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# Relaxation Methods in Control Theory

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**Abstract.** Two different "relaxed problems" associated with a problem of optimal control theory, governed by an ODE, are considered: the first is obtained by Young's methods and the second by semicontinuity arguments. A formula which relates the two relaxed functionals to each other is given.

## 1. Introduction

Consider the following minimum problem:

$$\min\{F(u), u \in X\}, \tag{1.1}$$

where $F$ is a real functional on the topological space $X$. If $X$ is not compact or $F$ is not lower semicontinuous, problem (1.1) may not have a solution. However, from the point of view of application, it is interesting to study the asymptotic behavior of the minimizing sequences. For this purpose another problem, the so-called "relaxed problem," has been associated with (1.1).

Let $W$ be a topological space and $G$ a functional on $W$ so that:

(i) $X$ is identified with a subspace of $W$, which is dense in $W$;

(ii) for every sequence $(u_j) \subset X$, which converges to $w$ in $W$, we have

$$G(w) \leq \liminf_j F(u_j);$$

(iii) for every $w \in W$ there exists a sequence $(u_j) \subset X$, which converges to $w$ and

$$G(w) = \lim_j F(u_j).$$

Then the following problem

$$\min\{G(u): w \in W\} \tag{1.2}$$

is the relaxed problem of (1.1).

From the definition it follows that (1.1) and (1.2) have the same infimum value. Moreover, every minimizing sequence of (1.1) has, as only cluster points, the possible solutions of (1.1) and for every solution $w$ of (1.2) there exists a minimizing sequence of (1.1) which converges to $w$.

We note that the construction of the relaxed problem is not unique. Relaxation in problems of variational calculus has been used many times for various purposes and under different names. In particular, in optimal control theory, two major approaches have been used: one based on the notion of relaxed control as a parametrized measure as intended by Young (see [6], [10]-[12], [14], and [15]), the other consists in considering the lower semicontinuous envelope of the original functional (see [2], [7], and [9]). Within optimal control theory, the application of the second method is very recent (see [1] and [3]). In particular, in [1], some sufficient conditions are given in order that the relaxed problem obtained by semicontinuity arguments should still be a control problem.

In this note, starting from an optimal control problem governed by an ordinary differential equation, we investigate the relationship between these two types of relaxed problem. In spite of the apparent difference in formulation, we demonstrate that they are strictly connected. More precisely we obtain a formula which relates the two relaxed functionals to each other.

## 2.

Let, $m$, $n$ be positive integers, $p \in ]1, +\infty]$, and $0 \leq T \leq +\infty$. Let $I = [0, T]$, consider the space of states $Y = W^{1,p}(I, R^m)$, endowed with the $L^\infty(I, R^m)$ topology, and the space $U$ of measurable functions $u: I \to K$, with $K$ compact set in $R^n$, endowed with the weak $L^p_{\mathrm{loc}}$-topology (weak-* if $p = +\infty$). Let $f: (s, y, \lambda) \in I \times R^m \times K \to [0, +\infty]$ be a function measurable in $s$ and continuous in $(y, \lambda)$.

Consider the following cost functional:

$$J(u, y) = \int_I f(s, y(s), u(s))\, ds; \tag{2.1}$$

the admissible set $\Lambda$ is defined by

$$\Lambda = \{(u, y) \in U \times Y: \dot{y}(s) = g(s, u(s), y(s)), y(0) = 0\}, \tag{2.2}$$

where $g: I \times R^m \times K \to R^m$ is a function measurable in $s$, continuous in $(y, \lambda)$. Set

$$\chi_\Lambda(u, y) = \begin{cases} 0 & \text{if } (u, y) \in \Lambda, \\ +\infty & \text{if } (u, y) \notin \Lambda. \end{cases}$$

The problem of optimal control is

$$(P) \quad \min\{F(u, y): (u, y) \in U \times Y\},$$

where $F$ denotes

$$F(u, y) = J(u, y) + \chi_\Lambda(u, y). \tag{2.3}$$

We assume that $f$ and $g$ satisfy

$$|f(s,y',\lambda)-f(s,y'',\lambda)|\le c|y'-y''|, \qquad |f(s,y,\lambda)|\le \alpha(s), \tag{2.4}$$

$$|g(s,y',\lambda)-g(s,y'',\lambda)|\le d|y'-y''|, \qquad |g(s,y,\lambda)|\le \beta(s), \tag{2.5}$$

for all $y', y''$ in a bounded subset of $R^m$, $y\in R^m$, $\lambda\in K$, $s\in I$ and with $\alpha,\beta\in L^1(I)$. Under the above assumptions, it is not obvious that $(P)$ has any solutions.

