) \le \liminf_j F(u_j).$$

Proof We argue as in Theorem 1.1: first using the blow-up type Lemma 2.11 to reduce to inequality (4.1). At this point we distinguish as before the two cases (a) and (b). The latter is dealt with as before, while for case (a) we argue as follows.

Fix $M > 0$ and consider the function $v_{k,M}$ in (4.3). Then,

$$\int_{Q_1} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_{k,M}, \det \nabla v_{k,M}) dy$$
$$\le \int_{Q_1} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k, \det \nabla v_k) dy + C\,\mathcal{L}^n(\{y \in Q_1 : |v_k| \ge M\}).$$

In turn, for any $\delta > 0$, by definition of h_δ we get

$$\liminf_k \int_{Q_1} f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_{k,M}, \det \nabla v_{k,M}) dy \ge h_\delta(x_0, u(x_0)),$$

and the conclusion then follows as $f(x_0, u(x_0), t) = \sup_\delta h_\delta(x_0, u(x_0))$ for all $t \in \mathbb{R}$. □

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