Everywhere regularity for a class of vectorial functionals under subquadratic general growth conditions

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Abstract

We consider the integral functional of the calculus of variations

$$\int_{\Omega} f(Du) \, dx,$$

where $f: \mathbb{R}^N \to \mathbb{R}$ satisfies $f(z) = g(|z|)$ and $g$ is an $N$-function with subquadratic $p-q$ growth. We prove that minimizers $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of such a functional are locally Lipschitz continuous, provided $g$ verifies some additional conditions.

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1. Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let us consider the variational integral

$$I(u) = \int_{\Omega} f(Du(x)) \, dx,$$

where $f:\mathbb{R}^nN \to \mathbb{R}$ is continuous and nonnegative, $u:\Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $Du(x) = (\partial u^\alpha / \partial x^j)_{\alpha=1,...,N, \, j=1,...,n}$.

We say that a function $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of $I$ if $f(Du) \in L^1_{\text{loc}}(\Omega)$ and, for every $\varphi \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ with $\text{supp} (\varphi) \subset \subset \Omega$ we have

$$\int_{\text{supp} (\varphi)} f(Du(x)) \, dx \leq \int_{\text{supp} (\varphi)} f(Du(x) + D\varphi(x)) \, dx.$$

Let us assume that $f$ satisfies the growth condition

$$|z|^p - m \leq f(z) \leq M(1 + |z|^q),$$

where $m, M$ are positive constants and $1 < p \leq q$. We are going to deal with Lipschitz regularity of vector-valued minimizers, under the special structure assumption

$$f(z) = g(|z|), \quad \forall z \in \mathbb{R}^n.$$

When handling vector-valued mappings and aiming at Lipschitz continuity, such a special assumption is not surprising: Uhlenbeck [10], Giaquinta and Modica [4] for $p = q \geq 2$, Acerbi and Fusco [1] for $1 < p = q < 2$. Recently Marcellini in [7] has proved a $C^{1,\alpha}$ regularity result for local minimizers of functionals when $g$ has a nonoscillating property and, at least, quadratic growth: such a result does not cover the case in which $g$ has subquadratic growth. Our present paper is concerned with this case.

We assume that $g:[0, +\infty) \to [0, +\infty)$ is an $N$-function, i.e., $g(t) = 0$ if and only if $t = 0$,

$$\lim_{t \to \infty} \frac{g(t)}{t} = +\infty, \quad \lim_{t \to 0} \frac{g(t)}{t} = 0.$$

We assume also that $g$ is strictly convex and the following conditions hold:

1. There exist $A_1, A_2 > 0$ and $1 < p < q < 2$ such that $g \in C^2((0, +\infty)) \cap C^1([0, +\infty))$, $g'(0) = 0$, $g''(t)/t$ is decreasing and

$$A_1 t^{p-2} \leq \frac{g'(t)}{t} \leq A_2 (t^{q-2} + t^{p-2});$$

2. There exists $\gamma > 1$ such that

$$g''(t) t \leq g'(t) \leq \gamma g''(t) t.$$

We remark that (1.2) implies

$$\frac{A_1}{p} t^p \leq g(t) \leq \frac{A_2}{p} (t^q + t^p), \quad \forall t \geq 0,$$
thus we are in the subquadratic \( p-q \) growth. Let us remark that we do not require \( p \) to be close to \( q \); on the contrary, many regularity results assume that \( p \) is near \( q \); see [3,5,6,8]. The main result of the paper is the following

**Theorem 1.1.** Let \( g \) satisfy (G1), (G2) and \( u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N) \) be a local minimizer of functional \( I \) in (1.1). Then \( u \) is locally Lipschitz continuous in \( \Omega \). Moreover, for \( 0 < \rho < R \) with \( B_{2R} \subseteq \Omega \), there exists a positive constant \( c \) such that

\[
\sup_{B_{\rho}} |Du| \leq c \int_{B_{R}} \left( 1 + g(|Du|) \right) dx,
\]

where \( c = c(n, N, \gamma, \rho, R, g'(\sqrt{2})) > 0 \).

We observe explicitly that the constant \( c \) does not depend on \( \Lambda_1, \Lambda_2 \) of (1.2).

Our result includes energy densities \( f \) with slow growth. For instance, it can be proved that the function

\[ g(t) = t^p \log^\alpha (a + t) \]

with \( 1 < p < 2, \alpha > 0 \) and \( a > 0 \) large enough is an \( N \)-functions satisfying conditions (G1) and (G2). The limit case \( g(t) = t \log(1 + t) \) has been studied by Mingione and Siepe in [9].

The proof of our regularity result is splitted into two parts.

First, we consider the standard growth case, i.e., when \( f(z) = g(|z|) \) and \( g \) satisfies (1.3) with \( q \) instead of \( p \) in the left-hand side. If \( v \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N) \) is a local minimizer for \( I \) in this case, by the results of Acerbi and Fusco in [1] we have that \( v \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N) \). By our special assumptions we are able to derive an estimate of \( \sup |Dv| \) like (1.4), by using only the properties of the \( N \)-function \( g \), so the constant \( c \) does not depend on \( \Lambda_1 \) and \( \Lambda_2 \) in (1.2).

Then we study the case of \( p-q \) growth by applying a double approximation procedure as in [2,6,9], combined with some techniques about functionals without explicit polynomial growth. More precisely, we start from a local minimizer \( u \) of (1.1), we define \( f_\sigma(z) = f(z) + \sigma |z|^q \) with \( \sigma > 0 \), so that the function \( f_\sigma \) satisfies the standard growth condition of order \( q \).

