Everywhere regularity for vectorial integrals with non–standard growth

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This lecture deals with the *everywhere* regularity for local minimizers of integral functionals of the type:

$$I(v,\Omega) = \int_\Omega f(x, v(x), Dv(x))dx,$$

where Ω is an open regular set of R^n, $f : \Omega \times R^N \times R^{nN} \to R$ is a Carathéodory function and v is a vector field and $Dv(x) = \left(\frac{\partial v^\alpha}{\partial x^i}\right)$, $1 \leq \alpha \leq N$, $1 \leq i \leq n$ is the Jacobian matrix of v.
The central problem of Calculus of Variations consists to find $\tilde{u}$ among all functions in a class U such that:

$$I(\tilde{u}, \Omega) \leq I(v, \Omega) \qquad \forall u \in U.$$

The Dirichlet boundary value problem is given when

$$U = \left\{ u \in W^{1,p}(\Omega, R^N) : u = u_0 \text{on} \partial\Omega \right\}.$$

The arguments of Direct Methods are based on the sequentially lower semicontinuity of the functional $I(v, \Omega)$, as example in the weak topology of $W^{1,p}$, and the coercivity of I with respect to the norm of $W^{1,p}$, which implies the weak compactness of the minimizing sequences.
The convexity of $f = f(x, s, z)$ with respect to the variable z is implies the lower semicontinuity in the weak topology of $W^{1,1}$. The following *natural* growth conditions

$$|z|^p - c_1 \leq f(x, s, z) \leq c_2(1 + |z|^p),$$

where $c_1, c_2 > 0$ and $p > 1$, allows to say that $I(v, \Omega)$ is well defined, indeed since f is a Carathéodory function, the upper bound implies that $f(x, Du(x))$ is integrable in Ω and $I(v, \Omega)$ is finite in $W^{1,p}$.

In 1952, Morrey [9] introduced the notion of *quasi convexity*: a function f is quasiconvex when for every $(x_0, s_0, z_0$ and $\varphi \in C_0^\infty$

$$|\Omega| f(x_0, s_0, z_0) \leq \int_\Omega f(x_0, s_0, z_0 + D\varphi) dx$$

and showed that if the functional I is sequentially lower semicontinuous with respect to the weak $*$ topology of $W^{1,\infty}$, then f is quasiconvex.
Acerbi and Fusco [10] and Marcellini [23] proved that if f is quasiconvex and satisfies the *natural* growth conditions, then $I(v, \Omega)$ is lower semicontinuous in the weak topology of $W^{1,p}$.
Moreover, the quasiconvex functions can not satisfy the *natural* growth conditions, but the following $p - q$ growth conditions: there exist $1 < p < q$ and $c_1, c_2 > 0$ such that

$$|z|^p - c_1 \leq f(x, s, z) \leq c_2(1 + |z|^q),$$

More general, we can also consider the *non standard* or *general* growth conditions: there exists g_1 and g_2 convex functions such that

$$g_1(z) - c_1 \leq f(x, s, z) \leq c_2(1 + g_2(z)),$$

In the last years a great interest has raised around the study of the semicontinuity for functionals $I(v, \Omega)$ satisfying general and $p-q$ growth, by obtaining some conditions on the mutual dependence on p and q and g_1 and g_2 respectively.
There are many functionals that can be considered in the Calculus of Variations, whose integrand functions don't satisfy *natural* growth condition

- small perturbation of polynomial growth

$$f(z) = |z|^p \log^\alpha(1 + |z|) \qquad p \geq 1, \ \alpha > 0$$

- large perturbation of polynomial growth (exponential)

$$f(z) = e^{|z|^\alpha} \qquad \alpha > 0$$

- anisotropic growth

$$f(z) = (1 + |z|^2)^{\frac{p}{2}} + \sum_{i\alpha} |z_{i,\alpha}|^{p_i} \qquad p_i \geq p \ \forall i = 1, \ldots, n$$

$$f(x, z) = |z|^q + a(x)|z|^p \qquad 0 \leq a(x) \leq M$$

- variable growth

$$f(z) = |z|^{p(x)}$$
$$f(z) = [h(|z|)]^{p(x)}$$

where $1 < p \leq p(x) \leq q$ and h is a convex function in $\mathbb{R}$.

The theory of regularity for minimizers of integral functionals has been widely studied in scalar case $N = 1$ under *natural* growth, $p = q$, starting by the paper of E. De Giorgi in 1957 [27].
The study of the regularity for the $p - q$ growth started by some papers of P. Marcellini ([52],[53],[54],[55]) and a restriction between p and q needs: $q \leq c(n)p$ with $c(n)$ close to 1. In the last years, there are many contributions on this subjects and below we present a (far from being complete) list of references.
In the vectorial case $N > 1$ there are some well known counterexamples to the continuity of the minimizers (see De Giorgi, Giusti and Miranda [34]). Under natural growth the regularity has been investigated and in general one can aspect only *partial* regularity i.e. the minimizers is smooth in some open subset $\Omega_0 \subset \Omega$ with an estimate of the measure of the singular set. Nevertheless, in the case $f(x, \xi) = |\xi|^p$, $(p \geq 2)$, Uhlenbeck [36] proved that the minimizers are in $C^{1,\alpha}_{loc}$, a result which was later extended to more general integrands which grow like $|z|^p$ by Giaquinta and Modica [33] for $p \geq 2$ and Acerbi and Fusco, when $1 < p < 2$.
In [56], Marcellini considers integrals without growth conditions and proves the everywhere Hölder continuity of the gradient for minimizers when $f(x, z) = g(|z|)$ with g positive and convex, satisfying: $\frac{g'(t)}{t}$ is positive and increasing in $(0, +\infty)$ and a *non oscillatory* condition at infinity, i.e. for every $\alpha > 1$ there exists a constant $c = c(\alpha)$ such that

