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A uniqueness result for a class of non-strictly convex variational problems ☆

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ABSTRACT

Let Ω be a smooth domain in $\mathbb{R}^2$, we prove that if $g\colon [0,+\infty) \to [0,+\infty]$ is convex with $g(0) < g(t)$ whenever $t > 0$ then there exists an unique minimizer $u \in C^{0,1}(\Omega)$ of the functional $u \mapsto \int_\Omega g(|\nabla u|)\,dxdy$ among all Lipschitz-continuous functions that assume the same value of u on $\partial\Omega$.

1. Introduction

Let us consider an integral of the Calculus of Variations

$$F(u) = \int_\Omega f(x, u(x), \nabla u(x))\,dx \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, u is a real function defined on Ω in a Sobolev space, say $W^{1,p}(\Omega)$, and $f(x, s, \xi)$ is a Caratheodory function, i.e. measurable in x and continuous in s, ξ. The study of the existence of minimizers of F in a Dirichlet class $u \in u_0 + W^{1,p}(\Omega)$ via Direct Methods is based on the (sequential) lower semicontinuity of F in the weak topology of $W^{1,p}(\Omega)$. It is well known, starting by the classical work of Tonelli, that the lower semicontinuity of F is linked to the convexity of the integrand f with respect to the variable ξ. However, for integrand function not strictly convex uniqueness is not guaranteed.

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In this paper we are interested to uniqueness of minimizers of functionals of the form

$$G(u) = \int_{\Omega} g(|\nabla u(x)|)\, dx \tag{1.2}$$

with suitable prescribed boundary conditions, when g is convex but not necessarily strictly convex. The problem of uniqueness of minimizers for non-strictly convex functionals (1.2) appears when one deals with a non-convex problem and applies the relaxation methods. In fact, the existence or not existence for non-convex integrals is related to the non-uniqueness of minimizers of the (not strictly) convexified problem. The mathematical literature on non-convex problems is quite large starting by the results of Bogolyubov [3] and later by Marcellini [12] in one dimension. For $n \geq 2$ we recall Aubert–Tahraoui [2], Mascolo–Schianchi [14–16], Cellina [7], Friesecke [9], Zagatti [19], Sychev [18], Celada–Perrotta [6] and Fonseca–Fusco–Marcellini [8] and Celada–Cupini–Guidorzi [5], through Lipschitz-continuous regularity results for minimizers. We refer to the previous articles for the detailed bibliography on the subject. On the other hand, the uniqueness for non-strictly convex functionals is a classical question and it is however interesting in his own right.

A first uniqueness result is due to Parks [17] which shows the uniqueness of mimimizer for the functional

$$\int_{\Omega} |\nabla u|\, dx \tag{1.3}$$

i.e. $g(t) = t$ provided that the boundary datum satisfies the bounded slope condition. The arguments of Parks's Theorem utilize the fact that since u has the least gradient property, the level sets $E_\lambda = \{x \in \Omega : u(x) \geq \lambda\}$ have the oriented boundary of least area, by the results of Bombieri–De Giorgi–Giusti in [4]. Unfortunately, the elegant approach of Parks does not work for general functionals of type (1.2). Indeed, the integral (1.3) can be reconstructed starting from what happens on the level sets by means of the coarea formula, but for more general non-linear functionals of the form (1.2) the coarea formula does not hold.

A very interesting uniqueness result for non-strictly convex functionals under the assumption

$$g(0) < g(t) \quad \text{for } t > 0 \tag{1.4}$$

is due to Marcellini [13]. In dimension $n \geq 2$, he proved that if G in (1.2) has a minimizer u such that

$$u \in C^1(\overline{\Omega}) \text{ and } Du \neq 0 \text{ everywhere on } \overline{\Omega} \tag{1.5}$$

then u is the unique minimizer of G in the class of all Lipschitz continuous functions that assume the same value of u on $\partial\Omega$. Let us remark that the strict inequality in (1.4) is crucial in order to have uniqueness of the minimizer. Indeed, let us consider the boundary condition constant, say c, then the constant function $u = c$ is a minimizer of G. But, if there exists $t_0 > 0$ such that $g(t) = g(0)$ for any $t \in [0, t_0]$ then for any $\phi \in C_c^\infty(\Omega)$ with $||\nabla\phi||_\infty < t_0$ we get $G(u) = G(u + \phi)$ so that the function $c + \phi$ is still a minimizer of G.

