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Existence Theorems in the Calculus of Variations

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Consider the problem

$$\operatorname{Min}\left\{\int_G f(x, Dv(x))\,dx,\ v\in H_0^{1,2}(G)+u_o\right\}, \tag{0.1}$$

where $f(x,p)$ is a continuous function on $\bar{G}\times R^N$, G bounded open set in R^N, satisfying

$$c_1\,|p|^2-c_2\leqslant f(x,p)\leqslant c_3\,|p|^2+c_4, \qquad p\in R^N, x\in G,$$

and $u_0\in H^{1,2}$, $c_i\geqslant 0$.

In general, if $f(x,p)$ is not convex in p, problems of type (0.1) may be studied by means of the relaxed problem

$$\operatorname{Min}\left\{\int_G g(x, Dv(x))\,dx,\ v\in H_0^{1,2}(G)+u_0\right\}, \tag{0.2}$$

where $g(x,p)=f^{**}(x,p)$ is the lower convex envelope of f with respect to p.

In our previous papers [14, 15, 16], (see also [7] for Neumann problems), we prove the existence of solutions for particular cases of non convex problems, by finding an a.e. differentiable solution w of the related relaxed problem such that $Dw(x)$ belongs to the set $\{p: f(x,p)=f^{**}(x,p)\}$. In [16] we state existence theorems in $C^{0,1}$ for problem (0.2) with hypotheses on the boundary data strictly related to the techniques used.

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In the present paper our aim is to find sufficient conditions on g to obtain $C^{0,1}$-solutions for problem (0.2) with general boundary data. More precisely, in Section 1 we consider problem (0.2) with $g(x, p) \in C^0(\bar{G} \times R^N)$ convex in p, which is, for $|p|$ large enough, a C^2-function in (x, p) and strictly convex in p.

We prove that every solution of (0.2) is in $C^{0,1}_{\text{loc}}(G)$ if $u_0 \in H^{1,2}(G) \cap L^\infty$ and in $C^{0,1}(\bar{G})$ if $u_0 \in C^{1,1}(\bar{G})$. The main tools are classical regularity results and the barrier-technique [3, 4, 5, 9, 17].

In Section 2 we consider problem (0.1) where, for p large enough, f is strictly convex in p and regular in (x, p). By applying the results of Section 1, every solution of the relaxed problem is differentiable a.e. in G. Then, if $f^{**}(x, p)$ is an affine function in p on the bounded connected set $K(x) = \{p \in R^N : f(x, p) > f^{**}(x, p)\}$, we prove that every solution w of the relaxed problem verifies $Dw(x) \in R^N - K(x)$ a.e. in G, so that w is also a solution of (0.1). If $K(x)$ is not a connected set, we can obtain existence with the same arguments, by supposing that f^{**} is an affine function on each connected component of $K(x)$.

For $N = 1$, f^{**} is always an affine function in p for $p \in K(x)$ and first Marcellini in [10] used this property in order to find solutions of the non-convex problem. For $N > 1$ we have to assume the affinity of f^{**} to get existence. On the other hand, in [11] and [12], Marcellini proves some nonexistence results when the affinity of f^{**} in p is lacking. For $N > 1$ non-convex problems have been investigated by Aubert and Taharaoui in [1].

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Let G be a bounded open subset of R^N, $N \geqslant 2$, with a smooth boundary ∂G. Let $f(x, p) \in C^0(\bar{G} \times R^N)$ be convex in p for every $x \in \bar{G}$. Moreover we assume that $f \geqslant 0$ satisfies:

$$C_1 |p|^2 - C_2 \leqslant f(x, p) \leqslant C_3 + C_4 |p|^2 \qquad \text{for a.e. } x \in G, p \in R^N \tag{1.1}$$

with C_i positive constants. Defining $S_t = \{p \in R^N : |p| > t\}$, $t \in R_+$, we suppose that:

(i) $f \in C^2(\bar{G} \times S_t)$; moreover[1]

$$|f_p(x, p)|, \qquad |f_{p,x}(x, p)| \leqslant \mu(1 + |p|, \quad |f_{pp}(x, p)| \leqslant \mu, \quad x \in \bar{G}, p \in S_t; \tag{1.2}$$