$$g''(t)t^{2\alpha} \leq c[g(t)]^{\alpha}, \quad \forall t > 1;$$

these conditions imply at least quadratic growth but they allow exponential behaviour. The subquadratic case is studied by Leonetti, Mascolo and Siepe in [51].
Here, we are interested in the *everywhere* regularity for local minimizers in the non homogeneous case $f = f(x, z)$ and then we need the special structure on the density energy:

$$f = g(x, |z|).$$

We recall that u is a local minimizer of $I(v, \Omega)$ if $u \in W^{1,1}_{loc}(\Omega, R^N)$, $f(x, Du(x)) \in L^1_{loc}(\Omega)$ and

$$\int_{spt(\psi)} f(x, Du)dx \leq \int_{spt(\psi)} f(x, Du + D\psi)dx,$$

for every $\psi \in W^{1,1}(\Omega, R^N)$ with $spt(\psi) \subset\subset \Omega$, therefore u is also a weak solution of an elliptic system of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{\alpha}(x, Du) = 0, \quad \forall \alpha = 1, \dots, N \tag{1}$$

where the vector field $a = (a_i^{\alpha}) : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is the gradient with respect to z of the function f.

We present two different situations, in some sense complementary each other: the case in which $f(x,z)$ can have fast behavior, for example exponential, with respect to z and a case of $p-q$ growth, and then the behaviour of f can be also slow ($q<2$).
In the article *Everywhere regularity for vectorial functionals with general growth*, [57], written in collaboration with A.P.Migliorini, we consider non homogeneous densities, with the *non oscillatory* condition at infinity and we obtain the following result:
Assume that $g=g(x,t):\Omega\times[0,+\infty)\to[0,+\infty)$ is of class C^2, convex in t, such that $\forall x\in\Omega$, $\frac{g_t(x,t)}{t}$ is positive and increasing with respect to t and for every $\Omega_0\subset\subset\Omega$ and $\alpha>1$ there exist two positive constants c_1 and c_2, depending on α and on Ω_0, such that $\forall x\in\Omega_0$ and $\forall t>1$

$$g_{tt}(x,t)t^{2\alpha}\le c_1[g(x,t)]^\alpha$$

and $\forall t>0$ and $\forall s=1,\dots,n$

$$|g_{tx_s}(x,t)|\le c_2 g_t(x,t)[1+g_t^{\alpha-1}(x,t)].$$

Then every local minimizer u of

$$I(\Omega,u)=\int_\Omega g\left(x,|Du|\right)dx$$

is in $W^{1,\infty}_{loc}\left(\Omega,\mathrm{I\!R}^N\right)$ and there exist $c>0$ and $\sigma>0$ such that for every $B_R\subset\subset\Omega$:

$$\sup_{B_{R/2}}|Du|\le c\left\{\int_{B_R}\left[1+g\left(x,|Du|\right)\right]dx\right\}^{1+\sigma}.$$

Actually we prove the theorem under weak assumptions on g. (Theorem 2.1 of [57]).
The most relevant fact is that the integrand $f(x,\xi)$ may have exponential growth with respect to ξ, which involves systems (1) non uniformly elliptic.
Our result includes energy densities with *variable growth* as

$$\int_\Omega a(x)[h\left(|Du|\right)]^{p(x)}dx,$$

where h is a $C^2([0,+\infty))$ positive convex function satisfying previous conditions with a, $p\in W^{1,\infty}_{loc}(\Omega)$, $a(x)$, $p(x)\ge c>0$ *a.e.* $x\in\Omega$. In particular we can take

$$h(t)\sim\exp(t^m)$$

for $t\to+\infty$ and $m>0$.
Recently, this kind of system, with variable growth, has been used by Rajagopal and Růžička ([62] and [61]) , in their model for the behaviour of special viscous, non-Newtonian, fluids with the ability to change their mechanical properties in dependence on an applied electric field, the so-called *electrorheological fluids.*