For completeness, we mention also the partial uniqueness result by Kawohl–Stara–Wittum [11].

Inspired by the fundamental contributions of Parks and Marcellini, we show, at least when $n = 2$, that it is possible to remove assumptions (1.5), thus we answer the long standing open question which Marcellini placed in [13]. Precisely, we prove the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and open set and let $g\colon [0, +\infty) \to [0, +\infty]$ be convex and such that (1.4) holds true. Let $G\colon C^{0,1}(\overline{\Omega}) \to \mathbb{R}$ be given by*

$$G(u) := \int_{\Omega} g(|\nabla u(x, y)|)\, dxdy \tag{1.6}$$

If $u \in C^{0,1}(\overline{\Omega})$ is a minimizer of G among all Lipschitz continuous functions that assume the same value of u on the boundary $\partial\Omega$, then u is the unique minimizer in that class.

We observe that in order to apply the Direct Method of the Calculus of Variations, $C^{0,1}(\Omega)$ is the proper competitor class for the convex functional G in (1.6). In fact, the well known theorem of Hartmann–Stampacchia [10] ensures that there is at least one minimizer of G in the class of Lipschitz-continuous functions, when the boundary datum satisfies the so-called *bounded slope condition*. Moreover, it is essential to assume that the boundary condition is continuous, at least if we look for solutions in $BV(\Omega)$ i.e. in the class of L^1-functions with derivatives which are measures with bounded total variation. Indeed, Marcellini in [13] gives the following example, that for completeness we briefly describe. Take $g(t) = t$ and extend the functional G to be the total variation functional. Then, consider $\Omega := B_1(0)$ in $\mathbb{R}^2$ and take $u_1, u_2 \colon \overline{\Omega} \to \mathbb{R}$ given by

$$u_1(x,y) := \begin{cases} 1 & \text{if } |x| \le \sqrt{2}/2 \\ -1 & \text{if } |x| > \sqrt{2}/2 \end{cases}, \quad u_2(x,y) := \begin{cases} 1 & \text{if } |y| \le \sqrt{2}/2 \\ -1 & \text{if } |y| > \sqrt{2}/2. \end{cases}$$

Then, u_1 and u_2 have the same discontinuous boundary condition and both u_1 and u_2 minimize the total variation among all BV-functions with the same boundary condition.

The arguments of the proof of Theorem 1.1 are similar to those of Marcellini in [13]. The structure of level sets of Lipschitz functions is crucial in order to obtain the proof without assuming (1.5). The assumption $n = 2$ permits us to apply directly the result obtained by Alberti–Bianchini–Crippa in [1], where are established significant and fine properties of the level sets of general Lipschitz-continuous map from $\mathbb{R}^d$ to $\mathbb{R}^{d-k}$, $d \ge 2$ and $k < d$.

Finally, we remark that to study the higher dimension case, it could be really necessary to insert the problem in the framework of rectifiable currents, this is under investigation and it will be the subject of a forthcoming paper in collaboration with G. Alberti.

2. Proof of Theorem 1.1

In what follows for any $A \subset \mathbb{R}^2$ open and bounded the notation $C^{0,1}(A)$ stands for the class of Lipschitz-continuous functions on A. For any $w \in C^{0,1}(\mathbb{R}^2)$ we will denote by $S(w)$ the set of all points $(x,y) \in \mathbb{R}^2$ where w is either not differentiable at (x,y) or $\nabla w(x,y) = 0$; notice that the set of all points where w is not differentiable is negligible by Rademacher's Theorem. For any $s \in \mathbb{R}$ we denote by w^s the s-level set of w, that is $w^s := w^{-1}(s)$, and by w^s_* the union of all connected components C of w^s such that $\mathcal{H}^1(C) > 0$; it turns out that w^s_* is a Borel set [1, Prop. 6.1].

