ESAIM: Control, Optimisation and Calculus of Variations Will be set by the publisher
URL: http://www.emath.fr/cocv/

EVERYWHERE REGULARITY FOR VECTORIAL FUNCTIONALS WITH GENERAL GROWTH

Elvira Mascolo and Anna Paola Migliorini[1]

Abstract. We prove Lipschitz continuity for local minimizers of integral functionals of the Calculus of Variations in the vectorial case, where the energy density depends explicitly on the space variables and has general growth with respect to the gradient. One of the models is

$$F(u)=\int_\Omega a(x)[h(|Du|)]^{p(x)}dx$$

with h a convex function with general growth (also exponential behaviour is allowed).

Résumé. On démontre la continuité Lipschitzienne des minima locaux des fonctionnelles intégrales du calcul des variations dans le cas vectoriel, où la densité de l'énergie dépend explicitement des variables d'espace et est à croissance générale par rapport au gradient. Un modèle est

$$F(u)=\int_\Omega a(x)[h(|Du|)]^{p(x)}dx$$

avec h une fonction convexe à croissance générale (on admet aussi un comportement exponentiel).

AMS Subject Classification. 49N60; 35J50.

.

1. Introduction

In this paper we study the local Lipschitz continuity for local minimizers of the integral functional

$$F(u)=\int_\Omega f(x,Du(x))\,dx, \tag{1.1}$$

where $\Omega\subset\mathbf{R}^n$ is an open set, $f=f(x,\xi):\Omega\times\mathbf{R}^{nN}\to\mathbf{R}$ is a Carathéodory function and $Du=(u^\alpha_{x_i})$ for $i=1,\dots,n$ $(n\geq 2)$ and $\alpha=1,\dots,N$ denotes the Jacobian matrix of the vector-valued function $u:\Omega\to\mathbf{R}^N$.

We say that $u\in W^{1,2}_{loc}(\Omega,\mathbf{R}^N)$ is a local minimizer of F if $f(x,Du)\in L^1_{loc}(\Omega)$ and for every $\varphi\in C^1_0(\Omega,\mathbf{R}^N)$

$$\int_{\mathrm{spt}\,\varphi} f(x,Du)\,dx\leq\int_{\mathrm{spt}\,\varphi} f(x,Du+D\varphi)\,dx;$$

Keywords and phrases: Minimizers, Regularity, Nonstandard Growth, Exponential Growth.

[1] Dipartimento di Matematica "U. Dini", Universita' di Firenze, viale Morgagni 67/A, 50134 - Firenze, Italy.

therefore u is also a weak solution of an elliptic system of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{\alpha}(x, Du) = 0, \quad \forall \alpha = 1, \dots, N \tag{1.2}$$

where the vector field $a = (a_i^{\alpha}) : \Omega \times \mathbf{R}^{nN} \to \mathbf{R}^{nN}$ is the gradient with respect to ξ of the function f.

The regularity properties for minimizers of vectorial integrals have been widely investigated under ellipticity and *natural growth conditions* and, in general, we can aspect only *partial regularity*, see [10, 12]. Nevertheless, in the case $f(x,\xi) = |\xi|^p$, $(p \geq 2)$, Uhlenbeck proved in [23] that the minimizers are in $C^{1,\alpha}_{loc}(\Omega, \mathbf{R}^N)$. Partial regularity is obtained when integrands have the form $g(x, u, |\xi|)$ with $|\xi|^p$ behaviour by Giaquinta-Modica [11] for $p \geq 2$ and Acerbi-Fusco [2] for $1 < p < 2$.
In the last years the interest in the study of regularity under *non natural growth conditions* has developed new approaches. In [15] Marcellini considers integrals without growth conditions and proves Hölder continuity of the gradient for minimizers when $f(x,\xi) = g(|\xi|)$ with g positive and convex, satisfying:

$$\frac{g'(t)}{t} \text{ is positive and increasing in } (0, +\infty) \tag{1.3}$$

and a *non oscillatory* condition at infinity, i.e. for every $\alpha > 1$ there exists a constant $c = c(\alpha)$ such that

$$g''(t)t^{2\alpha} \leq c[g(t)]^{\alpha}, \quad \forall t > 1; \tag{1.4}$$

these conditions imply at least quadratic growth but they allow exponential behaviour. The subquadratic case is studied by Leonetti-Mascolo-Siepe [9].

In this paper we consider the non homogeneous case

$$f(x,\xi) = g(x, |\xi|) \tag{1.5}$$

and we obtain the following regularity result.

Theorem 1.1. *Let $g = g(x,t) : \Omega \times [0,+\infty) \to [0,+\infty)$ be a function of class C^2, convex in t, such that $\forall x \in \Omega$, $\frac{g_t(x,t)}{t}$ is positive and increasing with respect to t. Assume that for every $\Omega_0 \subset\subset \Omega$ and $\alpha > 1$ there exist two positive constants c_1 and c_2, depending on α and on Ω_0, such that $\forall x \in \Omega_0$*

$$\begin{aligned} g_{tt}(x,t)t^{2\alpha} &\leq c_1[g(x,t)]^{\alpha}, \quad \forall t \geq 1, \\ |g_{tx_s}(x,t)| &\leq c_2 g_t(x,t)[1 + g_t^{\alpha-1}(x,t)], \quad \forall t \geq 0,\ \forall s = 1, \dots, n. \end{aligned}$$

Then every local minimizer u of the functional (1.1) with f given by (1.5) is in $W^{1,\infty}_{loc}(\Omega, \mathbf{R}^N)$ and there exist $c > 0$ and $\sigma > 0$ such that for every $B_R \subset\subset \Omega$

$$\sup_{B_{R/2}} |Du| \leq c \left\{ \int_{B_R} [1 + g(x, |Du|)]\, dx \right\}^{1+\sigma}. \tag{1.6}$$

Actually we prove the theorem under weak assumptions on g, (see (H_1)-(H_5) and theorem 2.1 of Section 2).
The most relevant fact is that the integrand $f(x,\xi)$ may have exponential growth with respect to ξ, which involves non uniformily elliptic systems. Our result includes *energy densities with variable growth* as

$$\int_{\Omega} a(x)[h(|Du|)]^{p(x)} dx, \tag{1.7}$$

where h is a $C^2([0,+\infty))$ positive convex function satisfying conditions (1.3) and (1.4) with a, $p \in W^{1,\infty}_{loc}(\Omega)$, $a(x)$, $p(x) \geq c > 0$ *a.e.* $x \in \Omega$; in particular we can take $h(t) \sim \exp(t^m)$ for $t \to +\infty$ and $m > 0$.

The interest in functionals (1.1) with general growth and non uniformily elliptic systems (1.2) is motivated by several models which arise from different problems in mathematical physics: for example, the exponential growth is present in combustion theory, see Mosely [20] and in reaction of gases, see Aris [1]. Recently, this kind of systems has been used by Rajagopal and Růžička [21, 22] in their model for the behaviour of special viscous fluids with the ability to change their mechanical properties in dependence on an applied electric field, the so-called *electrorheological fluids*. In fact, in the model proposed by Rajagopal and Růžička, the interaction between the electric field and the fluid in motion is expressed in the coefficients of the system by a variable exponent.

The particular case $f(x,\xi) = |\xi|^{p(x)}$ has been studied in the scalar case by ZhiKov [24], Mascolo-Papi [17] and Chiadò Piat-Coscia [5] (see also Marcellini [13, 14] and Dall'Aglio-Mascolo-Papi [7]). In the vectorial case, the regularity result is due to Coscia-Mingione [6] (see also Acerbi-Mingione [3, 4] for related results).

For functionals with integrand of the type (1.5) Migliorini in [18, 19] proves everywhere regularity of local minimizers in the context of $(p,\, q)$-growth conditions.

We improve these results to more general cases, like (1.7) and even to energies of the form

$$g(x,t) = \exp(t^{p(x)}) \quad \text{as } t \to +\infty,$$

by using different techniques. We do not control the stored energy $g(x,t)$ by means of power functions: indeed we use its particular structure and properties directly (see also Dall'Aglio-Mascolo [8] for L^∞-regularity).

The paper is organized as follows. Section 2 contains the statement of the general regularity theorem and some applications. In Section 3 we consider functionals with controllable growth, i.e. uniformly elliptic systems, and we prove for the gradient of minimizers an *a priori* estimate independent of the constants which appear in the controllability assumptions. In Section 4, we carry out the estimate to the general case by means of an approximation argument. More precisely, we construct a sequence of functions which converges to g such that the corresponding functionals have controllable growth. By applying the *a priori* estimate, a procedure of passage to the limit gives estimate (1.6) for the minimizer of the original functional.

2. Statement of the regularity theorem

Consider the integral functional

$$F(u) = \int_\Omega f(x, Du(x))\, dx, \tag{2.1}$$

where Ω is an open subset of $\mathbf{R}^n$ $(n \geq 2)$, Du is the gradient of a vector-valued function $u : \Omega \to \mathbf{R}^N$, thus $Du = (u^\alpha_{x_i})$ for $i = 1, \ldots, n$ and $\alpha = 1, \ldots, N$ is a matrix in $\mathbf{R}^{nN}$, and $f = f(x,\xi) : \Omega \times \mathbf{R}^{nN} \to \mathbf{R}$ is a Carathéodory integrand.

We consider the case in which the stored energy f depends on the modulus of the matrix Du and satisfies general growth conditions. More precisely, we assume that

$$f(x,\xi) = g(x, |\xi|), \tag{2.2}$$

where $g(x,t) : \Omega \times [0,+\infty) \to [0,+\infty)$ satisfies the following assumptions:

$(\mathbf{H_1})$ for *a.e.* $x \in \Omega$, $g(x,\cdot)$ is a positive convex function of class $C^2([0,+\infty))$ with $\frac{g_t(x,t)}{t}$ positive (strictly for $t > 0$) and increasing with respect to t for *a.e.* $x \in \Omega$.