In the model proposed by Rajagopal and Růžička, the interaction between the electric field and the fluid in motion is expressed in the coefficients of the system by a variable exponent.
The case $f(x,\xi) = |\xi|^{p(x)}$ has been studied in the scalar case by ZhiKov [63], Mascolo and Papi [58] and Chiadò Piat and Coscia [41] (see also Marcellini [54] and Dall'Aglio, Mascolo and Papi [44]). In the vectorial case, the regularity result is due to Coscia and Mingione [42] and regularity results on the systems related with of electrorheological fluids are contained in Acerbi and Mingione [37], [38].
Moreover, the result includes also more general cases as energies of the form

$$g(x,t) = \exp(t^{p(x)})$$

t near $+\infty$ and even every other finite composition of exponentials as for example

$$g(x,t) = \exp(\exp(t^{p_1(x)}))^{p_2(x)}$$

with $p_i(x) \geq 2$, $(i = 1, 2)$,
The interest in functionals with general exponential growth and non uniformly elliptic systems is also motivated by different models which arise from problems in mathematical physics as combustion theory and reaction of gases.
We are able to improve to fast behaviour the previous regularity results by using different technique, indeed we do not control the stored energy $g(x,t)$ by means of power functions but we use directly its particular structure and properties (in the same direction Dall'Aglio and Mascolo [45] for L^∞-regularity).
We give an idea of the proof. We consider first functionals

$$I(v,\Omega) = \int_\Omega f(x, Dv(x))dx,$$

with *controllable growth* i.e. there exist positive constants m, M and N, such that

$$m\,|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{z_i^\alpha z_j^\beta}\,(x,z)\,\lambda_i^\alpha\lambda_j^\beta \leq M\,|\lambda|^2\,,$$

and

$$|f_{\xi_i^\alpha x_s}(x,z)| \leq N(1+|\xi|^2)^{\frac{1}{2}}, \quad a.e.\ x \in \Omega_0, \quad \forall z \in \mathbb{R}^{nN}.$$

i.e. uniform elliptic systems (1). Previous inequalities imply that function g satisfies:

$$m \leq \frac{g_t(x,t)}{t} \leq g_{tt}(x,t) \leq M$$

and

$$|g_{tx_s}(x,t)| \leq N(1+t^2)^{\frac{1}{2}},$$

$\forall t > 0$ and for $a.e.\ x \in \Omega_0$.
Under these assumptions we prove that all local minimizers are in $W^{1,\infty}_{loc}$ and the following a-priori estimate

$$\sup_{B_\rho} |Du| \leq c\left\{\int_{B_R} [1 + g(x, |Du|)]\, dx\right\}^{1+\sigma}$$

where $0 < \rho < R < 1$ such that $B_R \subset\subset \Omega_0$, $\sigma = \sigma(n) > 0$ where c and β do not depend on m, M and N, which appear in the controllability assumptions.
In the second part we carry out the estimate to the general case by means of a suitable approximation argument. More precisely, we construct a sequence of functions $g^k \leq g$ which converges to g, with g^k satisfying the same assumptions of g and such that the corresponding functionals have controllable growth with constants , M_k,m_k and N_k, depending on k.
Let u be local minimizers of $I(v, \Omega)$. For each k, consider the Dirichlet problems in $B_R \subset\subset \Omega_0 \subset\subset \Omega$:

$$\inf \left\{ \int_{B_R} g^k \left(x, |Dv|\right) dx, \quad v \in u + W_0^{1,2} \left(B_R, \mathbb{R}^N\right) \right\}$$

and denote by u_k the unique solution.
We prove that, up to a subsequence, (u_k) converges weakly in $u + W_0^{1,2}\left(B_R, \mathbb{R}^N\right)$ to a function w. Moreover, by applying the *a priori* estimate to u_k, there exist $\sigma > 0$ and c independent of k, such that $\forall \rho < R$:

$$\sup_{B_\rho} |Du_k| \leq c \left\{ \int_{B_R} [1 + g^k \left(x, |Du_k|\right)] dx \right\}^{1+\sigma}.$$

By the minimality of u_k, we have

$$\sup_{B_\rho} |Du_k| \leq c \left\{ \int_{B_R} [1 + g \left(x, |Du|\right)] dx \right\}^{1+\sigma}.$$

The last inequality gives that (u_k), up to a subsequence, converges to the function w in the weak* topology of $W^{1,\infty}_{loc}(B_R, \mathbb{R}^N)$. By lower semicontinuity and using the dominated convergence theorem, as $k \to +\infty$ we have

$$\int_{B_R} g\left(x, |Dw|\right) dx \leq \int_{B_R} g\left(x, |Du|\right) dx.$$

Therefore w is a local minimizer of I and the strictly convexity of the functional gives $u = w$. A procedure of passage to the limit gives estimate for the local minimizer u.