[1] For a function $h = h(x, p)$ we denote by $h_x(x, p)$ and $h_p(x, p)$, respectively, the vector gradient of $h(x, p)$ with respect to x, i.e., $h_x(x, p) = (h_{x_1}(x, p), ..., h_{x_n}(x, p))$, $h_{x_i}(x, p) = \partial h / \partial x_i(x, p)$, and the vector gradient of $h(x, p)$ with respect to p, i.e., $h_p(x, p) = (h_{p_1}(x, p) ..., h_{p_n}(x, p))$, $h_{p_i}(x, p) = (\partial h / \partial p_i)(x, p)$. For a function $f = f(x)$ we denote by $Df(x)$ the vector gradient $Df = (D_1 f \cdots D_n f)$, $D_i f = \partial f / \partial x_i$.

there exists $\nu > 0$ such that

$$f_{p_ip_j}(x, p)\, \xi_i \xi_j \geqslant \nu\, |\xi|^2, \qquad x \in \bar{G}, p \in S_t, \xi \in R^N. \tag{1.3}$$

Let $\phi \in C_0^\infty(R^N)$, $0 \leqslant \phi(p) \leqslant 1$, $\phi(p) = 1$ in $R^N - S_{t+1}$ and $\phi(p) = 0$ in S_{t+2}. We define

$$f_n(x, p) = f(x, p)(1 - \phi(p)) + [(f\phi) * \eta_n](x, p),$$

where η_n are mollifiers. Obviously $f_n \in C^2(\bar{G} \times R^N)$, $\{f_n\}$ converges to f uniformly in $\bar{G} \times C$, with C any compact subset of R^N. Moreover f_n is convex in p for $p \in R^N - S_{t+1}$ and strictly convex in p for $p \in S_{t+2}$. We show now that there exists n_0 such that for $n > n_0$, f_n is strictly convex in $S_{t+1} - S_{t+2}$.

Since $f \in C^2(\bar{G} \times S_t)$, $f_{n,p_ip_j} = f_{p_ip_j} + (f\phi)_{p_ip_j} * \eta_n - (f\phi)_{p_ip_j}$ and f_{n,p_ip_j} converges to $f_{p_ip_j}$ uniformly in $\bar{G} \times C$ with C any compact subset of S_t. Thus for $\tau \in [0, \nu]$ there exists n_0 such that for $n > n_0$,

$$f_{n,p_ip_j}(x, p)\, \xi_i \xi_j \geqslant (\nu - \tau)\, |\xi|^2, \qquad x \in \bar{G}, p \in S_{t+1} - S_{t+2}.$$

So f_n is a convex function in $p \in R^N$.

Setting, for $n > n_0$, $F_n(x, p) = f_n(x, p) + (1/n)\, |p|^2$, $x \in \bar{G}, p \in R^N$, F_n is strictly convex in p and $F_n \in C^2(\bar{G} \times R^N)$. We consider the problem

$$\text{Min} \left\{ \int_G F_n(x, Dw(x))\, dx,\; w \in H_0^{1,2}(G) + u_0 \right\}, \tag{1.4}$$

where $u_0 \in H^{1,2}(G) \cap L^\infty(G)$. Via direct methods there exists a unique solution u_n of problem (1.4). By Theorem 3.2 of Chap. V in [9], there exists $M > 0$ such that

$$\sup_G |u_n| \leqslant M, \qquad n > n_0, \tag{1.5}$$

since, for n large enough, F_n satisfies a condition of type (1.1) with constants independent of n. Moreover from Theorem 3.1 in [4] the function u_n are locally equi-Hölder continuous, i.e., there exists α, $0 < \alpha < 1$, such that for any $G_0 \Subset G$ the norms $\|u_n\|_{C^{0,\alpha}(G_0)}$ are bounded independently of n.

Proposition 1.1. *Assume that* (1.1) *and* (i) *hold, let* u_n *be the solution of* (1.4). *Then for each* $x_0 \in G$ *there exists* R_0, $0 < R_0 < d(x_0) = \text{dist}(x_0, \partial G)$ *such that for* $R < R_0$, $R < \frac{1}{2} d$,

$$\sup_{B_{R/2}(x_0)} |Du_n| \leqslant \frac{\bar{C}_1}{R^N} \int_{B_{2R}(x_0)} (1 + |Du_n|^2)\, dx + \bar{C}_2, \tag{1.6}$$

where $B_r(x_0)$ denotes the ball of radius r, centered at x_0 and $\bar{C}_1$ and $\bar{C}_2$ are independent of n.