Observe that, since $\frac{g_t(x,t)}{t}$ is increasing, then necessarily $g_t(x,0) = 0$ for *a.e.* $x \in \Omega$. Moreover, without loss of generality, by adding a measurable bounded function of x to g, we can reduce to the case $g(x,0) = 0$ for

a.e. $x \in \Omega$.
Clearly from (H_1) it follows that

$$0 \leq g(x,t) \leq g_t(x,t)t, \tag{2.3}$$
$$0 \leq g_t(x,t) \leq g_{tt}(x,t)t, \tag{2.4}$$

$\forall t > 0$ and *a.e.* $x \in \Omega$.

$(\mathbf{H_2})$ For every $\Omega_0 \subset\subset \Omega$, there is a positive constant $\Lambda = \Lambda(\Omega_0)$ such that

$$g_{tt}(x,t) \leq \Lambda, \quad \forall t \in [0,1] \text{ and } a.e.\ x \in \Omega_0, \tag{2.5}$$

and a $t_0 \in (0,1)$ and $\lambda = \lambda(\Omega_0) > 0$ such that

$$g(x,t_0) \geq \lambda, \quad a.e.\ x \in \Omega_0. \tag{2.6}$$

The *non oscillatory* behaviour is included in the following assumption.

$(\mathbf{H_3})$ For every $\Omega_0 \subset\subset \Omega$ and $\alpha > 1$, there exists a positive constant $c_1 = c_1(\alpha, \Omega_0)$ such that

$$g_{tt}(x,t)t^{2\alpha} \leq c_1[g(x,t)]^\alpha, \quad \forall t \geq 1 \text{ and } a.e.\ x \in \Omega_0. \tag{2.7}$$

$(\mathbf{H_4})$ For every $t \in [0,+\infty)$, $g_t(x,t)$ admits weak derivatives $g_{tx_s}(x,t)$, $(\forall s = 1,\dots,n)$, which are Carathéodory functions in $\Omega \times [0,+\infty)$ and locally integrable in Ω. Moreover, for every $\Omega_0 \subset\subset \Omega$ and $\alpha > 1$ there exists a positive constant $c_2 = c_2(\alpha, \Omega_0)$ such that

$$|g_{tx_s}(x,t)| \leq c_2 g_t(x,t)[1 + g_t^{\alpha-1}(x,t)], \quad \forall t \geq 0 \text{ and } a.e.\ x \in \Omega_0. \tag{2.8}$$

$(\mathbf{H_5})$ For every $\Omega_0 \subset\subset \Omega$ and Q_0 compact subset of $[1,+\infty)$, $g_{tt}(x,t) \in L^\infty(\Omega_0 \times Q_0)$.

By using (2.2) and (2.4), the following inequality holds (see [14,15] for details):

$$\frac{g_t(x,|\xi|)}{|\xi|}|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\lambda_i^\alpha \lambda_j^\beta \leq g_{tt}(x,|\xi|)|\lambda|^2, \tag{2.9}$$

for *a.e.* $x \in \Omega$, $\forall \xi, \lambda \in \mathbf{R}^{nN}$.

In the sequel, fixed $\Omega_0 \subset\subset \Omega$ and $x_0 \in \Omega_0$, we denote by B_ρ and B_R balls with the same center x_0 of radii ρ and R respectively compactly contained in Ω_0, $(0 < \rho \leq R < \min\{dist(x_0, \partial\Omega_0), 1\})$.

Now we give the precise statement of our result.

Theorem 2.1. *Consider the functional F in (2.1) with $f(x,\xi) = g(x,|\xi|)$, where g satisfies (H_1) - (H_5). If u is a local minimizer of F, then u is of class $W^{1,\infty}_{loc}(\Omega, \mathbf{R}^N)$ and there exists $\sigma = \sigma(n) > 0$ such that*

$$\sup_{B_\rho} |Du| \leq c\left\{\int_{B_R}[1 + g(x,|Du|)]\,dx\right\}^{1+\sigma}, \tag{2.10}$$

where $c = c(n,N,c_1,c_2,\Lambda,\lambda,R,\rho)$.

Let now $h \in C^2([0,+\infty))$ be a strictly increasing convex function satisfying (1.3) and (1.4). Let $a(x), p(x) \in W^{1,\infty}_{loc}(\Omega)$ with $a(x), p(x) \geq c > 0$ for *a.e.* $x \in \Omega$. The function

$$g\left(x, |\xi|\right) = a(x) h\left(|\xi|\right)^{p(x)} \tag{2.11}$$

with h, a and p such that $g(x,t)$ is of class C^2 with respect to t, models in natural way the assumptions (H_1) - (H_5). It is easy to check that in (2.5) Λ depends on $h''(1)$ and on an upper bound for $a(x)$ and $p(x)$, while in (2.8) $c_2 = \max_{x\in\Omega_0}[|a_x(x)p(x)| + |a(x)p_x(x)|]$ where $|a_x(x)|$ and $|p_x(x)|$ denote the modulus of the gradient vectors of a and p.

We observe explicitly that if $h(t) = t^m$ or $h(t) = t^m \ln(t+1)$ all the assumptions are satisfied provided $mp(x) \geq 2$ for *a.e.* $x \in \Omega$.
On the other hand, if we consider exponential growth as

$$\begin{aligned} h(t) &\sim \exp(t^m) \quad \text{as } t \to +\infty, \text{ with } m > 0, \\ h(t) &\sim t^{\ln t} \quad \text{as } t \to +\infty, \\ h(t) &= \exp(t^m) \quad \text{with } m \geq 2, \end{aligned}$$

the variable exponent can be choosen such that $p(x) \geq \delta > 0$ for *a.e.* $x \in \Omega$.
Moreover,

$$g(x,t) = \exp(t^{p(x)}),$$

as $t \to +\infty$ or even every other finite composition of exponentials as for example

$$g(x,t) = \exp(\exp(t^{p_1(x)}))^{p_2(x)},$$

with $p_i(x) \geq 2$, $(i = 1, 2)$, satisfies (H_1) - (H_5).

3. A priori estimates

Marcellini in [15] proves some interesting inequalities in the case $g(x,t) = g(t)$ where g is a positive, convex function of class C^2 satisfying (1.3) and the *non oscillatory* condition (1.4). Using assumptions (H_1), (H_2) and (H_3), we can prove the same kind of inequalities for *a.e.* $x \in \Omega_0 \subset\subset \Omega$. Moreover, it is easy to check that the uniform boundedness assumptions in (H_2) imply that the constants in the pointwise inequalities are actually independent of $x \in \Omega_0$. These properties are contained in the following Lemma (see Lemmas 2.4, 2.6 and 2.7 of [15] for the proofs).

Lemma 3.1. *Let $\Omega_0 \subset\subset \Omega$ and g satisfy (H_1) - (H_2) - (H_3).*
(i) For every $\alpha > 1$ there exists a constant $c = c(\alpha, \Omega_0)$ such that

$$\begin{aligned} &g_t(x,t) t^{2\alpha-1} \leq c[g(x,t)]^\alpha, \quad g_{tt}(x,t) t^\alpha \leq c[g_t(x,t)]^\alpha \\ &\forall t \geq 1, \quad a.e.\ x \in \Omega_0. \end{aligned}$$

(ii) For every $\alpha > 1$ there exists a constant $c = c(\alpha, \Omega_0)$ such that

$$\begin{aligned} &1 + g_{tt}(x,t) t^{2\alpha} \leq c[1 + g(x,t)]^\alpha, \\ &\forall t \geq 0, \quad a.e.\ x \in \Omega_0. \end{aligned}$$

(iii) For every $\beta > 2$ there exists a constant $c = c(\beta, \Omega_0)$ such that $\forall \gamma \geq 0$

$$1 + g_{tt}(x,t)\left(\frac{t^{\gamma+1}}{\gamma+1}\right)^{\beta} \leq c\left[1 + \int_0^t s^{\gamma}\sqrt{\frac{g_t(x,s)}{s}}ds\right]^{\beta},$$
$$\forall t \geq 0, \quad a.e.\ x \in \Omega_0.$$

The constants in (i)-(iii) depend on Λ and λ in (H_2).

We make the following supplementary assumptions (which will be removed through the approximation method in Section 4).

Assume that there exist positive constants m, M and N, depending on $\Omega_0 \subset\subset \Omega$, such that

$$m \leq \frac{g_t(x,t)}{t} \leq g_{tt}(x,t) \leq M \tag{3.1}$$

and

$$|g_{tx_s}(x,t)| \leq N(1+t^2)^{\frac{1}{2}}, \tag{3.2}$$

$\forall t > 0$ and for $a.e.\ x \in \Omega_0$. By taking in account (2.9), (3.1) implies the uniform ellipticity condition, i.e.

$$m\,|\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\,\lambda_i^\alpha \lambda_j^\beta \leq M\,|\lambda|^2\,, \tag{3.3}$$

and, since

$$|f_{\xi_i^\alpha x_s}(x,\xi)| \leq |g_{tx_s}(x,|\xi|)|, \quad a.e.\ x \in \Omega_0, \quad \forall \xi \in \mathbf{R}^{nN}, \tag{3.4}$$

(3.2) gives

$$|f_{\xi_i^\alpha x_s}(x,\xi)| \leq N(1+|\xi|^2)^{\frac{1}{2}}, \quad a.e.\ x \in \Omega_0, \quad \forall \xi \in \mathbf{R}^{nN}.$$

First we present the following intermediate regularity result.

Proposition 3.2. *Consider the functional F in (2.1) with $f(x,\xi) = g(x,|\xi|)$ where g satisfies (H_1) - (H_4) and (3.1) - (3.2) and let u be a local minimizer of F. Then $u \in W^{1,\infty}_{loc}(\Omega, \mathbf{R}^N)$ and, for every $\Omega_0 \subset\subset \Omega$ and $0 < \rho < R < 1$ such that $B_R \subset\subset \Omega_0$, there exists $\sigma = \sigma(n) > 0$ such that the following estimate holds*

$$\sup_{B_\rho}|Du| \leq c\left\{\int_{B_R}[1 + g(x,|Du|)]\,dx\right\}^{1+\sigma}, \tag{3.5}$$

where c depends on n, N, R, ρ and on the constants in (H_1)-(H_4).