In the article ,[43], *Regularity of minimizers of vectorial integrals with $p-$growth*, in collaboration with G. Cupini and M. Guidorzi, the regularity of local minimizers is studied in the framework of the p-uniform convexity.
Define

$$I(v, \Omega) = \int_\Omega f(x, Dv(x)) dx,$$

Let $N = 1$ and f satisfy natural p-growth condition. In 1988 Manfredi in [35] proves that when $f = f(z)$ with $f \in C^2$ satisfies the ellipticity condition

$$\nu(1 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq (D^2 f(z)\lambda, \lambda) \leq L(1 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2 \tag{2}$$

for z and λ in $\mathbb{R}^{nN}$, the local minimizers are in $C^{1,\beta}$.
In 1997, Fonseca and Fusco [29] introduced, for $f = f(z)$, the following notion of *p-uniform convexity*: there exist $p > 1$ and $\nu > 0$ such that, for a.e. $x \in \Omega$ and for every $z_1, z_2 \in R^{nN}$

$$\frac{1}{2}[f(z_1) + f(z_2)] \geq f\left(\frac{z_1 + z_2}{2}\right) + \nu(1 + |z_1|^2 + |z_2|^2)^{(p-2)/2}|z_1 - z_2|^2, \qquad (3)$$

for all $z_1, z_2 \in \mathbb{R}^{nN}$.
They proved that a p-uniform convex and continuous function can be approximate by means a sequence of C^2 which satisfy (2). Moreover if $f \in C^2$ ellipticity condition (2) is equivalent to the p-uniformly convexity. For integrand p-uniformly convex, Fonseca and Fusco showed that the local minimizers are $W^{1,\infty}$.
The result was generalized by Fonseca, Fusco and Marcellini in [28] to non homogeneous densities $f = f(x, z)$, in order to study the existence of minimizers of some non convex variational problems, under the less restrictive assumption *p-uniform convexity at infinity*: there exist $p > 1$ and $\nu > 0$ such that

$$\frac{1}{2}[f(x, z_1) + f(x, z_2)] \geq f\left(x, \frac{z_1 + z_2}{2}\right) + \nu(1 + |z_1|^2 + |z_2|^2)^{(p-2)/2}|z_1 - z_2|^2, \quad (4)$$

for a.e. $x \in \Omega$ and for every $z_1, z_2 \in \mathbb{R}^{nN} \setminus B_R(0)$ endpoints of a segment contained in the complement of $B_R(0)$.
In the vectorial case $N > 1$ and $p - q$-growth, Esposito, Leonetti and Mingione [48], study the case f p-uniformly convex and $f = g(|z|)$. In [43], we consider the more general $f = f(x, z) = g(x, |z|)$ p-uniformly convex at infinity. In the both results, the $W^{1,\infty}$-regularity is obtained under the restriction $q > p\frac{(n+1)}{n}$.
We give now the precise statement of the regularity theorem contained in Theorem 1.1 of [43]:
Assume that $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ is a Carathéodory function, convex with respect to the last variable, satisfying the following conditions

(A1) there exist $R > 0$ and a function g such that for a.e. $x \in \Omega$ and every $z \in \mathbb{R}^{nN} \setminus B_R(0)$
$$f(x, z) = g(x, |z|)\,,$$

(A2) f is *p-uniformly convex at infinity*,

(A3) there exist $L > 0$ and q with $p < q < p\frac{(n+1)}{n}$, such that for a.e. $x \in \Omega$ and $z \in \mathbb{R}^{nN}$,
$$0 \leq f(x, z) \leq L(1 + |z|)^q,$$

(A4) for a.e. $x \in \Omega$ and every $z \in \mathbb{R}^{nN} \setminus B_R(0)$ let $D_t^+ \tilde{f}(x, |z|)$ be the right-side derivative of $\tilde{f}$ with respect to t and denote $D_{z_i^\alpha}^+ f(x, z) = D_t^+ \tilde{f}(x, |z|)\frac{z_i^\alpha}{|z|}$. Then for every $z \in \mathbb{R}^{nN} \setminus B_R(0)$, the vector field $x \mapsto D_z^+ f(x, z)$ is weakly differentiable and
$$|D_x D_z^+ f(x, z)| \leq L(1 + |z|)^{q-1}.$$

Let u be a local minimizer of

$$I(v,\Omega)=\int_\Omega f(x,Dv(x))dx,$$

whose integrand f satisfies the assumptions (A1)–(A4), $1<p\le q<p(n+1)/n$. Then u is locally Lipschitz continuous and for all $B_r(x_0)\subset\subset\Omega$

$$\sup_{B_{r/4}(x_0)}|Du|\le c\left[\int_{B_r(x_0)}(1+f(x,Du))\,dx\right]^\beta,$$

where $c=c(n,p,q,L,R,\nu)$, and $\beta=\beta(n,p,q)$.
Assumption (A2) imply that f is p-coercive i.e. there exist two positive constants c_0, c_1 such that

$$-c_0+c_1|z|^p\le f(x,z)$$

so f satisfies the $p-q$ growth condition.
Observe that the restriction between p and q is in some sense sharp. Indeed Esposito, Leonetti and Mingione in [48] provide an example where $q>p\frac{n+1}{n}$, and local minimizers may not belong to $W^{1,q}$.
The regularity conditions (A4) with respect to x are also needed. A functional I exists, with f only measurable with respect to x, such that its minimizers do not belong to $W^{1,q}(\Omega)$ (see Zhikov [63]).
The proof consists in two steps. First we consider regular elliptic and p-growth functionals((2 holds) and by known regularity results the local minimizers u are in $W^{1,\infty}$. A particular application of Moser's iteration method, permits to prove the following *sharp* a-priori estimate

$$\sup_{B_\rho(x_0)}|Du|^p\le c\left[\int_{B_r(x_0)}(1+f(x,Du)\,dx\right]^\beta$$

where c and β are indipendent of the constants ν and L of the ellipticity condition.
In the second part we consider a suitable sequence of approximating variational problems by means of the construction of a sequence of C^2-functions f_{kh} : $\Omega\times\mathbb{R}^{nN}\to[0,+\infty)$, with $f_{kh}\le f$ such that

$$\lim_{h,k\to\infty}f_{kh}=f$$

and

- f_{kh} satisfies the same assumptions of f ,(A1)–(A4), with constant independent of k and h;
- f_{kh} has p-growth, with constant depending on k;
- $f_{kh}\in C^2$ and satisfies the ellipticity condition 2 with constants depending on h.