Proof. The proof of (1.6) relies on classical arguments (see [5, 9]). However, we refer to [3] to give a sketch of the proof. Let u_n be the solution of (1.4), obviously u_n satisfies the Euler–Lagrange equation

$$D_j F_{n,p_j}(x, Du_n) = 0 \qquad \text{in } G. \tag{1.7}$$

Since F_n verifies conditions of type (1.2) and (1.3) in $\bar{G} \times R^N$, from the results of Sections 5, 6 of Chap. IV in [9], it follows that $u_n \in H^{2,2}_{\text{loc}} \cap C^{1,\beta}_{\text{loc}}$. Then, set $b_n(x) = \max\{|Du_n|^2 - l, 0\}$, for $l > (t+2)^2$, b_n is a continuous function and for $\mathscr{T} \in C_0^\infty(G)$ the functions

$$\phi_{n,rs}(x) = \begin{cases} \mathscr{T}^2 D_r u_n \min\{b_n(x), 1\} & \text{if } s = 0, \\ \mathscr{T}^2 D_r u_n b_n^s & \text{if } s = 1, 2, 3, \ldots, \end{cases}$$

for $r \in \{1, \ldots, N\}$, are in $H_0^{1,2}(G)$.

In the weak form of (1.7),

$$\int_G F_{n,p_j}(x, Du_n)\, D_j\phi(x)\, dx = 0, \qquad \phi \in H_0^{1,2}, \tag{1.8}$$

choose $\phi = \phi_{n,rs}$. Let us observe that, since $\phi_{n,rs} = 0$ and $D\phi_{n,rs} = 0$ where $|Du_n| \leqslant t+2$, the integral in (1.8) is just over the set where $f_n = f$.

So we can proceed as in the proof of Lemma 1.2 in [3]. We remark explicitly that the constants involved in the proof are, in the present case, independent of n, also taking into account the locally equi-Hölder-continuity of the functions u_n.

Definitively we get:

For every $x_0 \in G$ there exists $R_0 > 0$, $R_0 < d(x_0)$ such that for every $R < R_0$,

$$\int_{B_{R/2}} |Du_n|^{2+2s}\, dx \leqslant C \int_{B_R} (1 + |Du_n|^2)\, dx, \qquad s \geqslant 0, \tag{1.9}$$

where C is a constant independent of n. Now let be $x_0 \in G$, $R < \min\{R_0, \frac{1}{2}d(x_0)\}$ and $\xi \in C_0^\infty(B_R)$, $\xi = 1$ in $B_{R/2}$ and $|D\xi| \leqslant C/R$.

Set $\eta_{n,r}(x) = \xi^2 D_r u_n \max\{\xi^2 |Du_n|^2 - K, 0\}$, $K > (t+2)^2$, since $\xi |Du_n| \leqslant |Du_n|$, we have that $\eta_{n,r} = 0$, $D\eta_{n,r} = 0$ where $|Du_n| \leqslant t+2$. In (1.8) choose now $\phi = \eta_{n,r}$. Also in this case the integral is just over the set where $f = f_n$, so by using the properties of F_n and the well known inequality $ab \leqslant \varepsilon a^2 + 1/\varepsilon b^2$ we get

$$\int_{A_{n,K}} \left[\sum_{i,r} (D_{i,r} u_n)^2 (|Du_n|^2 \xi^2 - K)\, \xi^2 + \xi^4 \sum_i (D_i |Du_n|^2)^2 \right] dx$$

$$\leqslant \int_{A_{n,K}} (1 + |Du_n|)^4 (1 + |D\xi|)^2\, dx, \tag{1.10}$$

where $A_{n,K} = \{x \in G \colon \xi^2\, |Du_n|^2 > K\}$.