The proof follows by collecting Lemmas 3.3 and 3.4 below.

In the sequel, we denote by $1^* = \frac{n}{n-1}$ and by $2^* = \frac{2n}{n-2}$ if $n > 2$, while 2^* is any real number strictly greater than 1^*2, when $n = 2$.

Lemma 3.3. *Let (H_1)-(H_4) and (3.1)-(3.2) hold. If u is a local minimizer of F in (2.1), then $u \in W^{1,\infty}_{loc}(\Omega, \mathbf{R}^N)$ and there exists $c > 0$, depending on n, N and on the constants in (H_1)-(H_4), such that the following estimate* *holds*

$$\sup_{B_\rho}|Du| \le \frac{c}{(R-\rho)^{n-1}}\left\{\int_{B_R}\left[1+|Du|^{1^*2}g_{tt}(x,|Du|)\right]^{\frac{2^*}{1^*2}}dx\right\}^{\frac{1}{1^*}}.$$

Proof. Let u be a local minimizer of (2.1). By the left hand side of (3.3), u satisfies the Euler's first variation:

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\,\varphi^\alpha_{x_i}(x)\,dx = 0, \quad \forall \varphi = (\varphi^\alpha) \in W_0^{1,2}\left(\Omega,\mathbf{R}^N\right).$$

The technique of the difference quotient (see [10, 12] or in the context of non standard growth [15, 19]) gives that u admits second derivatives, precisely $u \in W^{2,2}_{loc}\left(\Omega,\mathbf{R}^N\right)$ and satisfies the second variation

$$\int_\Omega \left\{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\,\varphi^\alpha_{x_i}u^\beta_{x_sx_j} + \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\,\varphi^\alpha_{x_i}\right\}dx = 0, \tag{3.6}$$

$$\forall s = 1,\dots,n, \qquad \forall \varphi = (\varphi^\alpha) \in W_0^{1,2}\left(\Omega,\mathbf{R}^N\right).$$

Let $\Omega_0 \subset\subset \Omega$ and η be a positive function of class $C^1_0(\Omega_0)$; fixed $s \in \{1,\dots,n\}$, we choose $\varphi^\alpha = \eta^2 u^\alpha_{x_s}\Phi(|Du|)$ for every $\alpha = 1,\dots,N$, where Φ is a positive, increasing, bounded, Lipschitz continuous function defined in $[0,+\infty)$, (in particular Φ and Φ' are bounded, so that $\varphi = (\varphi^\alpha) \in W_0^{1,2}\left(\Omega,\mathbf{R}^N\right)$). Then

$$\varphi^\alpha_{x_i} = 2\eta\eta_{x_i}u^\alpha_{x_s}\Phi(|Du|) + \eta^2 u^\alpha_{x_sx_i}\Phi(|Du|) + \eta^2 u^\alpha_{x_s}\Phi'(|Du|)(|Du|)_{x_i}$$

and from (3.6) we obtain

$$\begin{aligned}
0 &= \int_\Omega \eta^2\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\,u^\alpha_{x_sx_i}u^\beta_{x_sx_j}\,dx \\
&+ \int_\Omega 2\eta\Phi \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\,\eta_{x_i}u^\alpha_{x_s}u^\beta_{x_sx_j}\,dx \\
&+ \int_\Omega \eta^2\Phi' \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\,u^\alpha_{x_s}u^\beta_{x_sx_j}(|Du|)_{x_i}\,dx \\
&+ \int_\Omega 2\eta\Phi \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\,\eta_{x_i}u^\alpha_{x_s}\,dx \\
&+ \int_\Omega \eta^2\Phi \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\,u^\alpha_{x_sx_i}\,dx \\
&+ \int_\Omega \eta^2\Phi' \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\,u^\alpha_{x_s}(|Du|)_{x_i}\,dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned} \tag{3.7}$$

(here and in the following we write only Φ and Φ' instead of $\Phi(|Du|)$ and $\Phi'(|Du|)$). We sum with respect to s from 1 to n the previous equation but we still indicate the integrals with I_1 - I_6. In the sequel we denote by c any constant which may take different values from line to line and depends on the constants in assumptions (H_1) - (H_4) and on the dimensions n and N.

Let us start with the estimate of the integral I_2. By Cauchy-Schwartz inequality, Young's inequality $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$, $\forall \epsilon > 0$, and (2.9)

$$\begin{aligned} |I_2| &= \left| \int_\Omega 2\eta\Phi \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, \eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j}\, dx \right| \qquad (3.8)\\ &\leq \int_\Omega 2\eta\Phi \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, \eta_{x_i} u^\alpha_{x_s} \eta_{x_j} u^\beta_{x_s} \right\}^{\frac{1}{2}} \\ &\cdot \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_s x_i} u^\beta_{x_s x_j} \right\}^{\frac{1}{2}} dx \\ &\leq c\epsilon_1 \int_\Omega \eta^2 \Phi \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\ &+ \frac{c}{4\epsilon_1} \int_\Omega |D\eta|^2 \Phi g_{tt}(x, |Du|)\, |Du|^2\, dx. \end{aligned}$$

Let us consider I_3. Since $f(x,\xi) = g(x, |\xi|)$, we have

$$\begin{aligned} f_{\xi_i^\alpha}(x,\xi) &= \frac{g_t(x,|\xi|)}{|\xi|} \xi_i^\alpha \\ f_{\xi_i^\alpha \xi_j^\beta}(x,\xi) &= \left(\frac{g_{tt}(x,|\xi|)}{|\xi|^2} - \frac{g_t(x,|\xi|)}{|\xi|^3} \right) \xi_j^\beta \xi_i^\alpha + \frac{g_t(x,|\xi|)}{|\xi|} \delta_{\xi_i^\alpha \xi_j^\beta}. \end{aligned}$$

Using (2.4) and the fact that $g_t(x,t)$ is positive, we can prove that

$$\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_s} u^\beta_{x_s x_j} (|Du|)_{x_i} \geq 0. \qquad (3.9)$$

In fact

$$\begin{aligned} &\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_s} u^\beta_{x_s x_j} (|Du|)_{x_i} \\ = &\left(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2} \right) \sum_{i,s,\alpha} \left[u^\alpha_{x_i} (|Du|)_{x_s} \right]^2 \\ + &\; g_t(x,|Du|) \sum_i (|Du|)^2_{x_i} \geq 0, \end{aligned}$$

since $(|Du|)_{x_i} = \frac{1}{|Du|}\sum_{s,\alpha} u^\alpha_{x_s} u^\alpha_{x_i x_s}$; hence (3.9) is proved and this easily implies that $I_3 \geq 0$. Consider now I_4: by assumption (H_4) and by (3.4) and (2.4), we have

$$\begin{aligned} |I_4| &= \left| \int_\Omega 2\eta\Phi \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\, \eta_{x_i} u^\alpha_{x_s}\, dx \right| \\ &\leq c \int_\Omega 2\eta\Phi g_{tt}(x,|Du|)|Du| \left[1 + g_t^{\alpha-1}(x,|Du|)\right] \sum_{i,s,\alpha} \left|\eta_{x_i} u^\alpha_{x_s}\right| dx \\ &\leq c \int_\Omega 2\eta|D\eta|\Phi g_{tt}(x,|Du|)|Du|^2 \left[1 + g_t^{\alpha-1}(x,|Du|)\right] dx. \end{aligned} \tag{3.10}$$

In order to estimate I_5, let us observe that, taking in account (2.4), (2.8) becomes

$$|g_{tx_s}(x,t)| \leq c_2 \left\{ \frac{g_t(x,t)}{t} \right\}^{\frac12} \left\{ g_{tt}(x,t)t^2 \left[1 + g_t^{2(\alpha-1)}(x,t)\right] \right\}^{\frac12}, \tag{3.11}$$

thus, by using Cauchy-Schwartz inequality and Young's inequality, we obtain

$$\begin{aligned} |I_5| &= \left| \int_\Omega \eta^2\Phi \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\, u^\alpha_{x_s x_i} dx \right| \\ &\leq \int_\Omega \eta^2\Phi \left\{ \sum_{i,s,\alpha} f^2_{\xi_i^\alpha x_s}(x, Du) \right\}^{\frac12} |D^2u| dx \\ &\leq c_2 \int_\Omega \eta^2\Phi \left\{ \frac{g_t(x,|Du|)}{|Du|} |D^2u|^2 \right\}^{\frac12} \left\{ g_{tt}(x,|Du|)|Du|^2 \left[1 + g_t^{2(\alpha-1)}(x,t)\right] \right\}^{\frac12} dx \\ &\leq c\epsilon_2 \int_\Omega \eta^2\Phi \frac{g_t(x,|Du|)}{|Du|} |D^2u|^2 dx \\ &+ \frac{c}{4\epsilon_2} \int_\Omega \eta^2\Phi g_{tt}(x,|Du|)|Du|^2 \left[1 + g_t^{2(\alpha-1)}(x,|Du|)\right] dx. \end{aligned} \tag{3.12}$$

Similarly

$$\begin{aligned} |I_6| &= \left| \int_\Omega \eta^2\Phi' \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x, Du)\, u^\alpha_{x_s} (|Du|)_{x_i}\, dx \right| \\ &\leq c_1 \int_\Omega \eta^2\Phi'|Du| \left\{ \frac{g_t(x,|Du|)}{|Du|} \sum_i (|Du|)^2_{x_i} \right\}^{\frac12} dx \\ &\cdot \left\{ g_{tt}(x,|Du|)|Du|^2 \left[1 + g_t^{2(\alpha-1)}(x,|Du|)\right] \right\}^{\frac12} dx \\ &\leq c\epsilon_3 \int_\Omega \eta^2\Phi'|Du| \frac{g_t(x,|Du|)}{|Du|} \sum_i (|Du|)^2_{x_i}\, dx \\ &+ \frac{c}{4\epsilon_3} \int_\Omega \eta^2\Phi'|Du| g_{tt}(x,|Du|)|Du|^2 \left[1 + g_t^{2(\alpha-1)}(x,|Du|)\right] dx. \end{aligned} \tag{3.13}$$