Let u be a local minimizer of the original functional I and for h and k consider the following problem in $B_r(x_0)$, with boundary datum u:

$$\min\left\{ I_{kh}(w; B_r(x_0)) := \int_{B_r(x_0)} f_{kh}(x, Dw)\,dx \;:\; w \in u + W_0^{1,p}(B_r(x_0)) \right\}.$$

Let v_{kh} be the unique solution, by applying the *a priori* estimate of the first part we obtain

$$\sup_{B_\rho(x_0)} |Dv_{kh}|^p \le c \left[\int_{B_r(x_0)} (1 + f_{kh}(x, Dv_{kh}))\,dx \right]^\beta .$$

with the same c and β for all h and k.
Then we show that v_{kh} converges locally weakly * in $W^{1,\infty}(B_r(x_0))$ to v, which is a minimizer of

$$\min\left\{ I(w; B_r(x_0)) := \int_{B_r(x_0)} f(x, Dw)\,dx \;:\; w \in u + W_0^{1,p}(B_r(x_0)) \right\}.$$

and moreover

$$\sup_{B_{r/4}(x_0)} |Dv|^p \le c \left[\int_{B_r(x_0)} (1 + f(x, Du))\,dx \right]^\beta$$

Since function f is not suppose to be stricly convex, in general $v \ne u$. However, by taking in account (A2) and (A3), a comparation methods gives that there exists $R_0(p, q, \nu, R, L) > R$ such that the Lebesgue measure of the set

$$\{x \in B_r(x_0) : |Du(x) + Dv(x)| > 2R_0 \text{ and } |Du(x) - Dv(x)| > 0\}$$

is zero. Therefore

$$\sup_{B_{r/4}(x_0)} |Du| \le \sup_{B_{r/4}(x_0)} |Dv| + \sup_{B_{r/4}(x_0)} |Du + Dv| \le 3 \sup_{B_{r/4}(x_0)} |Dv| + 2R_0$$

and the estimate holds also for the local minimizer u.

It would be interesting to ask if the convexity of f in B_R is necessary to get the $W^{1,\infty}$ regularity of the minimizer. In the recent paper [39], *Regularity of minimizers for non convex vectorial integrals with $p-q$ growth via Relaxation Methods*, in collaboration with I. Benedetti, we give a partial positive answer. More precisely, we prove a $W^{1,\infty}$-regularity result for a special class of non convex and non homogeneous density energies with $p - q$ growth.
Observe that when $f = f(x, z)$ is not convex to respect to z the functional $I(v, \Omega)$ is not lower semicontinuous, then the lower semicontinuous envelope of $I(v, \Omega)$ is introduced:

$$\begin{aligned} F(u, \Omega) \quad &= \inf_{u_k} \left\{ \liminf_{k\to\infty} \int_\Omega f(x, Du_k)\,dx, \right. \\ &\qquad \left. (u_k) \in W^{1,p}(\Omega, \mathbb{R}^m),\ u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^N) \right\} \end{aligned}$$

Functional F is called *relaxed* functional of I and it holds

$$\inf \left\{ F(v,\Omega) \right\} = \inf \left\{ I(v,\Omega) \right\}.$$

When $p = q$ and $N = 1$, F has an integral representation

$$F(v,\Omega) = \int_\Omega f^{**}(x, Dv)\, dx = I^{**}(v,\Omega)$$

where f^{**} is the convex envelope of f with respect to z (see for example [17], [60]). Therefore, all local minimizers of $I(v,\Omega)$ are also local minimizers of $I^{**}(v,\Omega)$. In this case it is sufficient to prove the regularity only for convex f, as in [28].
In the vectorial case $N > 1$ and $p-q$-growth the *relaxed* functional of I has the form

$$\begin{aligned} F^{p,q}(u,\Omega) \quad &= \inf_{u_k} \Big\{ \liminf_{k\to\infty} \int_\Omega f(x, Du_k)\, dx, \\ &(u_k) \in W^{1,q}(\Omega, \mathbb{R}^N), u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^N) \Big\} \end{aligned}$$

and no general result of integral representation holds.
In Theorem 2.6 of [39], under suitable assumption on $f = f(x,z)$, we give an integral reppresentation for $F^{p,q}$.
More precisely, assume

(H1) there exist $q > 1$ and $L > 0$ such that

$$c_1|z|^p - c_0 \le f(x,z) \le L(1 + |z|^q);$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^{nN}$ and $p < q < p\frac{(n+1)}{n}$

(H2) there exists a modulus of continuity $\lambda(t)$ (i.e. $\lambda(t)$ is a nonnegative increasing function that goes to zero as $t \to 0^+$) such that for every compact subset $\Omega_0 \subset \Omega$, there exists $x_0 \in \Omega_0$ such that:

$$|f(x_0, z) - f(x,z)| \le \lambda(|x - x_0|)(1 + f(x,z));$$

for all $x \in \Omega_0$ and $z \in \mathbb{R}^{nN}$.