Now we proceed as in the proof of Theorem 1.3 of [3]. From (1.10) the function $w_n(x) = \xi^2\, |Du_n|^2$ verifies

$$\int_{A_{n,K}} |Dw_n|^2\, dx \leqslant \frac{C}{R^2} \int_{A_{n,K}} (1 + |Du_n|)^4\, dx,$$

where C is independent of n; moreover by Hölder inequality

$$\int_{A_{n,K}} |Dw_n|^2\, dx \leqslant \frac{C}{R^2} \left\{ \int_{A_{n,K}} (1 + |Du_n|)^\sigma\, dx \right\}^{4/\sigma} (\text{meas } A_{n,K})^{1-(4/\sigma)}.$$

For σ such that $\sigma > 2N$, by applying a classical truncation lemma,[2] we get

$$\sup_{B_{R/2}} |Du_n|^2 \leqslant \frac{C}{R^2} \left(\int_{A_{n,K}} (1 + |Du_n|)^\sigma \right)^{4/\sigma} (\text{meas } A_{n,K})^{(2/N)-(4/\sigma)} + \text{Const.},$$

moreover, since $A_{n,K} \subset B_R$ for every n, we get

$$\sup_{B_{R/2}} |Du_n|^2 \leqslant \frac{C}{R^N} \int_{B_R} (1 + |Du_n|)^\sigma\, dx + \text{Const.}$$

This estimate together with (1.9) gives (1.6).

Now consider the problem

$$\text{Min} \left\{ \int_G f(x, Dv)\, dx,\ v \in H_0^{1,2} + u_0 \right\}. \tag{1.11}$$

THEOREM 1.2. *Assume* (1.1) *and* (i). *Then every solution of* (1.8) *is in* $C^{0,1}_{\text{loc}}(G)$.

Proof. Since, for n large enough, f_n verifies inequalities of type (1.1)

[2] Truncation Lemma. If $w \in H_0^{1,2}(G)$ and for $k > k_0$, $\varepsilon > 0$, $\int_{\{x: w(x) > k\}} |Dw|^2\, dx \leqslant \gamma(\text{meas}\{x\colon w(x) > k\}^{1-(2/N)+\varepsilon}$ then $w \in L^\infty(G)$ and $\|w\|_{L^\infty(G)} \leqslant k_0 + c[\gamma(\text{meas}\{w > k_0\})^\varepsilon]^{1/2}$. (See, e.g., [8, 9]).

with constants independent of n, we get, from the dominated convergence theorem,

$$\lim_n \int_G f_n(x, Dv)\, dx = \int_G f(x, Dv)\, dx, \qquad v \in H^{1,2}. \tag{1.12}$$

Now, for fixed $v \in H_0^{1,2} + u_0$, we have

$$\begin{aligned} C_1 \int_G |Du_n|^2\, dx &\leqslant \int_G f_n(x, Du_n)\, dx + \frac{1}{n} \int_G |Du_n|^2\, dx + C_2 \operatorname{meas} G \\ &\leqslant \int_G f_n(x, Dv)\, dx + \frac{1}{n} \int_G |Dv|^2\, dx + C_2 \operatorname{meas} G. \end{aligned}$$

Therefore, from (1.12)

$$\int_G |Du_n|^2\, dx \leqslant c, \qquad \text{for } n \text{ large enough.} \tag{1.13}$$

So, passing eventually to a subsequence, u_n converges weakly in $H^{1,2}$ to a function u. Inequalities (1.6) and (1.13), give for $G_0 \Subset G$,

$$\operatorname*{Sup}_{G_0} |Du_n| \leqslant c \qquad \text{for } n \text{ large enough,}$$

where the constant c depends only on G_0. Thus, from Sobolev embedding theorem, u_n converges uniformely in G_0 to u and $u \in C^{0,1}_{\mathrm{loc}}(G)$. From the definition of f_n we have, for $x \in \bar{G}$ and p in a compact subset of $\mathbb{R}^N$,

$$f_n(x, p) = f(x, p) - \varepsilon_n \qquad \text{with} \quad \varepsilon_n \to 0.$$