Collecting (3.8)-(3.13) and choosing ϵ_1 sufficiently small we have also

$$\begin{aligned} & \int_\Omega \eta^2 \Phi \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x, Du)\, u^\alpha_{x_s x_i} u^\beta_{x_s x_j} dx \\ \leq\ & c \int_\Omega |D\eta|^2 \Phi g_{tt}(x, |Du|)\, |Du|^2 \, dx \\ +\ & c \int_\Omega 2\eta |D\eta| \Phi g_{tt}(x, |Du|) |Du|^2 \left[1 + g_t^{\alpha-1}(x, |Du|)\right] dx \\ +\ & c\epsilon_2 \int_\Omega \eta^2 \Phi \frac{g_t(x, |Du|)}{|Du|} |D^2 u|^2 dx \\ +\ & \frac{c}{4\epsilon_2} \int_\Omega \eta^2 \Phi g_{tt}(x, |Du|) |Du|^2 \left[1 + g_t^{2(\alpha-1)}(x, |Du|)\right] dx \\ +\ & c\epsilon_3 \int_\Omega \eta^2 \Phi' |Du| \frac{g_t(x, |Du|)}{|Du|} \sum_i (|Du|)^2_{x_i} \, dx \\ +\ & \frac{c}{4\epsilon_3} \int_\Omega \eta^2 \Phi' |Du| g_{tt}(x, |Du|) |Du|^2 \left[1 + g_t^{2(\alpha-1)}(x, |Du|)\right] dx. \end{aligned} \tag{3.14}$$

By choosing ϵ_2 sufficiently small, the left inequality of (2.9), implies

$$\begin{aligned} & \int_\Omega \eta^2 \Phi \frac{g_t(x, |Du|)}{|Du|} |D^2 u|^2 dx \\ \leq\ & c \int_\Omega |D\eta|^2 \Phi g_{tt}(x, |Du|)\, |Du|^2 \, dx \\ +\ & c \int_\Omega 2\eta |D\eta| \Phi g_{tt}(x, |Du|) |Du|^2 [1 + g_t^{\alpha-1}(x, |Du|)] dx \\ +\ & c \int_\Omega \eta^2 \Phi g_{tt}(x, |Du|) |Du|^2 \left[1 + g_t^{2(\alpha-1)}(x, |Du|)\right] dx \\ +\ & c\epsilon_3 \int_\Omega \eta^2 \Phi' |Du| \frac{g_t(x, |Du|)}{|Du|} \sum_i (|Du|)^2_{x_i} \, dx \\ +\ & \frac{c}{4\epsilon_3} \int_\Omega \eta^2 \Phi' |Du| g_{tt}(x, |Du|) |Du|^2 \left[1 + g_t^{2(\alpha-1)}(x, |Du|)\right] dx. \end{aligned} \tag{3.15}$$

Now we allow only test function Φ satisfying

$$\Phi'(t)\, t \leq c_\Phi \Phi(t) \tag{3.16}$$

for a certain constant $c_\Phi \geq 0$. Recalling that $(|Du|)_{x_i} = \frac{1}{|Du|} \sum_{s,\alpha} u^\alpha_{x_s} u^\alpha_{x_i x_s}$, and using Cauchy-Schwartz inequality, we see that

$$|D(|Du|)|^2 = \sum_i (|Du|)^2_{x_i} \leq \sum_{i,s,\alpha} \left|u^\alpha_{x_s x_i}\right|^2 = \left|D^2 u\right|^2. \tag{3.17}$$

We use the last inequality to estimate the first member in (3.15) and for small ϵ_3 we get

$$\begin{aligned} & \int_\Omega \eta^2 \Phi \frac{g_t(x,|Du|)}{|Du|} \sum_i (|Du|)^2_{x_i}\, dx \\ \leq\ & c \int_\Omega |D\eta|^2\, \Phi g_{tt}\,(x,|Du|)\,|Du|^2\, dx \\ +\ & c \int_\Omega 2\eta |D\eta| \Phi g_{tt}(x,|Du|)|Du|^2 \left[1+g_t^{\alpha-1}(x,|Du|)\right] dx \\ +\ & c \int_\Omega \eta^2 \Phi g_{tt}(x,|Du|)|Du|^2 \left[1+g_t^{2(\alpha-1)}(x,|Du|)\right] dx \\ +\ & c(c_\Phi)^2 \int_\Omega \eta^2 \Phi g_{tt}(x,|Du|)|Du|^2 \left[1+g_t^{2(\alpha-1)}(x,|Du|)\right] dx. \end{aligned} \tag{3.18}$$

On the other hand, since $2\eta\,|D\eta|$, $|D\eta|^2$, η^2 are less then or equal to $\eta^2+|D\eta|^2$, using (3.17) we finally have

$$\begin{aligned} & \int_\Omega \eta^2 \Phi \frac{g_t(x,|Du|)}{|Du|}\, |D\,(|Du|)|^2\, dx \\ \leq\ & c(1+c_\Phi)^2 \int_\Omega [\eta^2+|D\eta|^2] \Phi g_{tt}(x,|Du|)|Du|^2 \left[1+g_t^{2(\alpha-1)}(x,|Du|)\right] dx, \end{aligned} \tag{3.19}$$

where $c=c(n,N,\Omega_0,\Lambda,\lambda,c_1,c_2)$. Let now Φ be a positive, increasing and locally Lipschitz continuous function in $[0,+\infty)$ satisfying (3.16). Then we can approximate Φ by a sequence of Lipschitz functions Φ_r bounded with Φ_r' bounded, in the following way:

$$\Phi_r\,(t) = \begin{cases} \Phi\,(t) & \text{for } t\in[0,r] \\ \Phi\,(r) & \text{for } t\in(r,+\infty) \end{cases} \quad r\in\mathbb{N}.$$

Since $\Phi_r'\,(t)\,t \leq c_\Phi \Phi\,(t)$, while $\Phi_r'\,(r^+)$ and $\Phi_r'\,(r^-)$ are uniformly bounded, the condition (3.16) holds for Φ_r with the same constant c_Φ, thus (3.19) holds Φ_r. By monotone convergence theorem, letting r tend to $+\infty$, we infer that (3.19) holds for such a Φ.
For $t\in[0,+\infty)$ and $x\in\Omega$ define

$$G\,(x,t) = 1 + \int_0^t \sqrt{\Phi(s)\frac{g_t(x,s)}{s}} ds;$$

since the integrand function is increasing and by (2.4), we get

$$\begin{aligned} [G\,(x,t)]^2 &\leq \left[1+t\sqrt{\Phi(t)\frac{g_t(x,t)}{t}}\right]^2 \\ &\leq 2\left[1+t^2\Phi(t)\frac{g_t(x,t)}{t}\right] \leq 2\left[1+\Phi(t)g_{tt}(x,t)t^2\right]. \end{aligned}$$

Moreover, by (H_4), $\forall i=1,\dots,n$ we have

$$\begin{aligned}\left[\frac{\partial}{\partial x_i}G(x,t)\right]^2 &= \left[\int_0^t\sqrt{\frac{\Phi(s)}{s}}\frac{g_{tx_i}(x,s)}{2\sqrt{g_t(x,s)}}ds\right]^2\\ &\le c\left[t\sqrt{\frac{\Phi(t)}{t}g_t(x,t)}\left[1+g_t^{\alpha-1}(x,t)\right]\right]^2\\ &\le c\Phi(t)g_{tt}(x,t)t^2\left[1+g_t^{2(\alpha-1)}(x,t)\right].\end{aligned}$$

We denote by D_xG the weak gradient of $G(x,t)$ with respect to x. The assumptions (H_1) and (H_4) ensure (see for instance Marcus-Mizel [16]) that the chain rule holds and the previous estimates yield:

$$\begin{aligned}&|D[\eta G(x,|Du|)]|^2\\ \le\;& c|D\eta|^2[G(x,|Du|)]^2+c\eta^2[G_t(x,|Du|)D(|Du|)]^2+c\eta^2[D_xG(x,|Du|)]^2\\ \le\;& c|D\eta|^2\left[1+\Phi g_{tt}(x,|Du|)|Du|^2\right]+c\eta^2\Phi\frac{g_t(x,|Du|)}{|Du|}|D(|Du|)|^2\\ +\;& c\eta^2\Phi g_{tt}(x,|Du|)|Du|^2\left[1+g_t^{2(\alpha-1)}(x,|Du|)\right].\end{aligned}$$

Therefore by (3.19), we deduce

$$\begin{aligned}&\int_\Omega|D[\eta G(x,|Du|)]|^2dx\\ &\le c(1+c_\Phi)^2\int_\Omega[\eta^2+|D\eta|^2]\Phi\left[1+g_{tt}(x,|Du|)|Du|^2\right]\left[1+g_t^{2(\alpha-1)}(x,|Du|)\right]dx\end{aligned}$$

where $c=c(n,N,\Omega_0,\Lambda,\lambda,c_1,c_2,\alpha)$. Let $2^*=\frac{2n}{n-2}$ for $n>2$, while 2^* equal to any fixed real number greater than 1^*2 if $n=2$. By Sobolev's inequality:

$$\begin{aligned}&\left\{\int_\Omega\eta^{2^*}[G(x,|Du|)]^{2^*}dx\right\}^{\frac{2}{2^*}}\\ \le\;& c(1+c_\Phi)^2\int_\Omega[\eta^2+|D\eta|^2]\left[1+\Phi g_{tt}(x,|Du|)|Du|^2\right]\left[1+g_t^{2(\alpha-1)}(x,|Du|)\right]dx.\end{aligned}\tag{3.20}$$