(H3) the quasiconvex envelope of f, with respect to the second variable, i.e.

$$Qf(x,z) = \sup \{ g \le f :\ g \text{ quasiconvex with respect to } z \}.$$

is a convex function with respect to z i.e. $Qf(x,z) = f^{**}(x,z)$.

Then, for all $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that $Qf(x, Du) \in L^1_{loc}(\Omega)$, we have the following relaxation identity:

$$F^{p,q}(u,U) = \int_U Qf(x, Du)\, dx \tag{5}$$

for all open set $U \subset\subset \Omega$.
The case $f(z)$ is contained in [19].
The previous relaxation result permits to obtain the $W^{1,\infty}$ regularity for local minimizers of non convex functionals. We give a sketch of the proof.
Let $f(x,z)$ be not convex in B_R and satisfy (H2)-(H3) and in $\mathbb{R}^{nm}/B_R$. the assumptions (A1)–(A4) of the regularity theorem of Cupini, Guidorzi and Mascolo.
Let u be a local minimizer of I and then, for all $U \subset\subset \Omega$, u is a solution of the following boundary value problem:

$$I(u) = \inf \left\{ I(v,U),\ v \in W_0^{1,p}(U,\mathbb{R}^N) + u \right\}.$$

Consider the associate relaxed problem:

$$\inf \left\{ I^{**}(v,U),\ v \in W_0^{1,p}(U,\mathbb{R}^m) + u \right\},$$

By the convexity of I^{**}, the problem has at least a solution $\bar{u} \in W_0^{1,p}(U,\mathbb{R}^m)+u$.
Observe that since f is p-uniformly convex at infinity (see[43]), there exists R_0 depending on ν, p, q, L such that:

$$f^{**}(x,z) = f(x,z),$$

so f^{**} satisfies $(A1)-(A4)$ (with R replaced by R_0).
Since $\bar{u}$ is also a local minimizer of I^{**}, by applying the regularity result of [43], we obtain that $\bar{u} \in W^{1,\infty}_{loc}(U,\mathbb{R}^N)$.
The regularity of $\bar{u}$, the method introduced by De Giorgi in [15] and the related arguments contained in Marcellini [52] permit to prove that for every $(u_k) \subset W^{1,q}(U,\mathbb{R}^N)$, with $u_k \rightharpoonup \bar{u}$ in the weak topology of $W^{1,p}(U,\mathbb{R}^N)$

$$\int_U f(x,Du)\,dx \le \liminf_{k\to\infty} \int_U f(x,Du_k)\,dx.$$

Taking the infimum over the sequences (u_k):

$$\int_U f(x,Du)\,dx \le\ F^{p,q}(\bar{u},U).$$

Since $Qf(x,z) = f^{**}(x,z)$, by the relaxation equality (5) and by the minimality of $\bar{u}$, we get

$$\int_U f(x,Du)\,dx \le \int_U f^{**}(x,D\bar{u})\,dx \le \int_U f^{**}(x,Du)\,dx.$$

and then:

$$\int_U f(x,Du)\,dx = \int_U f^{**}(x,D\bar{u})\,dx = \int_U f^{**}(x,Du)\,dx,$$

which implies that u is also a solution of the relaxed problem and again by the result of Cupini,Guidorzi Mascolo, $u \in W^{1,\infty}_{loc}(U, \mathbb{R}^N)$. For the arbitrariness of U, we get $u \in W^{1,\infty}_{loc}(\Omega, \mathbb{R}^N)$.
We exhibit a class of energy density f, for which $Qf(x, z) = f^{**}(x, z)$. More precisely: Let $g : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be such that for every $x \in \Omega$:

$$g(x, t) = g(x, -t),$$

and for every $t > 0$ and there exists a measurable, non negative function $\alpha : \Omega \longrightarrow \mathbb{R}_+$ such that for all t with $|t| < \alpha(x)$:

$$g^{**}(x, t) = g(x, \alpha(x)),$$

and for all $t \geq \alpha(x)$

$$g^{**}(x, t) = g(x, t),$$

Then, if

$$f(x, z) = g(x, |z|).$$

the following inequalities hold

$$g^{**}(x, |z|) = f^{**}(x, |z|) = Qf(x, |z|).$$

References

Calculus of Variations and Partial Differential Equations

[1] Brezis H., *Analyse Functionelle*, Masson, (1983).

[2] Dacorogna B.,*Direct methods in the Calculus of Variations*, Appl. Mat.78, Springer-Verlag, (1989).