In order to prove uniqueness of the minimizer following the idea of Marcellini [13] we apply the characterization of the level sets of Lipschitz-continuous functions of [1, Thm. 2.5]. More precisely:

Theorem 2.1. *[1, Thm. 2.5, (iii)–(iv)] Let $f \in C^{0,1}(\mathbb{R}^2)$ with compact support. For a.e. $s \in \mathbb{R}$ the following statements hold:*

(i) $\mathcal{H}^1(f^s \setminus f^s_*) = 0$;
(ii) *every connected component C of f^s is either a point or a closed simple curve with a Lipschitz parametrization $\gamma\colon [a,b) \to C$ which is injective and satisfies $\gamma(t) \notin S(f)$ and $|\gamma'(t)| = 1$ for a.e. $t \in [a,b)$.*

The following proposition is a very easy variant of Theorem 2.1.

Proposition 2.2. *Fix $w \in C^{0,1}(\mathbb{R}^2)$ with compact support and a negligible set $\Omega_0 \subset \mathbb{R}^2$. Then, for a.e. $s \in \mathbb{R}$ the following statements hold:*

(i) $\mathcal{H}^1(w^s \setminus w^s_*) = 0$;
(ii) *every connected component C of w^s is either a point or a closed simple curve with a Lipschitz and injective parametrization $\gamma\colon [a,b) \to C$ which satisfies $\gamma(t) \notin S(w) \cup \Omega_0$ and $|\gamma'(t)| = 1$ for a.e. $t \in [a,b)$.*

Proof. First of all (i) follows from Theorem 2.1. Let us denote by A the set of all $s \in \mathbb{R}$ such that every connected component C of w^s is either a point or a closed simple curve with a Lipschitz and injective parametrization $\gamma\colon [a,b) \to C$ which satisfies $\gamma(t) \notin S(w)$ and $|\gamma'(t)| = 1$ for a.e. $t \in [a,b)$. Then, (ii) of Theorem 2.1 says that $|\mathbb{R} \setminus A| = 0$. Now, notice that if χ denotes the characteristic function of $S(w) \cup \Omega_0$, namely $\chi(x,y) = 1$ if $(x,y) \in S(w) \cup \Omega_0$ and $\chi(x,y) = 0$ otherwise, then the coarea formula implies that

$$\int_{\mathbb{R}^2} \chi(x,y)|\nabla w(x,y)|\,dxdy = \int_{-\infty}^{+\infty} \mathcal{H}^1(w^s \cap (S(w) \cup \Omega_0))\,ds.$$

Since $\chi|\nabla w| = 0$ a.e. in $\mathbb{R}^2$ we can deduce that $\mathcal{H}^1(w^s \cap (S(w) \cup \Omega_0)) = 0$ for a.e. $s \in \mathbb{R}$. Let $B := \{s \in \mathbb{R} : \mathcal{H}^1(w^s \cap (S(w) \cup \Omega_0)) = 0\}$. We immediately have $|\mathbb{R} \setminus (A \cap B)| = 0$, and $A \cap B$ is the set of all $s \in \mathbb{R}$ such that every connected component C of w^s is either a point or a closed simple curve with a Lipschitz and injective parametrization $\gamma\colon [a,b) \to C$ which satisfies $\gamma(t) \notin S(w) \cup \Omega_0$ and $|\gamma'(t)| = 1$ for a.e. $t \in [a,b)$, which yields the conclusion. $\square$

From now on, let $u \in C^{0,1}(\overline{\Omega})$ be a minimizer of G among all Lipschitz-continuous functions $\overline{\Omega} \to \mathbb{R}$ that assume the same value of u on $\partial\Omega$.

Lemma 2.3. *Let $(x_0, y_0) \in \Omega$ and $s_0 := u(x_0, y_0)$. Assume that the connected component C of u^{s_0} which contains (x_0, y_0) is a closed simple curve. Then $C \cap \partial\Omega \neq \emptyset$.*

Proof. Assume by contradiction that $C \subset \Omega$. By Jordan's Theorem C is the boundary of a non-empty, open and bounded region $V \subset \Omega$; in particular $|V| > 0$. Let $w\colon \overline{\Omega} \to \mathbb{R}$ be given by

$$w(x,y) := \begin{cases} u(x,y) & \text{if } (x,y) \in \overline{\Omega} \setminus V \\ s_0 & \text{if } (x,y) \in V. \end{cases}$$

It is easy to see that w remains Lipschitz-continuous and by construction $u = w$ on $\partial\Omega$. But $\nabla w = 0$ on V and $|V| > 0$, hence $G(w) < G(u)$ because we are assuming that $g(0) < g(t)$ for every $t > 0$, and this is a contradiction since u is a minimizer for G. $\square$

Remark 2.4. We notice that if u is constant on $\partial\Omega$ then u must be constant on $\overline{\Omega}$ by condition (1.4). Thus, in what follows we can restrict the analysis to the case $|\Omega \setminus S(u)| > 0$.