Choose $\Phi(t)=t^{2\gamma}$ with $\gamma\ge0$, thus the condition (3.16) is satisfied with $c_\Phi=2\gamma$. With this choice of Φ, (3.20) reduces to

$$\begin{aligned}&\left\{\int_\Omega\eta^{2^*}[G(x,|Du|)]^{2^*}dx\right\}^{\frac{2}{2^*}}\\ &\le c(1+\gamma)^2\int_\Omega[\eta^2+|D\eta|^2]\left[1+|Du|^{2(\gamma+1)}g_{tt}(x,|Du|)\right]\left[1+g_t^{2(\alpha-1)}(x,|Du|)\right]dx.\end{aligned}\tag{3.21}$$

By (iii) of Lemma 3.1, for *a.e.* $x\in\Omega_0$ we get

$$\begin{aligned}[G(x,t)]^{2^*} &= \left[1+\int_0^t s^\gamma\sqrt{\frac{g_t(x,s)}{s}}ds\right]^{2^*}\\ &\ge c\left[1+\left(\frac{t^{\gamma+1}}{\gamma+1}\right)^{2^*}g_{tt}(x,t)\right],\end{aligned}$$

thus (3.21) becomes

$$\left\{\int_{\Omega}\eta^{2^*}\left[1+|Du|^{2^*(\gamma+1)}g_{tt}(x,|Du|)\right]dx\right\}^{\frac{2}{2^*}} \tag{3.22}$$
$$\leq c(1+\gamma)^4\int_{\Omega}[\eta^2+|D\eta|^2]\left[1+|Du|^{2(\gamma+1)}g_{tt}(x,|Du|)\right]\left[1+g_t^{2(\alpha-1)}(x,|Du|)\right]dx.$$

Fixed ρ_0 and R_0 such that $B_{\rho_0}\subset\subset B_{R_0}\subset\subset\Omega_0$, for $0<\rho_0<\rho<R<R_0$, let η be a positive test function equal to 1 in B_ρ, whose support is contained in B_R, such that $|D\eta|\leq\frac{2}{R-\rho}$. Set $\theta=\gamma+1$ and $\epsilon=2(\alpha-1)$, using (2.4) we have

$$\left\{\int_{B_\rho}\left[1+|Du|^{2^*\theta}g_{tt}(x,|Du|)\right]dx\right\}^{\frac{2}{2^*}} \tag{3.23}$$
$$\leq\ c\frac{\theta^4}{(R-\rho)^2}\int_{B_R}\left[1+|Du|^{2\theta}g_{tt}(x,|Du|)g_{tt}^{\epsilon}(x,|Du|)|Du|^{\epsilon}\right]dx.$$

For an arbitrary τ, $0<\tau<1$, using Hölder inequality we get

$$\int_{B_R}\left[1+|Du|^{2\theta}g_{tt}(x,|Du|)g_{tt}^{\epsilon}(x,|Du|)|Du|^{\epsilon}\right]dx$$
$$\leq\ c\int_{B_R}\left[1+|Du|^{2\theta}g_{tt}^{1-\tau}(x,|Du|)g_{tt}^{\epsilon+\tau}(x,|Du|)|Du|^{\epsilon}\right]dx$$
$$\leq\ c\left\{\int_{B_R}\left[1+|Du|^{\frac{2\theta}{1-\tau}}g_{tt}(x,|Du|)\right]dx\right\}^{1-\tau}$$
$$\cdot\ \left\{\int_{B_R}\left[1+g_{tt}^{\frac{\tau+\epsilon}{\tau}}(x,|Du|)|Du|^{\frac{\epsilon}{\tau}}\right]dx\right\}^{\tau}.$$

Moreover, by (H_2)

$$1+g_{tt}^{\frac{\tau+\epsilon}{\tau}}(x,t)t^{\frac{\epsilon}{\tau}}\leq c\left[1+g_{tt}(x,t)t^{\frac{\epsilon}{\tau+\epsilon}}\right]^{\frac{\tau+\epsilon}{\tau}}\leq c[1+g_{tt}(x,t)t^{1^*2}]^{\frac{\tau+\epsilon}{\tau}}$$

and then by (3.22)

$$\left\{\int_{B_\rho}\left[1+|Du|^{2^*\theta}g_{tt}(x,|Du|)\right]dx\right\}^{\frac{2}{2^*}} \tag{3.24}$$
$$\leq\ c\frac{\theta^4}{(R-\rho)^2}\left\{\int_{B_R}\left[1+|Du|^{\frac{2\theta}{1-\tau}}g_{tt}(x,|Du|)\right]dx\right\}^{1-\tau}$$
$$\cdot\ \left\{\int_{B_R}\left[1+g_{tt}(x,|Du|)|Du|^{1^*2}\right]^{\frac{\tau+\epsilon}{\tau}}dx\right\}^{\tau}.$$

To apply an iteration procedure, we need $\frac{2}{1-\tau}<2^*$, then it is sufficient that $\tau<\frac{2}{n}$. Choose $\tau=\frac{1}{n}$, thus $\frac{2}{1-\tau}=1^*2<2^*$ and let ϵ such that $\frac{\tau+\epsilon}{\tau}=1+\epsilon n=\frac{2^*}{1^*2}$. Since $u\in W^{2,2}_{loc}(\Omega,\mathbf{R}^N)$, then $Du\in L^{2^*}_{loc}(\Omega,\mathbf{R}^{nN})$ and recalling that g_{tt} satisfies the supplementary assumption (3.1), we deduce that following integral is finite:

$$\mathcal{A}=\int_{B_{R_0}}\left[1+g_{tt}(x,|Du|)|Du|^{1^*2}\right]^{\frac{2^*}{1^*2}}dx \tag{3.25}$$

and (3.24) becomes

$$\begin{aligned} &\left\{\int_{B_\rho}\left[1+|Du|^{2^*\theta}g_{tt}(x,|Du|)\right]dx\right\}^{\frac{1}{2^*}} \\ \leq\ & c\frac{\theta^2}{R-\rho}\left\{\int_{B_R}\left[1+|Du|^{1^*2\theta}g_{tt}(x,|Du|)\right]dx\right\}^{\frac{1}{1^*2}}\mathcal{A}^{\frac{1}{2n}}. \end{aligned} \tag{3.26}$$

We define a sequence of exponents θ_j in the following way:

$$\begin{aligned} \theta_0 &= 1 \\ \theta_j &= \frac{2^*}{1^*2}\theta_{j-1}, \quad \forall j=1,2,\dots \end{aligned} \tag{3.27}$$

or equivalently $\theta_0 = 1$ and $\theta_j = \left(\frac{2^*}{1^*2}\right)^j$, $\forall j = 1,2,\dots$.
Define also $\rho_j = \rho_0 + \frac{R_0-\rho_0}{2^j}$ for $j = 0,1,2,\dots$ and

$$A_j = \left\{\int_{B_{\rho_j}}\left[1+|Du|^{1^*2\theta_j}g_{tt}(x,|Du|)\right]dx\right\}^{\frac{1}{1^*2\theta_j}}$$

and insert in (3.26) $R = \rho_j$, $\rho = \rho_{j+1}$ and $\theta = \theta_j$. Since $R - \rho = \frac{R_0-\rho_0}{2^{j+1}}$, we obtain

$$A_{j+1} \leq \left[\frac{c\theta_j^2 2^j\mathcal{A}^{\frac{1}{2n}}}{R_0-\rho_0}\right]^{\frac{1}{\theta_j}} A_j.$$

By iteration we get

$$A_{j+1} \leq \left(\frac{c\mathcal{A}^{\frac{1}{2n}}}{R_0-\rho_0}\right)^{\sum_{k=0}^j\frac{1}{\theta_k}}\left(\prod_{k=0}^j\theta_k^{\frac{1}{\theta_k}}\right)^2 2^{\sum_{k=0}^j\frac{k}{\theta_k}}A_0,$$

(observe that $\mathcal{A}$ and A_0 are finite, thus every A_j is finite). The product is finite and the series in the exponents converge and after some calculation, using the definition (3.27) since

$$\sum_{k=0}^\infty\frac{1}{\theta_k} = \sum_{k=0}^\infty\left(\frac{1^*2}{2^*}\right)^k = n-1,$$

and

$$1+|Du|^{1^*2}g_{tt}(x,|Du|) \leq \left[1+|Du|^{1^*2}g_{tt}(x,|Du|)\right]^{\frac{2^*}{1^*2}},$$

by the definition of $\mathcal{A}$ we finally have

$$A_{j+1} \leq \frac{c}{(R_0-\rho_0)^{n-1}}\left\{\int_{B_{R_0}}\left[1+|Du|^{1^*2}g_{tt}(x,|Du|)\right]^{\frac{2^*}{1^*2}}dx\right\}^{\frac{1}{1^*}}. \tag{3.28}$$

We can easily prove that for every $\beta > 0$ and $t \geq 0$ there exists a constant $c = c(\Omega_0)$ such that

$$t^{\beta} \leq c\left[1 + t^{\beta} g_{tt}(x,t)\right], \quad \forall t \geq 0, \quad a.e.\ x \in \Omega_0. \tag{3.29}$$

In fact, (2.3), (2.4) and (H_2) imply that $g_{tt}(x,1) \geq g(x,t_0) > \lambda > 0$ for $a.e.\ x \in \Omega_0$. We can conclude

$$\begin{aligned} &\sup\{|Du(x)| : x \in B_{\rho_0}\} = \lim_{j\to+\infty}\left\{\int_{B_{\rho_0}} |Du(x)|^{2^*\theta_j}\,dx\right\}^{\frac{1}{2^*\theta_j}} \\ \leq\ & \lim_{j\to+\infty}\left\{c\int_{B_{\rho_{j+1}}}\left[1 + |Du|^{2^*\theta_j} g_{tt}(x,|Du|)\right]dx\right\}^{\frac{1}{2^*\theta_j}} \\ \leq\ & \frac{c}{(R_0-\rho_0)^{n-1}}\left\{\int_{B_{R_0}}\left[1 + |Du|^{1^*2} g_{tt}(x,|Du|)\right]^{\frac{2^*}{1^*2}}dx\right\}^{\frac{1}{1^*}}. \end{aligned}$$

The last inequality implies that $u \in W^{1,\infty}_{loc}\left(\Omega, \mathbf{R}^N\right)$ and Lemma 3.3 is proved. □

Lemma 3.4. *Let (H_1)-(H_4) and (3.1)-(3.2) hold. If u is a local minimizer of (2.1), then there exist $\sigma = \sigma(n) > 0$ and $\alpha = \alpha(n) > 0$ such that*

$$\int_{B_\rho}\left[1 + |Du|^{1^*2} g_{tt}(x,|Du|)\right]^{\frac{2^*}{1^*2}}dx \leq \frac{c}{(R-\rho)^{\alpha}}\left\{\int_{B_R}\left[1 + g(x,|Du|)\right]dx\right\}^{1^*+\sigma},$$

where c depends on n, N and on the constants in (H_1) - (H_4).