[3] Dacorogna B., *Introduction au Calcul des Variations*, Cahiers Mathematique de l'Ecole Politechnique de Lausanne, (1992).

[4] Giaquinta M., *Multiple Integrals in the Calculus of Variations and non linear elliptic systems*, Princeton University Press, (1983).

[5] Evans C.L.,*Partial Differential Equations*, American Mathematical Society,(1998).

[6] Gilbarg D., Trudinger N.S., *Elliptic Partial Differential Equations of second order*,Springer- Verlag, (197)7.

[7] Giusti E., *Metodi Diretti nel Calcolo delle Variazioni*, UMI,(1994).

[8] Ladyzhenskaya O., Uraltśeva N.,*Linear and quasilinear elliptic equations*, Vol 46, Accademic Press N.Y. (1968)

[9] Morrey C. B.,*Multiple integral Problems in the Calculus of Variations and related Topics*, University of California Press, Berkeley, (1966).

Some articles on the semicontinuity and relaxation under natural, p-q and general growth

[10] Acerbi E., Fusco N., *Semicontinuity problems in the calculus of variations*, Arch. Rat. Mech. Anal., 86 ,(1984),125–145.

[11] Ball J. M., *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal., 63, (1977), 337–403.

[12] Ball J.M., *Constitutive inequalities and existence theorem in non linear elastostatic*,Non linear Anal, Heriot Watt Sym, Vol.1,(1977) 187–241.

[13] Bianconi B., Focardi M., Mascolo E., *Existence of minimizers for a class of quasiconvex functionals with non standard growth*, Annali di Matematica Pura e Applicata, 180, (2002), 493–510.

[14] Dacorogna B., Marcellini P., *Semicontinuité pour des intégrandes polyconvexes sans continuité des déterminants*, C. R. Acad.Sci. Paris, 311,(1990), 393–396.

[15] De Giorgi E., *Teoremi di semicontinuità nel Calcolo delle Variazioni*, Appunti INdAM, Roma, 1968–69.

[16] De Giorgi E., *Sulla convergenza di alcune successioni di integrali del tipo area*, Rendic. Mat., 8, (1975), 277–294.

[17] I. Ekeland, R. Temam, *Analyse convex et problemes variationelles*, Paris, Dunod, Gauthier-Villars,(1974)

[18] Focardi M., Mascolo E., *Lower semicontinuty for integral functionals under non standard growth conditions*, Journal of Convex Analysis, Vol.8 (2002), No 2, 327–348.

[19] Fonseca I., Maly J., *Relaxation of multiple integrals in Sobolev spaces below the growth exponent for the energy density*, Ann. Inst. H. Poincaré, Analyse non lineaire, Vol.14, 3,(1997), 309–338.

[20] Fonseca I., Marcellini P., *Relaxation of multiple integrals in subcritical Sobolev spaces*, J. Geom. Anal.,7, (1997),57–81.

[21] Fusco F.,*On the convergence of integral functional depending on vector-valued functions*, Ricerche Mat., 32 (2), (1983), 321–339

[22] Maly J.,*Lower semicontinuity of quasi-convex integrals*, Manus. Math., 85,(1994), 419–428.

[23] Marcellini P.,*Approximation of quasiconvex functions and lower semicontinuity of multiple integrals*, Manuscripta Math., 51, (1985), 1–28.

[24] Marcellini P.,*On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré , Analyse Non Lineaire, 3, (1986), 391–409.

[25] Marcellini P., Sbordone C., *Semicontinuity problems in the Calculus of Variation*, Nonlinear Anal., 4 (2), (1980), 241–257

SOME ARTICLES ON THE REGULARITY OF MINIMIZERS OF FUNCTIONALS UNDER NATURAL GROWTH

[26] Acerbi E., Fusco N.,*Regularity for minimizers of non-quadratic functionals: the case* $1 < p < 2$, J. of Math.Anal.and Appl. 140 (1989), 115–135.

[27] De Giorgi E., *Sulla differenziabilit e l'analiticit degli esttremali degli integrali multipli regolari*, Memorie Acc. Sci. Torino, 3(3),(1957) 25–43.

[28] Fonseca I., Fusco N., Marcellini P.,*An existence result for non convex variational problem via regularity*, ESAIM: Control, Optim. Calc. Var., 7 (2002) 69–96

[29] Fonseca I., N. Fusco, *Regularity results for anisotropic image segmentation models*, Ann. Scuola Norm. Sup. Pisa ,24 (1997) 463–499.

[30] Giaquinta M.,Giusti E, *On the regularity of the minima of variationals integrals*, Acta Mathematica, 72,(1983) 31–46.

[31] Giaquinta M., Giusti E.,*Differentiability of the minima of non differentiable functionals*, Inventiones Math. 72, (1983),285–298.

[32] Giaquinta M., Giusti E.,*Quasi-minima, Ann.Inst.H. Poincaré*, 1, (1984) 79–107.

[33] Giaquinta M. Modica G.,/it Remarks on theregularity of the minimizers of certain degenerate functionals, Manuscripta Math., 57 (1986), 55–99.