From now on $v \in C^{0,1}(\overline{\Omega})$ denotes a different minimizer of G among all Lipschitz-continuous functions $\overline{\Omega} \to \mathbb{R}$ that assume the same value of u on $\partial\Omega$. The following lemma is contained in the proof of Theorem 3 of [13], for the sake of completeness we give the proof.

Lemma 2.5. *It turns out that*

$$\nabla v(x,y) = \lambda(x,y)\nabla u(x,y) \quad \textit{for a.e. } (x,y) \in \Omega \setminus S(u) \tag{2.1}$$

for some measurable function $\lambda\colon \Omega \to \mathbb{R}$.

Proof. Using the convexity of g for a.e. $(x,y) \in \Omega$ we have

$$g\left(\frac{|\nabla u(x,y) + \nabla v(x,y)|}{2}\right) \leq \frac{g(|\nabla u(x,y)|) + g(|\nabla v(x,y)|)}{2}.$$

Then from the minimality of u and v we get

$$G(u) \leq \int_\Omega g\left(\frac{|\nabla u(x,y) + \nabla v(x,y)|}{2}\right) dxdy \leq \frac{1}{2}\int_\Omega g(|\nabla u(x,y)|) + g(|\nabla v(x,y)|)\, dxdy = G(u).$$

Hence

$$\int_\Omega g\left(\frac{|\nabla u(x,y) + \nabla v(x,y)|}{2}\right) - \frac{g(|\nabla u(x,y)|) + g(|\nabla v(x,y)|)}{2}\, dxdy = 0$$

which means that

$$g\left(\frac{|\nabla u(x,y) + \nabla v(x,y)|}{2}\right) = \frac{g(|\nabla u(x,y)|) + g(|\nabla v(x,y)|)}{2}, \quad \text{for a.e. } (x,y) \in \Omega.$$

It follows that there exist real valued functions $m = m(x,y), q = q(x,y)$ such that

$$\frac{m(x,y)}{2}|\nabla u(x,y) + \nabla v(x,y)| + q(x,y) = \frac{1}{2}(m(x,y)|\nabla u(x,y)| + m(x,y)|\nabla v(x,y)| + 2q(x,y))$$

for a.e. $(x,y) \in \Omega$, that is

$$m(x,y)|\nabla u(x,y) + \nabla v(x,y)| = m(x,y)(|\nabla u(x,y)| + |\nabla v(x,y)|), \quad \text{a.e. } (x,y) \in \Omega. \tag{2.2}$$

Notice now that since g is convex and we are assuming $g(0) < g(t)$ whenever $t > 0$ we can also say that $m(x,y) > 0$. Hence, simplifying (2.2) we deduce that $\nabla u(x,y)$ and $\nabla v(x,y)$ are linearly dependent for a.e. $(x,y) \in \Omega \setminus S(u)$ and since $\nabla u(x,y) \neq 0$ everywhere on $\Omega \setminus S(u)$ we obtain (2.1). □

We are ready to prove the main theorem.

Proof of Theorem 1.1. Let $v \in C^{0,1}(\overline{\Omega})$ be a minimizer of G among all Lipschitz-continuous functions $\overline{\Omega} \to \mathbb{R}$ that assume the same value of u on $\partial\Omega$.

Step 1. It turns out that $u = v$ everywhere on $\Omega \setminus S(u)$.

For the sake of convenience, we extend both u and v to functions in $C^{0,1}(\mathbb{R}^2)$ with compact support and we do not relabel it: observe that this is always possible using, for instance, the McShane's Extension Theorem.