Proof. Consider the inequality (3.21) in the proof of the previous Lemma with $\gamma = 0$ (i.e. $\Phi = 1$):

$$\begin{aligned} &\left\{\int_{\Omega}\eta^{2^*}\left[G\left(x,|Du|\right)\right]^{2^*}dx\right\}^{\frac{2}{2^*}} \\ \leq\ & c\int_{\Omega}\left[\eta^2 + |D\eta|^2\right]\left[1 + |Du|^2 g_{tt}(x,|Du|)\right]\left[1 + g_t^{2(\alpha-1)}(x,|Du|)\right]dx. \end{aligned}$$

Let $1 < \delta \leq \frac{2^*}{1^*2}$ and apply (iii) of Lemma 3.1 with $\beta = \frac{2^*}{\delta} \geq 1^*2 > 2$:

$$\begin{aligned} [G(x,t)]^{2^*} &= \left[1 + \int_0^t\sqrt{\frac{g_t(x,s)}{s}}\,ds\right]^{\frac{2^*}{\delta}\delta} \\ &\geq c\left[1 + t^{\frac{2^*}{\delta}} g_{tt}(x,t)\right]^{\delta} \geq c\left[1 + t^{1^*2} g_{tt}(x,t)\right]^{\delta}. \end{aligned}$$

Therefore, choosing the test function η and ϵ as in the proof of Lemma 3.3, we obtain

$$\begin{aligned} &\left\{\int_{B_\rho}\left[1 + |Du|^{1^*2} g_{tt}(x,|Du|)\right]^{\delta}dx\right\}^{\frac{2}{2^*}} \\ \leq\ & \frac{c}{(R-\rho)^2}\int_{B_R}\left[1 + |Du|^2 g_{tt}(x,|Du|)\right]\left[1 + g_{tt}^{\epsilon}(x,|Du|)|Du|^{\epsilon}\right]dx \\ \leq\ & \frac{c}{(R-\rho)^2}\int_{B_R}\left[1 + |Du|^{1^*2} g_{tt}(x,|Du|)\right]^{1+\epsilon}dx. \end{aligned} \tag{3.30}$$

Set

$$V(x) = 1 + |Du(x)|^{1^*2} g_{tt}(x, |Du(x)|);$$

(3.30) can be written in the form:

$$\left\{ \int_{B_\rho} V^\delta dx \right\}^{\frac{2}{2^*}} \leq \frac{c}{(R-\rho)^2} \int_{B_R} V^{1+\epsilon} dx.$$

We fix $\delta = \frac{2^*}{1^*2} > 1$ and let $\gamma > \frac{2^*}{2} > \delta$. By using Hölder inequality with exponents γ and $\frac{\gamma}{\gamma-1}$, from (3.31) we have

$$\begin{aligned} \left\{ \int_{B_\rho} V^\delta dx \right\}^{\frac{2}{2^*}} &\leq \frac{c}{(R-\rho)^2} \int_{B_R} V^{1+\epsilon} dx \qquad (3.31) \\ &= \frac{c}{(R-\rho)^2} \int_{B_R} V^{\frac{\delta}{\gamma}} V^{1-\frac{\delta}{\gamma}+\epsilon} dx \leq \frac{c}{(R-\rho)^2} \left\{ \int_{B_R} V^\delta dx \right\}^{\frac{1}{\gamma}} \left\{ \int_{B_R} V^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}} dx \right\}^{\frac{\gamma-1}{\gamma}}, \end{aligned}$$

or equivalently

$$\int_{B_\rho} V^\delta dx \leq \frac{c}{(R-\rho)^{2^*}} \left\{ \int_{B_R} V^\delta dx \right\}^{\frac{2^*}{2\gamma}} \left\{ \left[\int_{B_R} V^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}} dx \right]^{\gamma-1} \right\}^{\frac{2^*}{2\gamma}}. \qquad (3.32)$$

Fixed R_0 and ρ_0 as before, we consider $\rho_j = R_0 - \frac{R_0-\rho_0}{2^j}$. We insert $R = \rho_j$ and $\rho = \rho_{j-1}$ in (3.32): since $R - \rho = \frac{R_0-\rho_0}{2^j}$, then we obtain

$$\int_{B_{\rho_{j-1}}} V^\delta dx \leq \left\{ \int_{B_{\rho_j}} V^\delta dx \right\}^{\frac{2^*}{2\gamma}} \frac{c 2^{2^* j}}{(R_0-\rho_0)^{2^*}} \left\{ \left[\int_{B_{R_0}} V^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}} dx \right]^{\gamma-1} \right\}^{\frac{2^*}{2\gamma}}. \qquad (3.33)$$

Denote by $A_j = \int_{B_{\rho_j}} V^\delta dx$: by (3.1) and Lemma 3.3, A_j are uniformly bounded with respect to j. Thus (3.33) becomes

$$A_{j-1} \leq A_j^{\frac{2^*}{2\gamma}} \frac{c 2^{2^* j}}{(R_0-\rho_0)^{2^*}} \left\{ \left[\int_{B_{R_0}} V^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}} dx \right]^{\gamma-1} \right\}^{\frac{2^*}{2\gamma}}.$$

Iterating:

$$\begin{aligned} A_0 &\leq A_j^{\left(\frac{2^*}{2\gamma}\right)^j} \prod_{j=1}^{\infty} \left[\frac{c 2^{2^* j}}{(R_0-\rho_0)^{2^*}} \right]^{\left(\frac{2^*}{2\gamma}\right)^j} \left\{ \left[\int_{B_{R_0}} V^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}} dx \right]^{\gamma-1} \right\}^{\left(\frac{2^*}{2\gamma}\right)^j} \qquad (3.34) \\ &\leq A_j^{\left(\frac{2^*}{2\gamma}\right)^j} \frac{c}{(R_0-\rho_0)^{2^* \frac{2^*}{2\gamma-2^*}}} \left\{ \int_{B_{R_0}} V^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}} dx \right\}^{(\gamma-1)\frac{2^*}{2\gamma-2^*}}, \end{aligned}$$

since $\sum_{j=1}^{\infty}\left(\frac{2^*}{2\gamma}\right)^j = \frac{2^*}{2\gamma-2^*}$. Use (ii) of Lemma 3.1 with exponent $1^* > 1$, i.e.

$$1 + g_{tt}(x,t)t^{1^*2} \le c[1 + g(x,t)]^{1^*}, \tag{3.35}$$

hence, in this case

$$V^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}} \le \left\{c\left[1 + g(x,|Du|)\right]^{1^*}\right\}^{\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1}}.$$

We can choose γ in such way

$$1^*\frac{\gamma-\delta+\epsilon\gamma}{\gamma-1} = 1. \tag{3.36}$$

Recalling that $\delta = \frac{2^*}{1^*2}$, an easy computation gives $\gamma = \frac{2\delta}{1+\epsilon n}$ and for ϵ sufficiently small (i.e. α sufficiently close to 1), $\gamma > \frac{2^*}{2}$ as required. With this choice of γ, from (3.34) we infer

$$\int_{B_{\rho_0}} V^\delta dx \le \left\{\int_{B_{\rho_j}} V^\delta dx\right\}^{\left(\frac{2^*}{2\gamma}\right)^j} \cdot \frac{c}{(R_0-\rho_0)^{\frac{(2^*)^2}{2\gamma-2^*}}}\left\{\int_{B_{R_0}}[1 + g(x,|Du|)]dx\right\}^{\frac{2^*(\gamma-1)}{2\gamma-2^*}}$$

and letting $j \to +\infty$, we conclude

$$\int_{B_{\rho_0}}\left[1 + |Du|^{1^*2}g_{tt}(x,|Du|)\right]^{\frac{2^*}{1^*2}} dx \le \frac{c}{(R_0-\rho_0)^{\frac{(2^*)^2}{2\gamma-2^*}}}\left\{\int_{B_{R_0}}[1 + g(x,|Du|)]dx\right\}^{\frac{2^*(\gamma-1)}{2\gamma-2^*}} \tag{3.37}$$

and the Lemma is proved with $\alpha(n) = \frac{(2^*)^2}{2\gamma-2^*}$ and $\frac{2^*(\gamma-1)}{2\gamma-2^*} = 1^* + \sigma$ with $\sigma > 0$. □

Remark 3.5. *We underline the fact that the constant c in Propositon 3.2 does not depend on m, M and N of (3.1) and (3.2).*

Remark 3.6. *It is not difficult to check that the result of Propositon 3.2 holds even if we assume g of class $W^{2,\infty}_{loc}$ with respect to t for a.e. $x \in \Omega$ instead of class C^2.*

4. Approximation and proof of the theorem 2.1

In this Section we will prove the estimate (3.5) of Proposition 3.2 for minimizers of our original functional F and then we have to remove the supplementary assumptions (3.1) and (3.2). The main ingredients are an approximation procedure and then a passage to the limit similar to the ones used by Marcellini in Sections 4 and 5 of [15], modified in order to handle the dependence on x of the integrand.