Let $G_h \Subset G$ such that $\bigcup_h G_h = G$, from lower semicontinuity, taking also into account the last inequality, we get

$$\begin{aligned} \int_{G_h} f(x, Du)\, dx &\leqslant \varliminf_n \int_{G_h} f(x, Du_n) \leqslant \varliminf_n \int_{G_h} (f_n(x, Du_n)\, dx + \varepsilon_n)\, dx \\ &\leqslant \varliminf_n \int_{G_h} f_n(x, Du_n)\, dx \leqslant \varliminf_n \int_G f_n(x, Du_n)\, dx, \end{aligned}$$

then for $h \to \infty$,

$$\begin{aligned} \int_G f(x, Du)\, dx &\leqslant \varliminf_n \left[\int_G f_n(x, Du_n)\, dx + \frac{1}{n} \int_G |Du_n|^2\, dx \right] \\ &\leqslant \varliminf_n \left[\int_G f_n(x, Dv)\, dx + \frac{1}{n} \int_G |Dv|^2\, dx \right], \qquad v \in H_0^{1,2} + u_0. \end{aligned}$$

From (1.12) we obtain

$$\int_G f(x, Du)\,dx \leqslant \int_G f(x, Dv)\,dx, \qquad v \in H_0^{1,2} + u_0.$$

So (1.11) has at least one solution u in $C^{0,1}_{\text{loc}}(G)$.

Now fixed $G_0 \Subset G$, let

$$\operatorname*{ess\,sup}_{G_0} |Du| \leqslant 2c(G_0).$$

Suppose that there exist a solution v of (1.11) and $\tilde{G}_0 \subseteq G_0$ with meas $\tilde{G}_0 > 0$ such that

$$|Dv| > 2(t + c(G_0)) \qquad \text{a.e.} \quad x \in \tilde{G}_0.$$

Setting $w = (u+v)/2$, we have $Dw(x) \in S_t$ a.e. in G_0, and so from the strict convexity of f in S_t, we get a contradiction. In fact,

$$\int_G f(x, Dx)\,dx = \int_{G_0} f(x, Dw)\,dx + \int_{G \setminus G_0} f(x, Dw)\,dx < \int_G f(x, Du)\,dx.$$

Consequently every solution v of (1.11) satisfies

$$\operatorname*{ess\,sup}_{G_0} |Dv| \leqslant 2(t + c(G_0)).$$

To obtain the regularity of the solution up to the boundary, we now construct the barriers relative to (1.4). With this objective, we need further assumptions:

(ii) G has boundary of class C^2,

(iii) $u_0 \in C^{1,1}(\bar{G})$.

Denoting by $d(x)$ the distance of x from ∂G, for $h > 0$ small enough, we set

$$G_h = \{x \in G: d(x) < h\},$$
$$\Gamma_h = \{x \in G: d(x) = h\}.$$

Note that (ii) implies that $d(x)$ is of class C^2 in G_h and for every $x \in G_h$ there exists only one y of least distance from ∂G.

Defining

$$L^n(w) = D_i F_{n,p_i}(x, Dw),$$

an upper (lower) barrier v^+ (v^-) is a $C^{0,1}$-function in some G_h, such that

$$\begin{array}{llll} v^+ = u_0, & (v^- = u_0) & \text{on} & \partial G, \\ v^+ \geqslant M, & (v^- \leqslant -M) & \text{on} & \Gamma_h, \\ L^n(v^+) \leqslant 0, & (L^n(v^-) \geqslant 0) & \text{in} & G_h, \end{array}$$

where M is the positive constant in (1.5). With classical methods of barriers (see [9, 17]) we can prove that

$$v(x) = u_0(x) + \psi(d(x)), \tag{1.14}$$

where ψ is a smooth function satisfying $\psi(0) = 0$, $\psi'(t) > 0$ and $\psi''(t) < 0$, is an upper barrier for each L^n.

Introduce the Bernstein function

$$E^n(x, p) = f_{n,p_ip_j}(x, p)\, p_i p_j, \qquad x \in \bar{G},\ p \in \mathbb{R}^N.$$

Obviously, for $x \in \bar{G}$ and $p \in S_{t+2}$, $E^n(x, p) = E(x, p) = f_{p_ip_j}(x, p)\, p_i p_j$. Set $A^n_{ij} = f_{n,p_ip_j}(x, Dv)$, $b^n_i = D_i f^n_{p_i}(x, Dv)$, we have

$$\begin{aligned} L^n(v) = {} & A^n_{ij} D_{ij} u_0 + \psi' A^n_{ij} D_{ij} d \\ & + \frac{\psi''}{(\psi')^2} \{E^n + A^n_{ij} D_{ij} u_0 - 2A^n_{ij} D_i v\, D_j u_0\} + b^n_i. \end{aligned}$$