Let us regularize v by convolution: given a family of mollifiers $\{\rho_\varepsilon\}_{\varepsilon>0}$ let $v_\varepsilon := v * \rho_\varepsilon$. Then, v_ε and ∇v_ε have compact support, $v_\varepsilon \to v$ uniformly and $\nabla v_\varepsilon \to \nabla v$ a.e. in Ω as $\varepsilon \to 0$. Let

$$\Omega_0 := \Omega \setminus \{(x,y) \in \Omega : \nabla v_\varepsilon(x,y) \to \nabla v(x,y)\}.$$

Observe that Ω_0 is negligible. Let Y be the set of all $(x,y) \in \Omega \setminus S(u)$ which are Lebesgue point for $|\nabla u|$. Since $\nabla u \in L^\infty(\mathbb{R}^2)$ we have $|(\Omega \setminus S(u)) \setminus Y| = 0$, hence it is sufficient to show that $u = v$ everywhere on Y. Take $(\bar{x}, \bar{y}) \in Y$. By definition,

$$\lim_{\rho \to 0} \fint_{B_\rho(\bar{x},\bar{y})} |\nabla u(x,y)| \, dxdy > 0. \tag{2.3}$$

Denote by L the Lipschitz constant of u. Fix $r > 0$ and take $s \in u(B_r(\bar{x},\bar{y}))$; then $s = u(x,y)$ for some $(x,y) \in B_r(\bar{x},\bar{y})$. Thus $|s - u(\bar{x},\bar{y})| = |u(x,y) - u(\bar{x},\bar{y})| \leq Lr$ which means that

$$u(B_r(\bar{x},\bar{y})) \subset (u(\bar{x},\bar{y}) - Lr, u(\bar{x},\bar{y}) + Lr). \tag{2.4}$$

Assume that for a.e. $s \in u(B_r(\bar{x},\bar{y}))$ the set $u^s \cap B_r(\bar{x},\bar{y})$ reduces to points. Then

$$u^s \cap B_r(\bar{x},\bar{y}) = (u^s \cap B_r(\bar{x},\bar{y})) \setminus u^s_*$$

and therefore by (i) of Theorem 2.2 we also have

$$\mathcal{H}^1(u^s \cap B_r(\bar{x},\bar{y})) = \mathcal{H}^1(u^s \cap B_r(\bar{x},\bar{y})) \setminus u^s_*) = 0$$

for a.e. $s \in u(B_r(\bar{x},\bar{y}))$. Thus, by the coarea formula

$$\int_{B_r(\bar{x},\bar{y})} |\nabla u(x,y)| \, dxdy = \int_{\infty}^{+\infty} \mathcal{H}^1(u^s \cap B_r(\bar{x},\bar{y})) \, ds = 0.$$

Therefore, $|\nabla u| = 0$ a.e. on $B_r(\bar{x},\bar{y})$ which contradicts (2.3). Combining Proposition 2.2 with (2.4) we have that there exists a sequence $s_h \to u(\bar{x},\bar{y})$ such that for all $h > 0$ at least one connected component C_h of u^{s_h} satisfies the following properties: C_h is a closed simple curve with a Lipschitz and injective parametrization $\gamma_h \colon [a_h, b_h) \to C_h$, $\gamma_h(t) \notin S(u) \cup \Omega_0$ and $|\gamma_h'(t)| = 1$ for a.e. $t \in [a_h, b_h)$, and $C_h \cap B_{1/h}(\bar{x},\bar{y}) \neq \emptyset$. In particular, we find $(x_h, y_h) \in C_h$ with $(x_h, y_h) \to (\bar{x},\bar{y})$. Applying Lemma 2.3 we get $C_h \cap \partial\Omega \neq \emptyset$.

Let $\alpha_h \in [a_h, b_h)$ be such that $\gamma_h(\alpha_h) = (x_h, y_h)$ and

$$\beta_h := \min\{t \in (\alpha_h, b_h) : \gamma_h(t) \in \partial\Omega\}$$

and define

$$\bar{\gamma}_h \colon [\alpha_h, \beta_h] \to \overline{\Omega}, \quad \bar{\gamma}_h := \gamma_h|_{[\alpha_h,\beta_h]}.$$

Then $\bar{\gamma}_h$ is a Lipschitz curve inside $\overline{\Omega}$ connecting (x_h, y_h) with $\partial\Omega$, with $\bar{\gamma}_h(t) \notin S(u) \cup \Omega_0$ and with $|\bar{\gamma}_h'(t)| = 1$ for a.e. $t \in [\alpha_h, \beta_h]$ and for any $h > 0$.