Let $\Omega_0 \subset\subset \Omega$ and g satisfy (H_1) - (H_5) of Section 2. We remember that, by (H_1) and (H_2):

$$g(x,0) = g_t(x,0) = 0 \quad \text{and} \quad g_t(x,1) \ge g(x,1) \ge \lambda > 0, \quad a.e.\ x \in \Omega_0.$$

For $t \in (0,+\infty)$ and $x \in \Omega$, set

$$a(x,t) = \frac{g_t(x,t)}{t} \tag{4.1}$$

which is positive, increasing and $a(x,1) \geq \lambda > 0$ *a.e.* $x \in \Omega_0$.
From assumption (H_1), it follows that $a(x,t) > 0$ if $t > 0$. For every $k \in \mathbb{N}$, let $t_k = \frac{1}{k}$ and define the sequence of functions

$$a^k(x,t) = \begin{cases} a(x,t_k) & \text{for } t \in [0,t_k) \\ a(x,t) & \text{for } t \in [t_k,k] \\ a(x,k) & \text{for } t \in (k,+\infty) \end{cases} .$$

For every $k \in \mathbb{N}$, $a^k(x,t)$ is continuous and increasing with respect to t and satisfies

$$a(x,t) \leq a^k(x,1) = a(x,1) \leq \Lambda, \quad a.e.\ x \in \Omega_0, \quad \forall t \in [0,1]. \tag{4.2}$$

Consider the function $g^k(x,t)$ given by

$$g^k(x,t) \quad = \quad \int_0^t a^k(x,s)sds, \quad a.e.\ x \in \Omega, \quad \forall t \in [0,+\infty). \tag{4.3}$$

By definition, it follows that fixed k_0, for every $t \in [0,k_0]$ and $k \geq k_0$ we have

$$0 \leq g^{k_0}(x,t) - g^k(x,t) \leq \frac{1}{2k_0^2} a(x,1), \quad a.e.\ x \in \Omega_0. \tag{4.4}$$

Moreover $g^k(x,t)$ converges pointwise to $g(x,t)$ for *a.e.* $x \in \Omega$ and $t \geq 0$.
Our next goal is to prove that g^k satisfies assumptions (H_1) - (H_4) with constants independent of k.

Lemma 4.1. *Let $g(x,t)$ satisfy (H_1) - (H_5) and let $g^k(x,t)$ defined as in (4.3). Then, for every $\Omega_0 \subset\subset \Omega$, g^k satisfies (H_1) and (H_2) for k sufficiently large, with constants independent of k. Moreover:*

(i) for every $k \in \mathbb{N}$, there exist m_k and $M_k > 0$ such that

$$m_k \leq \frac{g_t^k(x,t)}{t} \leq g_{tt}^k(x,t) \leq M_k \tag{4.5}$$

$\forall t > 0$ and a.e. $x \in \Omega_0$, where $g_{tt}^k(x,k)$ denotes the right second derivatives of g^k.

(ii) There exists a constant $L = L(\Omega_0)$ such that

$$g^k(x,t) \leq L[1 + g(x,t)] \tag{4.6}$$

$\forall k \in \mathbb{N}$, $t \geq 0$ and a.e. $x \in \Omega_0$.

(iii) For every $\alpha > 1$ there exists $C_1 = C_1(\alpha,\Omega_0)$ such that

$$g_{tt}^k(x,t)t^{2\alpha} \leq C_1[g^k(x,t)]^\alpha \tag{4.7}$$

$\forall k \in \mathbb{N}$, $t \geq 1$ and a.e. $x \in \Omega_0$.

(iv) For every $k \in \mathbb{N}$, there exists a constant N_k such that

$$|g_{tx_s}^k(x,t)| \leq N_k(1+t^2)^{\frac{1}{2}} \tag{4.8}$$

$\forall t \geq 0$ and a.e. $x \in \Omega_0$.
For every $\alpha > 1$, there exists $C_2 = C_2(\alpha, \Omega_0)$ such that

$$\left|g^k_{tx_s}(x,t)\right| \leq C_2 g^k_t(x,t)\left[1 + \left(g^k_t\right)^{\alpha-1}(x,t)\right] \tag{4.9}$$

$\forall k \in \mathbb{N}$, $t \geq 0$ and a.e. $x \in \Omega_0$.

Proof. Since $g^k_t(x,t) = a^k(x,t)t$ is increasing with respect to t, then $g^k(x,t)$ is convex with respect to t. Moreover $g^k(x,t)$ and $g^k_t(x,t)$ are Carathéodory functions in $\Omega \times [0,+\infty)$ and $g^k(x,t)$ is of class C^1 with respect to t. Since

$$g^k_{tt}(x,t) = \begin{cases} a(x,t_k) & \text{for } t \in [0,t_k) \\ g_{tt}(x,t) & \text{for } t \in [t_k,k] \\ a(x,k) & \text{for } t \in (k,+\infty) \end{cases} \tag{4.10}$$

we have, taking into account (H_5), that $g^k(x,\cdot) \in W^{2,\infty}_{loc}$ for *a.e.* $x \in \Omega$. By construction $a^k(x,t) = \frac{g^k_t(x,t)}{t}$ is increasing, thus (H_1) is satisfied . It is very easy to show that $g^k_{tt}(x,t) \leq \Lambda'$ for *a.e.* $x \in \Omega_0$ and $\forall t \in [0,1]$ with Λ' independent of k; moreover, for k sufficiently large $g^k(x,t_0) = g(x,t_0)$, thus (H_2) holds. Let us prove (i). Fixed $x \in \Omega_0$, since $a^k(x,t) = \frac{g^k_t(x,t)}{t}$ is increasing and from the definition of t_k we have

$$0 < m_k = \min_{x\in\Omega_0} a(x,t_k) \leq a^k(x,t) = \frac{g^k_t(x,t)}{t} \leq g^k_{tt}(x,t)$$

$\forall t > 0$. By taking in account (H_5), set

$$M_k = \max\left\{\|a(x,1)\|_{L^\infty(\Omega_0)}, \|g_{tt}(x,t)\|_{L^\infty(\Omega_0\times[1,k])}, \|a(x,k)\|_{L^\infty(\Omega_0)}\right\},$$

thus (4.5) holds.
In order to prove (ii) and (iii), let us show that $\forall k \in \mathbb{N}$ and *a.e.* $x \in \Omega_0$ the following inequalities hold:

$$g(x,1) \leq g^k(x,1); \tag{4.11}$$
$$g^k(x,t) \leq a(x,1) + g(x,t) \quad \forall t \in [0,+\infty); \tag{4.12}$$
$$g^k(x,t) \geq g(x,t) \quad \forall t \in [1,k); \tag{4.13}$$

If $t \in [0,1]$, it is clear that $a(x,t) \leq a^k(x,t) \leq a(x,1)$. By using (4.1) and (4.3), we obtain

$$\begin{aligned} g^k(x,1) &= g^k(x,1) - g^k(x,0) = \int_0^1 a^k(x,t)t\,dt \\ &\geq \int_0^1 a(x,t)t\,dt = g(x,1) - g(x,0) = g(x,1) \end{aligned}$$

and (4.11) is proved. If $t \in [0,1]$ we have

$$g^k(x,t) = \int_0^t a^k(x,s)s\,ds \leq a(x,1).$$

If $t \geq 1$ we have

$$g^k_t(x,t) = a^k(x,t)t \leq a(x,t)t = g_t(x,t)$$

and thus $\forall t \in [0, +\infty)$

$$\begin{aligned} g^k(x,t) &= g^k(x,1) + \int_1^t a^k(x,s)sds \\ &\leq a(x,1) + g(x,1) + \int_1^t a(x,s)sds = a(x,1) + g(x,t) \end{aligned}$$

and (4.12) is proved. By collecting (4.11) and (4.12), we have

$$g^k(x,t) \quad \leq \quad 2[1 + a(x,1)][1 + g(x,t)]$$

which implies (ii) since $1 + a(x,1) = 1 + g_t(x,1) \leq 1 + \Lambda(\Omega_0) = L$.
In order to prove (4.13) we observe that if $t \in [1,k)$, by (4.11), we have

$$\begin{aligned} g^k(x,t) &= \int_0^1 a^k(x,s)sds + \int_1^t a(x,s)sds \\ &= g^k(x,1) + g(x,t) - g(x,1) \geq g(x,t). \end{aligned}$$

Let us prove (iii): when $t \in [1,k]$ we use (H_3) and (4.13)

$$g^k_{tt}(x,t)t^{2\alpha} = g_{tt}(x,t)t^{2\alpha} \leq c[g(x,t)]^\alpha \leq c[g^k(x,t)]^\alpha,$$

while for $t \in (k, +\infty)$, by (i) of Lemma 3.1 we have

$$g^k_{tt}(x,t)t^{2\alpha} = g_t(x,k)k^{2\alpha-1}\frac{t^{2\alpha}}{k^{2\alpha}} \leq c[g(x,k)]^\alpha \frac{t^{2\alpha}}{k^{2\alpha}}.$$

By proceeding as in the proof of Lemma 4.3 of Marcellini [15], it is possible to show that $g(x,k)\frac{t^2}{k^2} \leq 2g^k(x,t)$. Thus

$$g^k_{tt}(x,t)t^{2\alpha} \quad \leq \quad c2^\alpha[g^k(x,t)]^\alpha,$$

and (4.7) is proved.
Now we prove (iv). For each fixed $t > 0$, the functions $g^k_t(x,t)$ have weak derivatives with respect to x_s, $g^k_{tx_s}(x,t)$, which are Carathéodory functions in $\Omega \times [0,+\infty)$ and locally summable in Ω. If $t \in [0,t_k)$, by (H_4)

$$\begin{aligned} |g^k_{tx_s}(x,t)| &= |a_{x_s}(x,t_k)|t = |g_{tx_s}(x,t_k)\frac{t}{t_k}| \leq c_2\frac{g_t(x,t_k)}{t_k}t\left[1 + g_t^{\alpha-1}(x,t_k)\right] \\ &\leq c_2 a^k(x,t)t\left[1 + g_t^{\alpha-1}(x,1)\right] \leq c g^k_t(x,t)\left[1 + \left(g^k_t\right)^{\alpha-1}(x,t)\right], \end{aligned}$$

where c depends on Λ. If $t \in [t_k, k]$

$$|g^k_{tx_s}(x,t)| \quad = \quad |g_{tx_s}(x,t)| \leq c_2 g_t(x,t)\left[1 + g_t^{\alpha-1}(x,t)\right] \leq c_2 g^k_t(x,t)\left[1 + \left(g^k_t\right)^{\alpha-1}(x,t)\right].$$