We regularize the original minimizer \( u \) by means of mollifiers, thus obtaining the sequence \( \{u_\varepsilon\} \). Then we consider the Dirichlet problem in \( B_R \subseteq \Omega \),

\[
\min_{B_R} \left\{ I_\varepsilon (v) = \int_{B_R} f_\varepsilon(Dv) dx : v \in u_\varepsilon + W^{1,q}_0(B_R, \mathbb{R}^N) \right\}.
\]

Let \( v_{\varepsilon, \sigma} \) be the unique solution of (1.5). By the previous results we can estimate

\[
\sup_{B_R} |Dv_{\varepsilon, \sigma}| \leq c \left\{ \int_{B_{R+\varepsilon}} \left[ 1 + g(|Du|) \right] dx + \sigma \int_{B_R} |Du_\varepsilon|^q dx \right\},
\]

where \( c \) is independent of \( \sigma \) and \( \varepsilon \). Then, by letting first \( \sigma \to 0 \) and then \( \varepsilon \to 0 \), estimate (1.4) follows.
2. Regularity under standard growth conditions

In this section we start from Acerbi–Fusco regularity result (see [1]), for the minimizers of subquadratic functionals and we give an estimate for sup |Du|, in which we carefully prove how the constant depends on the assumptions of the energy density. This will allow us to deal with the case of general growth.

Definition 2.1. We say that \( h : [0, +\infty) \to [0, +\infty) \) is an \( N \)-function if \( h \) is convex and increasing, \( h(t) = 0 \) if and only if \( t = 0 \),

\[
\lim_{t \to +\infty} \frac{h(t)}{t} = +\infty, \quad \lim_{t \to 0^+} \frac{h(t)}{t} = 0.
\]

Moreover we say that an \( N \)-function \( h \) is of class \( \Delta^m_2 \) if there exists \( m > 1 \) such that

\[
h(\lambda t) \leq \lambda^m h(t), \quad \forall t \geq 0, \quad \forall \lambda > 1.
\]

As it can be easily checked, if \( h \) is of class \( C^1 \), the latter is equivalent to require that

\[
h'(t)t \leq mh(t), \quad \forall t \geq 0.
\]

Let \( 1 < p < q < 2 \) and let \( h \) be an \( N \)-function strictly convex in \( [0, +\infty) \). We will assume that \( h \) satisfies the following assumptions:

(H1) \( h \in C^2((0, +\infty)) \cap C^1([0, +\infty)) \). Moreover \( h'(0) = 0 \) and, for every \( t > 0 \), \( h'(t)/t \) is decreasing and two positive constants \( \Lambda_1, \Lambda_2 \) exist such that

\[
\Lambda_1 t^{q-2} \leq \frac{h'(t)}{t} \leq \Lambda_2 (t^{q-2} + t^{p-2});
\]

(2.1)

(H2) There exists \( \gamma > 1 \) such that

\[
h''(t) t \leq h'(t) \leq \gamma h''(t)t, \quad \forall t > 0.
\]

(2.2)

Remark 2.1. We observe that the left inequality in (2.2) implies without other assumptions that \( h \in \Delta^2_2 \).

Moreover by (2.1) it easily follows that

\[
\frac{\Lambda_1}{q} t^q \leq h(t) \leq \frac{\Lambda_2}{p} (t^q + t^p).
\]

(2.3)

Let us consider the integral functional

\[
I(u) = \int_{\Omega} f(Du) \, dx,
\]

(2.4)

where \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a bounded open set, \( f : \mathbb{R}^n \to \mathbb{R} \) \((N > 1)\) is such that

\[
f(z) = h(|z|)
\]

(2.5)

and \( u : \Omega \to \mathbb{R}^N \) is a weakly differentiable function.

We remark also that, under such conditions, \( f \) turns out to be strictly convex in \( \mathbb{R}^n \).

The main result of this section is the following
Proposition 2.1. Let $u$ be a local minimizer of functional (2.4), where $f$ is as in (2.5) and $h$ satisfies conditions (H1) and (H2). Then $u \in W_{\text{loc}}^{1,\infty}(\Omega, \mathbb{R}^N)$. Moreover, for every $0 < \rho < R$ such that $B_R \subseteq \Omega$, there exists a positive constant $c$ such that the following inequality holds:

$$\sup_{B_{\rho}} |Du| \leq c \int_{B_R} \left[ 1 + h(|Du|) \right] dx,$$

(2.6)

where $c = \tilde{c}[V(h'(\sqrt{2}))]^{2/(2^*-2)}$, $\tilde{c} = \tilde{c}(n, N, \rho, R) > 0$, $V(t) = 1 + t + c_0 t^{-\vartheta}$, $c_0 = c_0(n, \gamma) > 0$ and $2^* = 2n/(n-2)$ if $n \geq 3$, while $2^* = 3$ if $n = 2$.

Remark 2.2. We note in particular that the constant $c$ in (2.6) does not depend on $\Lambda_1$ and $\Lambda_2$.

Proof of Proposition 2.1. We divide the proof into three steps.

Step 1 (Approximation by means of nondegenerate densities). Let us fix $\mu \in (0, 1]$ and define

$$H_{\mu}(t) = h(\sqrt{\mu^2 + t^2}) - h(\mu).$$

(2.7)

It is easy to check that $H_{\mu}$ is an $N$-function of class $\Delta^2_2$. Moreover, by properties (2.1)–(2.3) of $h$ it follows that

$$\frac{A_1}{q} (\mu^2 + t^2)^{q/2} - \frac{A_1}{q} \mu^q \leq H_{\mu}(t) \leq \frac{A_2}{p} ((\mu^2 + t^2)^{q/2} + (\mu^2 + t^2)^{p/2})$$

$$\leq \frac{2A_2}{p \mu^q} (\mu^2 + t^2)^{q/2}, \quad \forall t \geq 0,$$

(2.8)

$$H_{\mu} \in C^2(\mathbb{R}), \quad H_{\mu}'(0) = 0, \quad H_{\mu}'(t)/t \text{ is decreasing in } (0, +\infty) \text{ and}$$

$$A_1 (\mu^2 + t^2)^{(q-2)/2} \leq \frac{H_{\mu}''(t)}{t} \leq A_3 (\mu^2 + t^2)^{(q-2)/2}, \quad \forall t > 0,$$