We introduce two problems $(\bar{P})$ and $(\tilde{P})$ that are relaxed problems of $(P)$, in the sense specified above.

**Relaxed Problem** $(\bar{P})$. Let $\bar{F}$ be the lower semicontinuous envelope of $F$ in the topology of $U\times Y$, i.e.,

$$\bar{F}(u,y)=\operatorname{Inf}\{\liminf_j F(u_j,y_j)\},$$

where the infimum is in the set of all sequences $(u_j, y_j)$ converging to $(u, y)$ in $U\times Y$. Consider

$$(\bar{P}) \quad \operatorname{Min}\{\bar{F}(u,y): (u,y)\in U\times Y\}.$$

From the definition of $\bar{F}$ it follows that (i)–(iii) of Section 1 are verified and therefore $(\bar{P})$ is a relaxed problem of $(P)$.

In [1] it is proved that, for a large class of control problems, the functional $\bar{F}$ can be split into a new cost functional $\bar{J}$ and the characteristic function of a new set $\bar{\Lambda}$ of admissible pairs.

**Relaxed Problem** $(\tilde{P})$. Assume $T<+\infty$. For the construction of $(\tilde{P})$ we follow [14] (see also [5], [10], and [14]). We denote by $\mathcal{M}(K)$ the space of Random measures on $K$. Identifying $\mathcal{M}(K)$ with the dual space $(C(K))^*$ of $C(K)$, which is the space of the continuous functions on $K$, we can endow $\mathcal{M}(K)$ with the weak-* topology of $(C(K))^*$. Let $\tilde{U}$ be the space of measurable mappings $\sigma: s\in I\to\sigma^s\in\mathcal{M}(K)$ such that the value of $\sigma$ in $s$, $\sigma^s$, is a probability measure for almost every $s\in I$.

We call the elements of $\tilde{U}$ relaxed controls. We observe that $U\subset\tilde{U}$. In fact, any $u\in U$ can be identified with the relaxed control $\sigma^u: s\in I\to\delta_{u(s)}$, where $\delta_\lambda$ denotes the Dirac measure concentrated at $\lambda$. Moreover, we identify $\sigma\in\tilde{U}$ with the following functional on $L^1(I, C(K))$,

$$\sigma: \varphi\in L^1(I,C(K))\to\langle\sigma,\varphi\rangle=\int_I ds\int_K \varphi(s,\lambda)\,d\sigma^s(\lambda),$$

and endow $\tilde{U}$ with the weak-* topology of $L^1(I, C(K))^*$. Consequently, a sequence $(\sigma^j)$ converges to $\sigma$ in $\tilde{U}$ if and only if

$$\lim\langle\sigma^j,\varphi\rangle=\langle\sigma,\varphi\rangle \qquad \text{for} \quad \varphi\in L^1(I,C(K)). \tag{2.6}$$

From Theorems IV.2.1 and IV.2.6 of [14] it follows that $\tilde{U}$ is a compact space and is the closure of $U$. Moreover, for any sequence $(u_j)\subset U$ there exists $\sigma\in\tilde{U}$ and a subsequence $(u_{j_h})$ such that $\sigma^{u_{j_h}}$ converges to $\sigma$, i.e.,

$$\lim_h\int_I \varphi(s,u_{j_h}(s))\,dt=\int_I ds\int_K \varphi(s,\lambda)\,d\sigma^s(\lambda). \tag{2.7}$$

Consider the following functional in $\tilde{U} \times Y$,

$$\tilde{J}(\sigma, y) = \int_I ds \int_K f(s, y(s), \lambda)\, d\sigma^s(\lambda),$$

and the following set of admissible pairs,

$$\tilde{\Lambda} = \left\{ (\sigma, y) \in \tilde{U} \times Y \colon y(t) = \int_0^t ds \int_K g(s, y(s), \lambda)\, d\sigma^s(\lambda) \right\}.$$