Since $v_\varepsilon \circ \bar{\gamma}_h$ is still Lipschitz-continuous and $v_\varepsilon \in C_c^\infty(\mathbb{R}^2)$, we have

$$v_\varepsilon(x_h, y_h) - v_\varepsilon(\bar{\gamma}_h(\beta_h)) = \int_{\beta_h}^{\alpha_h} \nabla v_\varepsilon(\bar{\gamma}_h(t)) \cdot \bar{\gamma}_h'(t) \, dt. \tag{2.5}$$

Observe that $|\nabla v_\varepsilon(\bar{\gamma}_h(t)) \cdot \bar{\gamma}_h'(t)| \leq c$ for some constant $c > 0$ since v_ε are uniformly Lipschitz and $|\bar{\gamma}_h'(t)| = 1$ for a.e. $t \in [\alpha_h, \beta_h]$. Therefore, using the Dominated Convergence's Theorem and (2.1) since $\bar{\gamma}_h(t) \notin S(u) \cup \Omega_0$ a.e. on $[\alpha_h, \beta_h]$ we can pass to the limit as $\varepsilon \to 0$ in (2.5) obtaining

$$\begin{aligned} v(x_h, y_h) - v(\bar{\gamma}_h(\beta_h)) &= \int_{\beta_h}^{\alpha_h} \nabla v(\bar{\gamma}_h(t)) \cdot \bar{\gamma}_h'(t)\, dt \\ &= \int_{\beta_h}^{\alpha_h} \lambda(\bar{\gamma}_h(t)) \nabla u(\bar{\gamma}_h(t)) \cdot \bar{\gamma}_h'(t)\, dt \\ &= \int_{\beta_h}^{\alpha_h} \lambda(\bar{\gamma}_h(t)) \frac{d}{dt} u(\bar{\gamma}_h(t))\, dt = 0 \end{aligned}$$

where the last equality follows since u is constant along $\bar{\gamma}_h$. Hence, for any $h > 0$ we get $v(x_h, y_h) = v(\bar{\gamma}_h(\beta_h)) = u(\bar{\gamma}_h(\beta_h)) = u(x_h, y_h)$ because $\bar{\gamma}_h(\beta_h) \in \partial\Omega$, and $u = v$ on $\partial\Omega$ and again u is constant along $\bar{\gamma}_h$. Passing to the limit as $h \to +\infty$ we conclude that $u(\bar{x}, \bar{y}) = v(\bar{x}, \bar{y})$.

Step 2. It turns out that $u = v$ everywhere on $S(u)$.

First of all we claim that also $\nabla v = 0$ a.e. on $S(u)$. Indeed, on the contrary assume that $|\{(x, y) \in S(u) : \nabla v(x, y) \neq 0\}| > 0$. Then, since $g(0) < g(t)$ whenever $t > 0$ and since by step 1 we have $u = v$ on $\Omega \setminus S(u)$, we obtain

$$\begin{aligned} G(v) &= \int_\Omega g(|\nabla v(x, y)|)\, dxdy \\ &= \int_{\Omega \setminus S(u)} g(|\nabla u(x, y)|)\, dxdy + \int_{S(u)} g(|\nabla v(x, y)|)\, dxdy \\ &> \int_{\Omega \setminus S(u)} g(|\nabla u(x, y)|)\, dxdy + \int_{S(u)} g(|\nabla u(x, y)|)\, dxdy = G(u) \end{aligned}$$

which is a contradiction since both u and v are minimizers of G.

By Step 1 we thus have that $\nabla u = \nabla v$ a.e. on Ω and this implies that $u - v$ is constant a.e. on each connected component of Ω. As u and v are Lipschitz-continuous functions which coincide at the boundary of Ω, they must be equal and this ends the proof. □

Remark 2.6. Observe that the uniqueness is still true if $g = g(x, y, t)$ is a Caratheodory's function with $g(x, y, \cdot)$ convex and $g(x, y, 0) < g(x, y, t)$ for a.e. $(x, y) \in \Omega$ and any $t > 0$. However, in this case there are no general results of existence of Lipschitz-continuous minimizers.

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