(2.9)

where $A_3 = A_3(A_2, p, q, \mu)$ and finally

$$H_{\mu}''(t) t \leq H_{\mu}'(t) \leq \gamma H_{\mu}''(t) t, \quad \forall t \geq 0.$$

(2.10)

Let us consider the functionals

$$I_{\mu}(v) = \int_{\Omega} f_{\mu}(Dv) dx,$$

(2.11)

where we set $f_{\mu}(z) = H_{\mu}(|z|)$ for every $z \in \mathbb{R}^n$. By (2.8) and (2.9) we have

$$\frac{A_1}{q} (\mu^2 + |z|^2)^{q/2} - \frac{A_1}{q} \mu^q \leq f_{\mu}(z) \leq \frac{A_3}{q} (\mu^2 + |z|^2)^{q/2}.$$

(2.12)

Moreover $f_{\mu} \in C^2(\mathbb{R}^n)$, $f_{\mu}(z) \leq A_3 \sqrt{nN} (\mu^2 + |z|^2)^{(q-2)/2}$

(2.13)
and

\[ \{D^2 f_{\mu}(z, \lambda, \lambda) \geq \frac{A_1}{\gamma} (\mu^2 + |z|^2)^{(q-2)/2} |\lambda|^2 \]  

(2.14)

for every \( z, \lambda \in \mathbb{R}^n \). Let us check (2.13) and (2.14).

To simplify our notations, from now on we will write \( f \) and \( H \) to denote the functions \( f_{\mu} \) and \( H_{\mu} \). First we observe that for \( z \neq 0 \),

\[ f_{\mu}^{\sigma} (z) = H'(\|z\|) \frac{\zeta^\sigma}{|z|}, \]

\[ f_{\mu}^{\alpha, \beta} (z) = \left( \frac{H''(|z|)}{|z|^2} - \frac{H'(|z|)}{|z|^3} \right) \zeta^\alpha z^\beta + \frac{H'(|z|)}{|z|} \delta_{ij} \zeta^\alpha \zeta^\beta \]

(2.15)

and then

\[ |D^2 f(z)| = \sum_{\alpha, \beta, i, j} (f_{\mu}^{\alpha, \beta} (z))^2 \leq \sqrt{nN} \frac{H'(|z|)}{|z|} \leq A_3 \sqrt{nN} (\mu^2 + |z|^2)^{(q-2)/2}, \]

that is (2.13). Finally we have

\[ \sum_{\alpha, \beta, i, j} f_{\mu}^{\alpha, \beta} (z) \zeta^\alpha \zeta^\beta \geq \left( \frac{H''(|z|)}{|z|^2} - \frac{H'(|z|)}{|z|^3} \right) |\zeta, \lambda|^2 + \frac{H'(|z|)}{|z|} |\lambda|^2, \]

from which, by (2.10) it follows that

\[ \sum_{\alpha, \beta, i, j} f_{\mu}^{\alpha, \beta} (z) \zeta^\alpha \zeta^\beta \geq \left( \frac{H''(|z|)}{|z|^2} - \frac{H'(|z|)}{|z|^3} \right) |\zeta, \lambda|^2 + \frac{H'(|z|)}{|z|} |\lambda|^2 \]

(2.16)

and then by (2.9), (2.14) follows. Moreover by (2.10) we have

\[ \sum_{\alpha, \beta, i, j} f_{\mu}^{\alpha, \beta} (z) \zeta^\alpha \zeta^\beta \leq \frac{H'(|z|)}{|z|} |\lambda|^2. \]

(2.17)

**Step 2** (Estimates for minimizers of nondegenerate densities). Let \( v \) be a local minimizer of the functional \( I_{\mu} \) defined in (2.11), with \( H_{\mu} \) as in (2.7). By Lemma 2.5 and Proposition 2.7 in [1], taking into account (2.12)–(2.14) we deduce that \( v \in W^{2,2}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N) \). Furthermore \( f_{\mu, \alpha} (Dv) \in W^{1,2}_{\text{loc}}(\Omega) \) and the chain rule can be used for computing \( D_{v} (f_{\mu, \alpha} (Dv)) \).

If \( 0 < \rho < R \) are such that \( B_R \subseteq \Omega \), then we claim that there exists a positive constant \( c \) such that

\[ \sup_{B_R} |Dv| \leq c \int_{B_R} \left[ 1 + H_{\mu} (|Dv|) \right] dx, \]

(2.18)

where \( c = c_4 [V(h(\sqrt{\rho}))]^{2/(2^*-2)} \) with \( c_4 (n, |\Omega|, \rho, R) \) and \( |\Omega| \) is the \( n \)-dimensional Lebesgue measure of \( \Omega \).
Let us drop again $\mu$ from $H_\mu$ and $f_\mu$. We will prove our claim starting from the second variation of our functional $I_\mu$,

$$\int_\Omega \sum_{\alpha, \beta, i, j} f_{\alpha, \beta}^\mu (Dv) v_{i}^\beta v_{j}^\alpha \phi_{i}^\mu \phi_{j}^\alpha \, dx = 0 \quad (2.19)$$

for every function $\phi \in C_0^\infty(\Omega, \mathbb{R}^N)$ and any fixed $s \in \{1, \ldots, n\}$. Since $f_{\alpha, \beta}^\mu (Dv) v_{i}^\beta v_{j}^\alpha \in L^2_{\text{loc}}(\Omega)$, equality (2.19) holds true for every $\phi \in W^{1, 2}(\Omega)$ with $\text{supp}(\phi) \subset \Omega$.