[34] Giusti E., Miranda M., *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del Calcolo delle Variarioni*, Boll.UMI 2 (1968) 1–8

[35] Manfredi L., *Regularity for a classminima of functional with p-growth*, J Differental Equations, 76 (1988) 203–212

[36] Uhlenbeck K., *Regularity for a class of non-linear elliptic systems*, Acta Math. 138, (1977), 219–240.

REGULARITY OF MINIMIZERS OF FUNCTIONALS UNDER P-Q AND GENERAL GROWTH

[37] Acerbi E., Mingione G.,*Regularity results for a class of functionals with nonstandard growth*, Arch. Rat. Mech. Anal.,156,(2001) 121–140.

[38] Acerbi E., Mingione G.,*Regularity results for quasiconvex functionals with nonstandard growth*, Ann. Scuola Norm. Sup.Pisa, 30,(2001).

[39] Benedetti I., Mascolo E.,*Regularity forr non convex functionals with* $p - q$ *growth via relaxations methods*, Abstract Applied Analysis,(2004).

[40] Choe H.J.,*Interior behavior of minimizers for certain functionals with non standard growth conditions*, Nonlin. Anal. TMA 19,10 (1992), 933–945.

[41] Chiadò Piat V., Coscia A.,*Hölder continuity of minimizers of functionals with variable growth exponent*, Manuscripta Math. 93, (1997) 283–299.

[42] Coscia A., Mingione G.,*Hölder continuity of the gradient of $p(x)$-harmonic mappings*, C.R. Acad. Sci.Paris 328 (1999) 363–368.

[43] Cupini G., Guidorzi M., Mascolo E.,*Regularity of minimizers of vectorial integrand with p-q growth*, Non Linear Analysis TMA, 4,(2003), 591–616.

[44] Dall'Aglio A., Mascolo E., Papi G., *Local boundedness for minima of functionals with non standard growth conditions*, Rendiconti di Matematica, 18 (1998) 305–326.

[45] Dall'Aglio A., Mascolo E., *L^∞-estimates for a class of nonlinear elliptic systems with non standard growth*, Atti del Seminario Mat e Fisico Univ. Modena,

[46] Esposito L., Leonetti F., Mingione G. ,*Regularity for minimizers of functionals with $p-q$ growth conditions*, NoDEA Nonl. Diff. Equat. and Appl. 6 (1999), 133–148.

[47] Esposito L., Leonetti F., Mingione G., *Higher integrability for minimizers of integral functionals with (p,q) growth*, J. Diff. Equat., 157 (1999), 414–438.

[48] Esposito L., Leonetti F.Mingione G. ,*Regularity results for minimizers of irregular integrals with p, q growth*, Forum Math.

[49] Giaquinta M., *Growth conditions and regularity, a counterexample*, Manuscripta Math., 59, (1987), 245–248.

[50] Leonetti F., Mascolo E., Siepe F. ,*Gradient regularity for minimizers of functionals under $p-q$ subquadratic growth*, Boll. UMI, 4-B (2001), 571–586.

[51] Leonetti F., Mascolo E.,Siepe F.,*Everywhere regularity for a class of vectorial functionals under subquadratic general growth conditions*, Jour Math Anal and Appl., 287, (2003),593–608.

[52] Marcellini P., *Un example de solution discontinue d'un probleme variationel dans le cas scalaire*, (1987).

[53] Marcellini P. , *Regularity of minimizers of integrals in the calculus of variations with non standard growth conditions*, Arch. Rat. Mech. and Analysis, 105 (1989), 267–284.

[54] Marcellini P., *Regularity and existence of solutions of elliptic equations with (p,q)-growth conditions*, J. Diff. Equations, 90 (1991) 1–30.

[55] Marcellini P.,*Regularity for elliptic equations with general growth*, J. Diff. Equat., 105 (1993), 296–333.

[56] Marcellini P., *Everywhere regularity for a class of elliptic systems without growth conditions*, Annal. Sc. Norm. Sup. Pisa , 23 (1996), 1–25.

[57] Mascolo E., Migliorini A.P., *Everywhere regularity for vectorial functionals under general growth*, ESAIM: Control Optim. Calc. Var. 9 (2003), 399–418

[58] Mascolo E., Papi G.,*Local boundedness of integrals of Calculus of Variations*, Annali di Mat. Pura e Appl.,167 (1994) 323–339.

[59] Mascolo E., Papi G.,*Harnack inequality for minimizes of integrals functionals with general growth conditions*, NoDEA, 3, (1996), 231–244.

[60] Mascolo E., Siepe F., *Functionals of the Calculus of Variations with non standard growth conditions*, Universitad Austral, Rosario 1 (2000), 1–22.

[61] Růžička M., *Flow of shear dependent electrorheological fluids*, C. R. Acad. Sci. Paris, 329 (1999) 393–398.

[62] Rajagopal K. R., Růžička M., *On the modeling of electrorheological materials*, Mech. Res. Commun. 23 (1996) 401–407.

[63] Zhikov V. V., *On the Lavrentiev pheonomenon* Russian Jour. Math. Physics, 3 (1995) 249–269.

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