If $t \in [k, +\infty)$

$$|g^k_{tx_s}(x,t)| \quad = \quad |a_{x_s}(x,k)|t = |g_{tx_s}(x,k)|\frac{t}{k} \leq c_2 g_t(x,k)\frac{t}{k}\left[1 + g_t^{\alpha-1}(x,k)\right] \tag{4.14}$$

thus

$$\begin{aligned} |g^k_{tx_s}(x,t)| &\leq c_2 g_t(x,k)\frac{t}{k}\left[1+\left(\frac{g_t(x,k)t}{k}\right)^{\alpha-1}\left(\frac{k}{t}\right)^{\alpha-1}\right] \\ &\leq c_2 g^k_t(x,t)\left[1+\left(g^k_t\right)^{\alpha-1}(x,t)\right] \end{aligned}$$

and (4.9) is proved.
Finally, fixed $\alpha_0>1$, for $t\in[0,t_k)$,

$$|g^k_{tx_s}(x,t)|\leq ca(x,1)\left[1+g_t^{\alpha_0-1}(x,1)\right]\leq\overline{C},$$

for $t\in[t_k,k]$,

$$|g^k_{tx_s}(x,t)|\leq c\max_{x\in\Omega_0}\left\{g_t(x,k)\left[1+g_t^{\alpha_0-1}(x,k)\right]\right\}=N_k,$$

for $t\in[k,+\infty)$, (4.14) gives

$$|g^k_{tx_s}(x,t)|\leq c\frac{g_t(x,k)}{k}t\left[1+g_t^{\alpha_0-1}(x,k)\right]\leq\frac{N_k}{k}(1+t^2)^{\frac{1}{2}}$$

and (4.8) holds. □

Proof of theorem 2.1.
Let u be a local minimizer of (2.1). For every $k\in\mathbb{N}$ we consider the functional

$$\int_\Omega g^k\left(x,|Du|\right)dx, \tag{4.15}$$

with g^k defined as in (4.3). Let $B_R\subset\subset\Omega_0\subset\subset\Omega$: the Dirichlet problem

$$\inf\left\{\int_{B_R}g^k\left(x,|Dv|\right)dx,\quad v\in u+W_0^{1,2}\left(B_R,\mathbf{R}^N\right)\right\}$$

has one solution u_k, i.e.

$$\int_{B_R}g^k\left(x,|Du_k|\right)dx\leq\int_{B_R}g^k\left(x,|Dv|\right)dx$$

for every $v\in u+W_0^{1,2}\left(B_R,\mathbf{R}^N\right)$. In particular

$$\int_{B_R}g^k\left(x,|Du_k|\right)dx\leq\int_{B_R}g^k\left(x,|Du|\right)dx. \tag{4.16}$$

By assumption (H_3) (see (iii) of Lemma 3.1) we have that

$$t^2\leq c[1+g^k(x,t)],\quad\forall t\geq0,\quad a.e.\,x\in\Omega_0$$

and then (4.16) and (4.6) give

$$\int_{B_R}|Du_k|^2dx\leq c\int_{B_R}\left[1+g^k(x,|Du|)\right]dx\leq c\int_{B_R}\left[1+g(x,|Du|)\right]dx,$$

which implies that, up to a subsequence, (u_k) converges weakly in $u + W_0^{1,2}\left(B_R, \mathbf{R}^N\right)$ to a function w.
By Lemma 4.1, the functional in (4.15) satisfies the assumptions of Proposition 3.2 and then there exist $\sigma > 0$ and c independent on k such that $\forall \rho < R$

$$\sup_{B_\rho} |Du_k| \leq c \left\{ \int_{B_R} [1 + g^k(x, |Du_k|)] dx \right\}^{1+\sigma}.$$

Moreover, by (4.16) and (4.6), we have that for every $k \in \mathbb{N}$

$$\sup_{B_\rho} |Du_k| \leq c \left\{ \int_{B_R} [1 + g^k(x, |Du|)] dx \right\}^{1+\sigma} \leq c \left\{ \int_{B_R} [1 + g(x, |Du|)] dx \right\}^{1+\sigma}. \tag{4.17}$$

The last inequality gives that (u_k), up to a subsequence, converges to the function w in the weak* topology of $W_{loc}^{1,\infty}(B_R, \mathbf{R}^N)$. Let k_0 be such that $\|Du_k\|_{L^\infty} \leq k_0$. By (4.4) and (4.16), we infer that for $k \geq k_0$

$$\begin{aligned} \int_{B_\rho} g^{k_0}(x, |Du_k|)\, dx &\leq \int_{B_\rho} g^k(x, |Du_k|)\, dx + \frac{1}{2k_0^2} \int_{B_\rho} a(x, 1)\, dx \\ &\leq \int_{B_R} g^k(x, |Du|)\, dx + \frac{c}{k_0^2}. \end{aligned}$$

By lower semicontinuity and using the dominated convergence theorem, as $k \to +\infty$ we have

$$\int_{B_\rho} g^{k_0}(x, |Dw|)\, dx \leq \int_{B_R} g(x, |Du|)\, dx + \frac{c}{k_0^2}$$

and then as $k_0 \to +\infty$ and $\rho \to R$ we get

$$\int_{B_R} g(x, |Dw|)\, dx \leq \int_{B_R} g(x, |Du|)\, dx.$$

Therefore w is a local minimizer of F and the strictly convexity of the functional gives $u = w$. Finally (4.17) gives

$$\|Du\|_{L^\infty(B_\rho, \mathbf{R}^{nN})} \leq c \left\{ \int_{B_R} [1 + g(x, |Du|)] dx \right\}^{1+\sigma}$$

and thus the theorem is proved.

References

[1] R. Aris, *The mathematical theory of diffusion and reaction of permeable catalysts.* Clarendon Press, Oxford (1975).

[2] E. Acerbi and N. Fusco, Regularity for minimizers of non-quadratic functionals: the case $1 < p < 2$. *Journal of Math. Anal. and Appl.* **140** (1989) 115-135.

[3] E. Acerbi and G. Mingione, Regularity results for a class of functionals with nonstandard growth. *Arch. Rat. Mech. Anal.* **156** (2001) 121-140.

[4] E. Acerbi and G. Mingione, Regularity results for quasiconvex functionals with nonstandard growth. *Ann. Scuola Norm. Sup. Pisa* **30** (2001).

[5] V. Chiadò Piat and A. Coscia, Hölder continuity of minimizers of functionals with variable growth exponent. *Manuscripta Math.* **93** (1997) 283-299.

[6] A. Coscia and G. Mingione, Hölder continuity of the gradient of p(x)-harmonic mappings. *C.R. Acad. Sci. Paris* **328** (1999) 363-368.

[7] A. Dall'Aglio, E. Mascolo and G. Papi, Local boundedness for minima of functionals with non standard growth conditions. *Rendiconti di Matematica* **18** (1998) 305-326.

[8] A. Dall'Aglio and E. Mascolo, L^{∞}-estimates for a class of nonlinear elliptic systems with non standard growth. *Atti del seminario matematico e fisico dell'Università di Modena* (to appear).

[9] F. Leonetti, E. Mascolo and F. Siepe, Everywhere regularity for a class of vectorial functionals under subquadratic general growth, Preprint Dipartimento di Matematica "U. Dini", University of Florence.

[10] M. Giaquinta, *Multiple integrals in the calculus of variations and non linear elliptic systems.* Annals of Math. Studies **105** Princeton Univ. Press, Princeton NJ (1983).

[11] M. Giaquinta and G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals. *Manuscripta Math* **57** (1986) 55-99.

[12] E. Giusti, *Metodi diretti nel calcolo delle variazioni.* UMI, Bologna (1994).

[13] P. Marcellini, Regularity and existence of solutions of elliptic equations with (p,q)-growth conditions. *J. Diff. Equations* **90** (1991) 1-30.

[14] P. Marcellini, Regularity for elliptic equations with general growth conditions. *J. Diff. Equations* **105** (1993) 296-333.

[15] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions. *Ann. Sc. Norm. Sup. Pisa* **23** (1996) 1-25.

[16] M. Marcus and V.J. Mizel, Continuity of certain Nemitsky operators on Sobolev spaces and chain rule. *Journal Analyse Math.* **28** (1975) 303-334.

[17] E. Mascolo and G. Papi, Local boundedness of integrals of Calculus of Variations. *Annali di Mat. Pura e Appl.* **167** (1994) 323-339.

[18] A.P. Migliorini, Everywhere regularity for a class of elliptic systems with p, q growth conditions. *Rend. Istit. Mat. Univ. Trieste* **XXXI** (1999) 203-234.

[19] A.P. Migliorini, *Everywhere regularity for a class of elliptic systems with general growth conditions.* PhD Thesis, University of Florence, Italy (2000).

[20] J. Mosely, A two dimensional Dirichlet problem with an exponential nonlinearity. *SIAM Journal Math. Analysis* **14** 5 (1983), 719-735.

[21] M. Růžička, Flow of shear dependent electrorheological fluids. *C.R. Acad. Sci. Paris* **329** (1999) 393-398.

[22] K.R. Rajagopal and M. Růžička, On the modeling of electrorheological materials. *Mech. Res. Commun.* **23** (1996) 401-407.

[23] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems. *Acta Math.* **138** (1977) 219-240.

[24] V.V. ZhiKov, On Lavrentiev phenomenon. *Russian Jour.* Math. Physics **3** (1995) 249-269.