Let $\psi : [0, +\infty) \to [0, +\infty)$ be continuous, bounded, piecewise of class $C^1$ with only a finite number of corner points, such that $\psi'$ is bounded and $\psi' \geq 0$; moreover let $\eta \in C_0^1(\Omega)$ and set $\phi^\alpha = \eta^2 u_{i}^\alpha \psi(|Dv|)$ for every $\alpha = 1, \ldots, N$. Then $\phi \in W^{1, 2}(\Omega, \mathbb{R}^N)$ and

$$\phi_{i}^\alpha = 2\eta \eta_{i}^\alpha \psi(|Dv|) + \eta^2 v_{i}^\alpha \psi(|Dv|) + \chi_{|Dv| \notin L} \eta^2 v_{i}^\alpha \psi'(|Dv|)(|Dv|)_{s_i},$$

where we denote by $L$ the set of the corner points of $\psi$. Now we insert $\phi_{i}^\alpha$ in (2.19) and we add up over $s$,

$$\sum_s \int_\Omega 2\eta \psi(|Dv|) \sum_{\alpha, \beta, i, j} f_{\alpha, \beta}^\mu (Dv) v_{i}^\beta v_{j}^\alpha \eta_{i}^\alpha v_{i}^\alpha \, dx$$

$$+ \sum_s \int_\Omega \eta^2 \psi(|Dv|) \sum_{\alpha, \beta, i, j} f_{\alpha, \beta}^\mu (Dv) v_{i}^\beta v_{j}^\alpha \eta_{i}^\alpha v_{i}^\alpha \, dx$$

$$+ \int_{\Omega \cap \{|Dv| \notin L\}} \eta^2 \psi'(|Dv|) \sum_{\alpha, \beta, i, j, s} f_{\alpha, \beta}^\mu (Dv) v_{i}^\beta v_{j}^\alpha \psi(|Dv|)(|Dv|)_{s_i} \, dx$$

$$= \sum_s I_{1,s} + I_2 + I_3 = 0. \quad (2.20)$$

By Cauchy–Schwartz inequality and since $ab \leq (a^2 + b^2)/2$ for every $a, b \geq 0$ we have

$$|I_{1,s}| \leq 2 \int_\Omega \psi(|Dv|) \left( \eta^2 \sum_{\alpha, \beta, i, j} f_{\alpha, \beta}^\mu (Dv) v_{i}^\beta v_{j}^\alpha \eta_{i}^\alpha v_{i}^\alpha \right)^{1/2}$$

$$\times \left( \sum_{\alpha, \beta, i, j} f_{\alpha, \beta}^\mu (Dv) \eta_{i}^\alpha v_{i}^\alpha \eta_{j}^\alpha v_{j}^\alpha \right)^{1/2} \, dx$$

$$\leq \frac{1}{2} \int_\Omega \eta^2 \psi(|Dv|) \sum_{\alpha, \beta, i, j} f_{\alpha, \beta}^\mu (Dv) v_{i}^\alpha v_{i}^\beta \, dx$$

$$+ 2 \int_\Omega \psi(|Dv|) \sum_{\alpha, \beta, i, j} f_{\alpha, \beta}^\mu (Dv) \eta_{i}^\alpha v_{i}^\alpha \eta_{j}^\alpha v_{j}^\alpha \, dx.$$
\[
(H''(|Dv|) - H'(|Dv|) |Dv|^2) \sum_{\alpha, \beta, i, s} v^\alpha x_i v^\beta x_j x_s (|Dv|)_{x_i} \\
+ H'(|Dv|) |Dv| \sum_{i} ((|Dv|)_{x_i})^2 \\
\geq (H''(|Dv|) - H'(|Dv|) |Dv|^2) |Dv|^2 |D(|Dv|)|^2 + H'(|Dv|)|D(|Dv|)|^2 \\
= H''(|Dv|)|Dv||D(|Dv|)|^2 \geq 0.
\]

Moreover, since
\[
\sum_{\beta, j} v^\beta x_j v^\beta x_j x_s = (|Dv|)_{x_s} |Dv|,
\]
we have
\[
A = \left( \frac{H''(|Dv|) - H'(|Dv|) |Dv|^2}{|Dv|} \right) \sum_{\alpha, i, s} v^\alpha x_i (|Dv|)_{x_i} v^\alpha x_i (|Dv|)_{x_i} \\
+ H'(|Dv|) \sum_{i} ((|Dv|)_{x_i})^2 \\
\geq \left( \frac{H''(|Dv|) - H'(|Dv|) |Dv|^2}{|Dv|} \right) |Dv|^2 |D(|Dv|)|^2 + H'(|Dv|)|D(|Dv|)|^2 \\
= H''(|Dv|) |Dv| |D(|Dv|)|^2 \geq 0.
\]

Since \( \psi' \geq 0 \) and \( A \geq 0 \), it turns out that \( I_3 \geq 0 \), so that (2.20) gives
\[
\int_{\Omega} \eta^2 \psi(|Dv|) \sum_{\alpha, \beta, i, j, s} f_{\alpha, \beta}(Dv) v^\alpha x_i v^\beta x_j x_s \, dx \\
\leq 4 \int_{\Omega} \psi(|Dv|) \sum_{\alpha, \beta, i, j, s} f_{\alpha, \beta}(Dv) \eta x_i v^\alpha x_i \eta x_j v^\beta x_s \, dx.
\]

Now since \(|D(|Dv|)|^2 \leq |D^2 v|^2\), by (2.17) and (2.16) we obtain
\[
\int_{\Omega} \eta^2 \psi(|Dv|) H''(|Dv|)|D(|Dv|)|^2 \, dx \\
\leq 4 \int_{\Omega} \psi(|Dv|) \sum_{\alpha, \beta, i, j, s} f_{\alpha, \beta}(Dv) \eta x_i v^\alpha x_i \eta x_j v^\beta x_s \, dx \\
\leq 4 \int_{\Omega} |D\eta|^2 \psi(|Dv|) H'(|Dv|)|Dv| \, dx.
\]  

(2.21)

We use (2.21) with \( \psi \equiv 1 \),
\[
\int_{\Omega} \eta^2 H''(|Dv|)|D(|Dv|)|^2 \, dx \leq 4 \int_{\Omega} |D\eta|^2 H'(|Dv|)|Dv| \, dx < +\infty.
\]