Since for $x \in G$ and $p \in S_{t+2}$, $f_n = f$, (1.2) and (1.3) imply that there exists $\Lambda \in R_+$ such that

$$\begin{aligned} & \nu\, |\xi|^2 \leqslant f_{n,p_ip_j}(x, p)\, \xi_i \xi_j \leqslant (\Lambda + 1)\, |\xi|^2, \\ & |Df_{n,p}(x, p)| \leqslant \mu(1 + |p|) \qquad \text{uniformly with respect to } n. \end{aligned}$$

Then if ψ' is large enough,

$$L^n(v) \leqslant (1 + |Dv|)(1 + \Lambda) + \frac{\psi''}{(\psi')^2} \left\{ \frac{1}{2} E^n - 2c\Lambda \right\},$$

and therefore, because $E(x, p) \geqslant c\, |p|\, (\Lambda + 1)$, for $x \in \bar{G}$ and $|p|$ sufficiently large we have that $L^n(v) \leqslant 0$.

Choosing $\psi(s) = c \log(1 + \Theta s)$, $\Theta \in R_+$, it is easy to check that v, defined as in (1.14) is $\forall L^n$ an upper barrier in G_{h_0}, for a suitable h_0. In the same way one can prove the existence of a lower barrier.

THEOREM 1.3. *Assume* (1.1) *and* (i)–(iii). *Then every solution of* (1.11) *is in* $C^{0,1}(\bar{G})$.

Proof. First we prove that there exists $L > 0$ such that

$$\sup_{G} |Du_n| \leq L \qquad \text{for } n \text{ large enough.} \tag{1.15}$$

Let r and s be the functions

$$r(x) = \begin{cases} \inf(v^+, M), & x \in G_{h_0} \\ M, & x \in G - G_{h_0}, \end{cases}$$

$$s(x) = \begin{cases} \sup(v^-, -M), & x \in G_{h_0}, \\ -M, & x \in G - G_{h_0}, \end{cases}$$

where v^+ and v^- are the upper and lower barrier and M is the constant in (1.5).

Since for n large enough, from the maximum principle,

$$s(x) \leq u_n(x) \leq r(x), \qquad x \in \bar{G},$$

we get

$$|u_n(x) - u_n(y)| \leq K\,|x - y|, \qquad x \in \bar{G},\ y \in \partial G. \tag{1.16}$$

On the other hand, the Cacciopooli inequality holds (see, e.g., [4]),

$$\int_{B_R} |Du_n|^2\,dx \leq c\left\{\frac{1}{R^2}\int_{B_{2R}} |u_n - u_{n,R}|^2\,dx + \text{meas } B_{2R}\right\},$$

where $B_R = B_R(x_0) \subset G$ and $u_{n,R} = (1/\text{meas } B_R)\int_{B_R} u_n(y)\,dy$.

From (1.6) and (1.16) we get

$$|Du_n(x_0)|^2 \leq \frac{c}{R^{N+2}}\int_{B_{4R}} |u_n - u_{n,R}|^2\,dx + c. \tag{1.17}$$

Let $R = d(x_0)/4$, from (1.16) we have

$$|u_n(x) - u_{n,R}| \leq k'R, \qquad x \in B_{4R}. \tag{1.18}$$

Then (1.17) and (1.18) imply (1.15). Consequently u_n converges uniformly to a function u in $C^{0,1}(\bar{G})$. Now we can complete the proof by proceeding as in Theorem 1.2.

2

In this section we apply the previous results to obtain existence theorems for a class of nonconvex problems.

Let $f(x, p) \in C^0(\bar{G} \times \mathbb{R}^N)$ satisfy (1.1). Consider the problem

$$\text{Min}\left\{J(v)=\int_G f(x, Dv(x))\,dx,\ v \in H_0^{1,2}+u_0\right\}, \tag{2.1}$$

where $u_0 \in H^{1,2} \cap L^\infty$.