The set $\tilde{\Lambda}$ is the closure of $\Lambda$ in $\tilde{U} \times Y$. In fact, let us consider $(\sigma_j, y_j) \subset \tilde{\Lambda}$ converging to $(\sigma, y)$. By (2.6), in particular, we obtain

$$\lim_j \int_I dt \int_K g(s, y(s), \lambda)\, d\sigma_j^s(\lambda) = \int_I dt \int_K g(s, y(s), \lambda)\, d\sigma^s(\lambda). \tag{2.8}$$

Hence, by assumption (2.5) and by the uniform convergence of $y_j$ to $y$, we have

$$\lim \int_I dt \int_K g(s, y_j(s), \lambda)\, d\sigma_j^s(\lambda) = \lim \int_I dt \int_K g(s, y(s), \lambda)\, d\sigma_j^s(\lambda), \tag{2.9}$$

consequently $(\sigma, y) \in \tilde{\Lambda}$ and so $\tilde{\Lambda}$ is closed. Furthermore, by the properties of $\tilde{U}$, there exists $(u_j) \subset U$ such that $(\sigma^{u_j}) \subset \tilde{U}$ converges to $\sigma$ in $\tilde{U}$. Define

$$y_j(t) = \int_0^t g(s, y_j(s), u_j(s))\, ds,$$

obviously $(u_j, y_j) \subset \Lambda$. By (2.5) and the Ascoli–Arzela compactness theorem, we can extract, from $(y_j)$, a subsequence which converges uniformly in $I$ to some continuous function $\bar{y}$. By proceeding as above we obtain $y(t) = \bar{y}(t)$ almost everywhere in $I$. Moreover, if $(\sigma_j, y_j) \subset \tilde{U} \times X$ converges to $(\sigma, y)$, by proceeding as in (2.8) and (2.9), with $f$ instead of $g$, we obtain

$$\lim_j \tilde{J}(\sigma, y_j) = \tilde{J}(\sigma, y). \tag{2.10}$$

Let us consider

$$(\tilde{P}) \quad \min\{\tilde{F}(\sigma, y) \colon (\sigma, y) \in \tilde{U} \times Y\},$$

where

$$\tilde{F}(\sigma, y) = \tilde{J}(\sigma, y) + \chi_{\tilde{\Lambda}}(\sigma, y).$$

We point out that, for every $(u, y) \in U \times Y$,

$$\tilde{F}(\sigma^u, y) = F(u, y). \tag{2.11}$$

The functional $\tilde{F}$ is lower semicontinuous in $\tilde{U} \times Y$. In fact, let $(\sigma_j, y_j) \subset \tilde{U} \times Y$ converge to $(\sigma, y)$. If $\liminf_j F(\sigma_j, y_j) < +\infty$ there exists a subsequence $(\sigma_{j_h}, y_{j_h}) \subset \tilde{\Lambda}$ for which

$$\lim_h \tilde{F}(\sigma_{j_h}, y_{j_h}) = \liminf_j F(\sigma_j, y_j),$$

then $(\sigma, y) \in \tilde{\Lambda}$ and (2.10) implies the semicontinuity of $\tilde{F}$.

Now we establish that $(\tilde{P})$ is a relaxed problem of $(P)$. Part (i) comes from the properties of $\tilde{U}$. Part (ii) is a direct consequence of the lower semicontinuity of $\tilde{F}$. Finally, part (iii) comes from the density of $\Lambda$ in $\tilde{\Lambda}$ and from (2.10) and (2.11) if $(\sigma, y) \in \Lambda$; otherwise it comes from the semicontinuity of $\tilde{F}$.