Now let \( M > 1 \) such that \( |Dv| \leq M \) on \( \text{supp}(\eta) \). For \( \delta > 0 \) we define
\[
\psi(t) = \begin{cases} 
\delta^2 & \text{if } t \in [0, M], \\
M^{2\delta} & \text{if } t \in (M, +\infty). 
\end{cases}
\]
In the case of $\delta \in [1/2, +\infty)$ we can use such a function $\psi$ in (2.21) in order to get
\[
\int_{\Omega} \eta^2 H''(|Dv|)|Dv|^{2\delta} |D(|Dv|)|^2 \, dx \leq 4 \int_{\Omega} |D\eta|^2 H'(|Dv|)|Dv|^{2\delta+1} \, dx. \tag{2.22}
\]
When $\delta \in (0, 1/2)$, such $\psi$ does not have bounded derivative near 0. So we linearize between 0 and $1/k$, for every integer $k \geq 1$ and we get the following sequence of functions:
\[
\psi_k(t) = \begin{cases} 
\left(\frac{1}{k}\right)^{2\delta-1} t & \text{if } t \in [0, \frac{1}{k}], \\
\psi(t) & \text{if } t \in \left(\frac{1}{k}, +\infty\right).
\end{cases}
\]
It can be easily shown that $0 \leq \psi_k \leq \psi_{k+1} \leq \psi$ and $\psi_k(t) \to \psi(t)$ in $[0, +\infty)$. Then we can use estimate (2.21) with $\psi_k$ and monotone convergence Theorem gives (2.22) for $\delta \in (0, 1/2)$ too.

Let us define
\[
G(t) = 1 + \int_0^t \sqrt{s^{2\delta} H''(s)} \, ds.
\]
Then, by means of inequality $(a + b)^2 \leq 2(a^2 + b^2)$ we have
\[
|D(\eta G(|Dv|))|^2 \leq 2|D\eta|^2 |G(|Dv|)|^2 + 2\eta^2 |Dv|^{2\delta} H''(|Dv|)|D(|Dv|)|^2
\]
so that, by (2.22),
\[
\int_{\Omega} |D(\eta G(|Dv|))|^2 \, dx \leq 2 \int_{\Omega} |D\eta|^2 |G(|Dv|)|^2 \, dx + 8 \int_{\Omega} |D\eta|^2 H'(|Dv|)|Dv|^{2\delta+1} \, dx. \tag{2.23}
\]
Since $H'$ is increasing, by (2.10),
\[
G(t) \leq 1 + \int_0^t \sqrt{s^{2\delta-1} H''(s)} \, ds \leq 1 + 2\sqrt{H'(t)} \frac{t^{(2\delta+1)/2}}{2\delta + 1},
\]
which implies, since $\delta \geq 0$,
\[
|G(t)|^2 \leq 2 \left(1 + \frac{4H'(t)}{(2\delta + 1)^2} t^{2\delta+1}\right) \leq 8 \left(1 + H'(t)t^{2\delta+1}\right). \tag{2.24}
\]
By (2.23) and (2.24) it follows that
\[
\int_{\Omega} |D(\eta G(|Dv|))|^2 \, dx \leq 24 \int_{\Omega} |D\eta|^2 \left[1 + H'(|Dv|)|Dv|^{2\delta+1}\right] \, dx.
\]
Set $2^* = 2n/(n - 2)$ if $n > 2$ and $2^* = 3$ if $n = 2$. By Sobolev inequality, there exists a positive constant $C_1 = C_1(n, |\Omega|)$ such that
\[
\left(\int_{\Omega} |\eta G(|Dv|)|^{2^*} \, dx\right)^{2/2^*} \leq C_1 \int_{\Omega} |D\eta|^2 \left[1 + H'(|Dv|)|Dv|^{2\delta+1}\right] \, dx. \tag{2.25}
\]
Now by (2.10), since $H'(t)/t$ is decreasing, we observe that

$$
\left[ G(t) \right]^{2^*} \geq 1 + \left( \int_0^t \sqrt{s^{2^*}H''(s) \, ds} \right)^{2^*} 
= 1 + \left( \frac{H'(t)}{t} \right)^{2^*/2} \frac{1}{\gamma^{2^*/2}} \left( \int_0^t s^{\delta} \, ds \right)^{2^*/2} 
\geq 1 + \left[ H'(t) \right]^{2^*/2} \frac{1}{\gamma^{2^*/2}} \left( \delta + 1 \right)^{2^*/2}
$$

for every $t \in [0, +\infty)$.

Let us assume that $t \geq 1$. By (2.7) we have

$$
H'(t) = \frac{h'\left(\sqrt{\mu^2 + t^2}\right)}{\sqrt{\mu^2 + t^2}} - t
$$

and then, since $H'$ is increasing and $h'(t)/t$ is decreasing, by assuming $\mu \leq 1$ we have

$$
H'(t) \geq H'(1) = \frac{h'\left(\sqrt{\mu^2 + 1}\right)}{\sqrt{\mu^2 + 1}} \geq \frac{h'\left(\sqrt{2}\right)}{\sqrt{2}}.
$$

Then

$$
\left[ G(t) \right]^{2^*} \geq \frac{C_2}{(\delta + 1)^{2^*/2}} \left[ 1 + H'(1) t^{(2\delta + 1)2^*/2} \right],
$$

where

$$
C_2 = \min \left\{ 1, \gamma^{-2^*/2} \left( \frac{h'\left(\sqrt{2}\right)}{\sqrt{2}} \right)^{2^*/2-1} \right\}.
$$

For $t \in [0, 1)$, since $h'$ is increasing, $G(t) \geq 1$ and $\delta \geq 0$ we have

$$
1 + H'(t) t^{(2\delta + 1)2^*/2} \leq 1 + H'(1) \leq \left[ 1 + h'\left(\sqrt{2}\right) \right] \left[ G(t) \right]^{2^*}.
$$

Then for every $t \geq 0$ the inequality

$$
\left[ G(t) \right]^{2^*} \geq \frac{C_3}{(\delta + 1)^{2^*/2}} \left[ 1 + H'(1) t^{(2\delta + 1)2^*/2} \right] \tag{2.26}
$$

holds with

$$
C_3 = \min \left\{ C_2, \frac{1}{1 + h'\left(\sqrt{2}\right)} \right\}.
$$

Let $0 < \rho < R$ be such that $B_R \subset \Omega$ and let us fix $\eta$ in such a way that $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, $\eta = 1$ in $B_\rho$, supp($\eta$) $\subset B_R$ and $|D\eta| \leq 2/(R - \rho)$ in $\mathbb{R}^n$. Then by (2.25) and (2.26) we get
\[
\int_{B_{\rho}} \left[ 1 + H'(|Dv|)|Dv|^{(2\delta+1)2^*/2} \right] dx^{2^*/2} \leq \frac{C_4(\delta + 1)^2}{(R - \rho)^2} \int_{B_{R}} \left[ 1 + H'(|Dv|)|Dv|^{2^*/2} \right] dx,
\]

(2.27)

where \( C_4 = 4C_1C_4^{-2/2^*} \).

Let us set \( \vartheta = 2\delta + 1 \). Then (2.27) becomes

\[
\left( \int_{B_{\rho}} \left[ 1 + H'(|Dv|)|Dv|^{2^*/2} \right] dx \right)^{2^*/2} \leq \frac{C_4\vartheta^2}{(R - \rho)^2} \int_{B_{R}} \left[ 1 + H'(|Dv|)|Dv|^{\vartheta} \right] dx.
\]

(2.28)

Now we define a sequence of radii and another one of numbers as follows:

\[
\rho_i = \rho + \frac{R - \rho}{2^i}, \quad \vartheta_i = \left( \frac{2^*}{2} \right)^i
\]

for \( i = 0, 1, 2, \ldots \). Moreover we set

\[
A_i = \left( \int_{B_{\rho_i}} \left[ 1 + H'(|Dv|)|Dv|^{\vartheta_i} \right] dx \right)^{1/\vartheta_i}.
\]

Using this notation, and putting \( \rho = \rho_{i+1}, R = \rho_i \) and \( \vartheta = \vartheta_i \) in (2.28) we easily have

\[
A_{i+1} \leq \left[ \frac{C_4(\delta + 1)^2\vartheta^2}{(R - \rho)^2} \right]^{1/\vartheta_i} A_i,
\]

thus, if we iterate this estimate,

\[
A_{i+1} \leq \left\{ \prod_{k=0}^{i} \left[ \frac{C_4(\delta + 1)^2\vartheta^2}{(R_0 - \rho_0)^2} \right]^{1/\vartheta_k} \right\} A_0 \leq C_5 A_0,
\]

(2.29)

where \( C_5 = C_5(n, \gamma, |\Omega|, h'(\sqrt{2}), \rho, R) \) and, in particular,

\[
C_5 = \left( 1 + \frac{4C_4}{(R - \rho)^2} \right)^{2^*/(2^* - 2)} (2^* + 1)^{2^*/(2^* - 2)}.
\]

Then (2.29) leads to

\[
\left( \int_{B_{\rho_0}} \left[ 1 + H'(|Dv|)|Dv|^{(2\delta + 1)2^*/2} \right] dx^{2^*/2} \right)^{2^*/2} \leq C_5 \int_{B_{R_0}} \left[ 1 + H'(|Dv|)|Dv| \right] dx < +\infty.
\]

(2.30)
Now we observe that, since $H'$ is increasing and $H'(t)/t$ is decreasing, for every $\tau > 1$ and every $t \geq 1$ we have

$$H'(t)t^\tau \geq H'(1)t^\tau \geq \frac{h'(\sqrt{2})}{\sqrt{2}}t^\tau.$$ 

Then we can say that, for every $t > 0$ and every $\tau > 1$,

$$t^\tau \leq C_6(1 + H'(t)t^\tau),$$

where

$$C_6 = \max \left\{ 1, \frac{\sqrt{2}}{h'(\sqrt{2})} \right\}.$$ 

Therefore by (2.30) it follows that

$$\sup_{B^R} |Dv| = \lim_{i \to +\infty} \left( \int_{B^R} |Dv|^{(2^*/2)^{i+1}} \right)^{(2^*/2)^{i+1}} \leq \lim_{i \to +\infty} \left( C_6 \int_{B^R} \left[ 1 + H'(|Dv|) |Dv|^{(2^*/2)^{i+1}} \right] \right)^{(2^*/2)^{i+1}} \leq C_5 \int_{B^R} \left[ 1 + H'(|Dv|) |Dv| \right] dx \leq 2C_5 \int_{B^R} \left[ 1 + H(|Dv|) \right] dx,$$

where we used the $\Delta_2^2$ property in the last inequality. Thus (2.18) holds true if we check the way $C_5$ depends on $h'(\sqrt{2})$.

A careful inspection shows that

$$C_5 \leq C_7 \left[ V(h'(\sqrt{2})) \right]^{2/(2^*-2)},$$

where $V(t) = 1 + t + c_0 t^{-\vartheta}$, $c_0 = c_0(n, \gamma) > 0$, $\vartheta = \vartheta(n) > 0$ and $C_7 = C_7(n, |\Omega|, \rho, R) > 0$. This ends the second step of the proof.

Step 3 (Let $\mu$ go to 0).

We proceed as in Lemma 2.13 of [1].