The relationship between $(\bar{P})$ and $(\tilde{P})$ is contained in the following:

**Theorem.** *Assume that f and g satisfy* (2.4) *and* (2.5). *For all* $(u, y) \in U \times Y$, *we have*

$$\bar{F}(u, y) = \min\{\tilde{F}(\sigma, y) : \sigma \in B(u)\}, \tag{2.12}$$

*where* $B(u)$ *denotes the set of measures* $\sigma \in \tilde{U}$ *which have u as barycenter, i.e.,*

$$B(u) = \left\{\sigma \in \tilde{U} : u(s) = \int_K \lambda d\sigma^s(\lambda)\right\}.$$

*Proof.* First we prove that, for every $(u, y) \in U \times Y$ and $\sigma \in B(u)$,

$$\bar{F}(u, y) \leq \tilde{F}(\sigma, y). \tag{2.13}$$

For measures such that $(\sigma, y) \notin \tilde{\Lambda}$, (2.13) is obvious. Let $(\sigma, y) \in \tilde{\Lambda}$ and $\sigma \in B(u)$. By taking into account the density of $\Lambda$ in $\tilde{\Lambda}$, there exists $(u_j, y_j) \subset \Lambda$ such that $(\sigma^{u_j}, y_j)$ converges to $(\sigma, y)$. By considering in (2.7), $\varphi(s, \lambda) = \varphi(s)\lambda$ with $\varphi \in L^1(I)$, we obtain

$$\lim_j \int_I \varphi(s) u_j(s)\, dt = \int_I \varphi(s) \int_K \lambda\, d\sigma^s(\lambda)\, ds,$$

i.e., $(u_j)$ converges to the barycenter $u$ of $\sigma$. Since, from (2.10) and (2.11), we obtain

$$\lim_j F(u_j, y_j) = \lim_j \tilde{F}(\sigma^{u_j}, y_j) = \tilde{F}(\sigma, y),$$

inequality (2.13) easily follows from the definition of $\bar{F}$. Now let $(u, y) \in U \times Y$ so that $\bar{F}(u, y) \neq +\infty$, there exists $(u_j, y_j) \subset \Lambda$ which converges to $(u, y)$ and

$$\bar{F}(u, y) = \lim_j F(u_j, y_j). \tag{2.14}$$

Since definitely $(u_j, y_j) \subset \Lambda$, there exists $\sigma \in \tilde{U}$ such that $(\sigma, y) \in \tilde{\Lambda}$ and a subsequence of $(\sigma^{u_j}) \subset \tilde{U}$ which converges to $\sigma$ in $\tilde{U}$ with $\sigma \in B(u)$. From (2.10), (2.11), and (2.14) we get

$$\bar{F}(u, y) = \lim_j F(u_j, y_j) = \tilde{F}(\sigma, y)$$

and the proof is completed. □

Formula (2.13) also gives the relationship between the minima of $(\bar{P})$ and $(\tilde{P})$. In fact, if $(u, y)$ is a solution of $(\bar{P})$ there exists $\sigma \in B(u)$ such that $(\sigma, y)$ is a solution of $(\tilde{P})$. Conversely, if $(\sigma, y)$ is a solution of $(\tilde{P})$, the barycenter $u$ of $\sigma$ is such that the pair $(u, y)$ is a solution of $(\bar{P})$.

We observe that, if $T = +\infty$, the result of the theorem is still true if we make some suitable modifications. In particular, defining $\tilde{U}$ and $Y$ as before, we say that $\sigma_j$ converges to $\sigma$ if, for each $\varphi \in L^1([0, T], C(K))$ and $T > 0$,

$$\lim_j \int_0^T ds \int_K \varphi(s, \lambda)\, d\sigma_j^s(\lambda) = \int_0^T ds \int_K \varphi(s, \lambda)\, d\sigma_s(\lambda),$$

and that $y_j$ converges to $y$ in $Y$ if $y_j$ converges uniformly on the compact set of $[0, +\infty[$.

Finally, we would like to point out that $\tilde{F}$ is the lower semicontinuous envelope of the following functional in $\tilde{U} \times Y$:

$$\mathscr{F}(\sigma, y) = \begin{cases} F(u, y) & \text{if} \quad \sigma = \sigma^u, \quad u \in U, \\ +\infty & \text{otherwise.} \end{cases}$$

After this paper was completed, T. Zolezzi brought paper [10] to our knowledge. When particularized to problems in the calculus of variations the theorem stated above gives a result which is comparable to those of [10] (see Example 2.5).

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