Let $h$ satisfy conditions (H1) and (H2). We recall that $u$ is a local minimizer of $I$ defined by (2.4) and (2.5). Let $B_R$ be a ball such that $B_R \subseteq \Omega$ and, for every $\mu \in (0, 1)$, let us define the function

$$H_\mu(t) = h(\sqrt{\mu^2 + t^2}) - h(\mu).$$

We consider the variational problem in $B_R$,

$$\min_{I_\mu(v)} = \int_{B_R} f_\mu(Dv) dx: v \in u + W^{1,q}_0(B_R, \mathbb{R}^N).$$

(2.31)

where $f_\mu(z) = H_\mu(|z|)$. Because of (2.12) and (2.14), there exists a unique solution $v_\mu$ of (2.31). Then we have
Now, let us consider a sequence \( \{\mu_k\} \in (0,1) \), with \( \mu_k \to 0 \). Then, up to a subsequence, \( Dv_{\mu_k} \rightharpoonup Du_0 \) in \( L^q(B_R) \), for some function \( u_0 \in W^{1,q}_0(B_R, \mathbb{R}^N) \) and eventually, by lower semicontinuity and (2.32),

\[
\int_{B_R} f(Du_0) \, dx = \int_{B_R} h(|Du_0|) \, dx \leq \liminf_{k \to +\infty} \int_{B_R} h(|Dv_{\mu_k}|) \, dx \\
\leq \liminf_{k \to +\infty} \int_{B_R} h(\sqrt{\mu_k^2 + |Du|^2}) \, dx = \int_{B_R} h(|Du|) \, dx \\
= \int_{B_R} f(Du) \, dx.
\]

Thus \( u \) and \( u_0 \) are minimizers with the same boundary datum; since \( f \) is strictly convex, it follows that \( u_0 = u \).

Let \( 0 < \rho < R \); we use Step 2 with balls \( B_{\rho} \) and \( B_{(\rho+R)/2} \), so that the minimality of \( v_{\mu_k} \) with respect to \( u \) gives

\[
\sup_{B_\rho} |Dv_{\mu_k}| \leq c \int_{B_R} [1 + H_{\mu_k}(|Dv_{\mu_k}|)] \, dx \\
\leq c \int_{B_R} [1 + h(\mu_k^2 + |Du|^2)] \, dx \leq c \int_{B_R} [1 + h(\sqrt{1 + |Du|^2})] \, dx,
\]

thus

\[
\sup_{B_\rho} |Dv_{\mu_k}| \leq c, \quad \forall k,
\]

for some constant \( c \) independent of \( \mu_k \). Then, up to a subsequence, \( \{Dv_{\mu_k}\} \) converges in the weak-* topology of \( L^\infty(B_\rho) \), to some function \( w \in L^\infty(B_\rho) \) that turns out to be \( Du \).

The lower semicontinuity of the \( L^\infty \)-norm gives

\[
\sup_{B_\rho} |Du| \leq c \int_{B_R} [1 + h(|Du|)] \, dx,
\]

where \( c = \tilde{c}(V(h'(\sqrt{2})))^{2/(2^* - 2)} \) and \( \tilde{c} = \tilde{c}(n, N, \rho, R) > 0 \).
3. Proof of Theorem 1.1

In this section we study the regularity of minimizers of integral functionals assuming this time that the integrand \( f \) satisfies nonstandard growth. For convenience of the reader, we recall the assumptions of Theorem 1.1

Let \( 1 < p < q < 2 \) and \( g \) be an \( N \)-function. We assume that there exist two constants \( \Lambda_1, \Lambda_2 > 0 \) such that

\[
\text{(G1)} \quad g \in C^2((0, +\infty)) \cap C^1([0, +\infty)), \quad g'(0) = 0, \quad g'(t)/t \text{ is decreasing and}
\]

\[
\Lambda_1 t^{p-2} \leq \frac{g'(t)}{t} \leq \Lambda_2 (t^{q-2} + t^{p-2}); \quad (3.1)
\]

\[
\text{(G2)} \quad \text{There exists } \gamma > 1 \text{ such that}
\]

\[
g''(t)t \leq g'(t) \leq \gamma g''(t)t. \quad (3.2)
\]

As we already observed, by (3.1) and (3.2) it follows that

\[
\frac{\Lambda_1}{p} t^p \leq g(t) \leq \frac{\Lambda_2}{p} (t^q + t^p)
\]

and \( g \in \Delta^2_2 \).

Consider, for \( \sigma \in (0, 1) \), the functions

\[
g_\sigma(t) = g(t) + \sigma t^q.
\]

As it can be easily checked, \( g_\sigma : [0, +\infty) \rightarrow [0, +\infty) \) is an \( N \)-function strictly convex, \( g_\sigma \in C^2((0, +\infty)) \cap C^1([0, +\infty)) \), \( g_\sigma'(0) = 0 \) and \( t \to g_\sigma'(t)/t \) is decreasing in \((0, +\infty)\). Furthermore we have that

\[
\sigma q t^{q-2} \leq \frac{g_\sigma'(t)}{t} \leq (\Lambda_2 + q)(t^{q-2} + t^{p-2}),
\]

\[
g_\sigma''(t)t \leq g_\sigma'(t) \leq \max\left\{\gamma, \frac{1}{q-1}\right\} g_\sigma''(t)t.
\]

Now let

\[
f_\sigma(z) = g_\sigma(|z|)
\]

for every \( \varepsilon \in \mathbb{R}^{nN} \) and consider the functional

\[
I_\sigma(v) = \int_{\Omega} f_\sigma(Dv) \, dx. \quad (3.3)
\]

Let \( 0 < \varepsilon < \min\{1, R\} \), where \( R > 0 \) is such that \( B_{2R} \subseteq \Omega \). Moreover let \( \{u_i\}_i \) be a sequence of smooth functions obtained from \( u \) by means of standard mollifiers, then \( u_i \in W^{1,q}(B_R, \mathbb{R}^N) \).

Since \( I_\sigma \) has \( q \)-growth, we consider the following variational problem:

\[
\min\left\{I_\sigma(v): \quad v \in u_\varepsilon + W_0^{1,q}(B_R, \mathbb{R}^N)\right\} \quad (3.4)
\]
and let \( v_{\epsilon, \sigma} \in u_{\epsilon} + W^{1,q}_{0}(B_{R}, \mathbb{R}^{N}) \) be the (unique) minimizer. We are now in condition to apply Proposition 2.1 for \( 0 < \rho < R \),

\[
\sup_{B_{\rho}} |Dv_{\epsilon, \sigma}| \leq \tilde{c} \left[ V(g_{\sigma}'(\sqrt{2})) \right]^{2/(2^{*}-2)} \int_{B_{R}} \left[ 1 + g_{\sigma}(|Du_{\epsilon}|) \right] dx = (I).
\]

Let us point out that

\[
0 < g'(\sqrt{2}) \leq g_{\sigma}'(\sqrt{2}) \leq g'(\sqrt{2}) + 4
\]

thus

\[
V(g_{\sigma}'(\sqrt{2})) \leq 5V(g'(\sqrt{2}))
\]

and

\[
(I) \leq \tilde{c} \left[ 5V(g'(\sqrt{2})) \right]^{2/(2^{*}-2)} \int_{B_{R}} \left[ 1 + g_{\sigma}(|Du_{\epsilon}|) \right] dx.
\]

Now we use the minimality of \( v_{\epsilon, \sigma} \) with respect to \( u_{\epsilon} \) and Jensen inequality,

\[
\frac{A_{1}}{p} \int_{B_{R}} |Du_{\epsilon, \sigma}|^{p} dx \leq \int_{B_{R}} g(|Du_{\epsilon, \sigma}|) dx \leq \int_{B_{R}} g_{\sigma}(|Du_{\epsilon, \sigma}|) dx
\]

\[
\leq \int_{B_{R}} g_{\sigma}(|Du_{\epsilon}|) dx = \int_{B_{R}} g(|Du_{\epsilon}|) dx + \sigma \int_{B_{R}} |Du_{\epsilon}|^{q} dx
\]

\[
\leq \int_{B_{R+\epsilon}} g(|Du|) dx + \sigma \int_{B_{R+\epsilon}} |Du|^{q} dx \leq c(\epsilon) \quad (3.5)
\]

and

\[
\sup_{B_{\rho}} |Dv_{\epsilon, \sigma}| \leq \tilde{c} \left[ 5V(g'(\sqrt{2})) \right]^{2/(2^{*}-2)} \left\{ \int_{B_{R+\epsilon}} \left[ 1 + g(|Du|) \right] dx + \sigma \int_{B_{R+\epsilon}} |Du|^{q} dx \right\}.
\]

\[
\sup_{B_{\rho}} |Du_{\epsilon}| \leq c \left[ 5V(g'(\sqrt{2})) \right]^{2/(2^{*}-2)} \left\{ \int_{B_{R+\epsilon}} \left[ 1 + g(|Du|) \right] dx + \sigma \int_{B_{R+\epsilon}} |Du|^{q} dx \right\}. \quad (3.6)
\]

Then for every fixed \( \epsilon \), (3.5) gives us weak compactness in \( L^{p}(B_{R}) \) as \( \sigma \to 0 \). So, up to a subsequence \( Du_{\epsilon, \sigma} \rightharpoonup Du_{\epsilon} \) in \( L^{p}(B_{R}) \) as \( \sigma \to 0 \), for some \( w_{\epsilon} \in u_{\epsilon} + W^{1,p}_{0}(B_{R}, \mathbb{R}^{N}) \).

Moreover, by (3.6), \( \sup_{B_{R}} |Du_{\epsilon, \sigma}| \) is equibounded with respect to \( \sigma \). Hence \( \{Du_{\epsilon, \sigma}\}_{\sigma} \) converges in the weak-* topology of \( L^{\infty} \) to \( Du_{\epsilon} \) and

\[
\sup_{B_{\rho}} |Du_{\epsilon}| \leq c \left[ 5V(g'(\sqrt{2})) \right]^{2/(2^{*}-2)} \left\{ \int_{B_{R+\epsilon}} \left[ 1 + g(|Du|) \right] dx + \sigma \int_{B_{R+\epsilon}} |Du|^{q} dx \right\}. \quad (3.7)
\]

for some \( c = c(n, N, \rho, R) > 0 \). By lower semicontinuity in (3.5) we get

\[
\int_{B_{R}} g(|Du_{\epsilon}|) dx \leq \liminf_{\sigma \to 0} \int_{B_{R+\epsilon}} g(|Du_{\epsilon, \sigma}|) dx \leq \int_{B_{R+\epsilon}} g(|Du|) dx.
\]

(3.8)

Now, (3.8) gives weak compactness in \( L^{p}(B_{R}) \) as \( \epsilon \to 0 \), thus up to a subsequence, \( Du_{\epsilon} \rightharpoonup Dw \) in \( L^{p}(B_{R}) \) for some function \( w \in u + W^{1,p}_{0}(B_{R}, \mathbb{R}^{N}) \).
Lower semicontinuity and (3.8) allow us to write
\[
\int_{B_R} g(|Dw|) \, dx \leq \liminf_{\varepsilon \to 0} \int_{B_R} g(|Dw_{\varepsilon}|) \, dx \leq \int_{B_R} g(|Du|) \, dx.
\]
The minimality of \( u \) and the strict convexity of \( g \) imply \( w \equiv u \).

Finally, using (3.7) we obtain that also \( Dw_{\varepsilon} \) converges to \( Dw = Du \) as \( \varepsilon \to 0 \), in the weak-* topology of \( L^\infty \) and, letting \( \varepsilon \to 0 \) in (3.7) we easily get
\[
\sup_{B_\rho} |Du| \leq c \left[ 5V\left(g'(\sqrt{2})\right) \right]^{2/(2^*-2)} \int_{B_R} \left[ 1 + g(|Du|) \right] \, dx
\]
for some \( c = c(n, N, \rho, R) \). \( \square \)

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