Let $f^{**}(x, p)$ be the lower convex envelope of $f(x, p)$ with respect to p. The relaxed problem of (2.1) is

$$\text{Min}\left\{J^{**}(v)=\int_G f^{**}(x, Dv(x))\,dx,\ v \in H_0^{1,2}+u_0\right\}. \tag{2.2}$$

Define for $x \in G$

$$K(x)=\{p \in \mathbb{R}^N: f^{**}(x, p)<f(x, p)\}.$$

We assume

(iv) For $x \in G$, $K(x)$ is a connected bounded subset of $\mathbb{R}^N$ and there exist $N+1$ functions defined on $\bar{G}$, m_i, $i=1,\dots,N$, and $q(x)$ such that

$$f^{**}(x, p)=\sum_{i=1}^N m_i(x)\,p_i+q(x), \qquad \forall p \in K(x). \tag{2.3}$$

(v) For $i=1,\dots,N$, $m_i \in C^1(G)$ and $\text{meas}\{x \in G: \sum_i D_i m_i(x)=0\}=0$.

THEOREM 2.1. *Assume* (1.1)(iv) *and* (v). *Every solution of* (2.2) *which is a.e. differentiable in G is also a solution of* (2.1)

Proof. Let u be a solution of (2.2) a.e. differentiable in G. We shall prove that $Du(x) \in \mathbb{R}^N - K(x)$ a.e. in G, so $J^{**}(u)=J(u)$, then u is also a solution of (2.1). Let u be differentiable in x_0 and $Du(x_0) \in K(x_0)$. From lemma 4 of the Appendix in [2], there exist two functions ψ_+ and ψ_- in $C_0^1(G)$ such that

$$D\psi_\pm(x_0)=Du(x_0), \qquad \psi_\pm(x_0)=u(x_0); \qquad \psi_-(x)<u(x),\ \psi_+(x)>u(x),$$
$$\forall x \in B_r(x_0)-\{x_0\} \text{ for some } r>0. \tag{2.4}$$

In the following we use ψ_- when we assume $\sum_i D_i m_i(x_0)>0$ and $\psi_+(x)$ when we assume $\sum_i D_i m_i(x_0)<0$.

If $\sum_i D_i m_i(x_0)>0$, from (v) there exists $\tilde{\delta}$ such that

$$\sum_i D_i m_i(x)>0, \qquad \forall x \in B_{\tilde{\delta}}(x_0)=B_{\tilde{\delta}}. \tag{2.5}$$

Moreover $K(x)$ is an open set for a.e. $x \in G$, in fact, since $f \in C^0(\bar{G} \times \mathbb{R}^N)$ and satisfies (1.1) we have $f^{**} \in C^0(\bar{G} \times \mathbb{R}^N)$ (see, e.g., [12]).

Consequently there exists $\delta \in]0, \tilde{\delta}[$ such that

$$D\psi_-(x) \in K(x), \qquad \forall x \in B_\delta. \tag{2.6}$$

Let $\varphi \in C_0^\infty(B_\delta)$ satisfy

$$0 \leqslant \varphi \leqslant 1, \qquad \varphi(x_0) = 1 \tag{2.7}$$

and consider the function $\psi_- + \varepsilon\varphi$. By (2.6), for ε small, $D(\psi_- + \varepsilon\varphi)(x) \in K(x)$, $\forall x \in B_\delta$.

Moreover there exists an open subset $A \subset B_\delta$ such that $\psi_- + \varepsilon\varphi = u$ on ∂A. Define

$$\chi(x) = \begin{cases} \psi_-(x) + \varepsilon\varphi(x), & x \in A \\ u(x), & x \in G - A. \end{cases}$$

Now we prove that $J^{**}(u) > J^{**}(\chi)$. In fact, by using the inequality $f^{**}(x, p) \geqslant \sum_{i=1}^N m_i(x)\, p_i + q(x)$, $\forall p \in \mathbb{R}^N$ and by applying the divergence theorem, we get

$$\begin{aligned} J^{**}(u) - J^{**}(\chi) &= \int_A f^{**}(x, Du) - f^{**}(x, D\chi)\, dx \\ &\geqslant \int_A \sum_i m_i(x)\, D_i(u-\chi)\, dx = \int_{B_\delta} \sum_i m_i(x)\, D_i(u-\chi)\, dx \\ &= \int_{\partial B_\delta} \sum_i m_i(x)(u-\chi)\, v_i\, ds - \int_{B_\delta} \sum_i D_i m_i(x)(u-\chi)\, dx, \end{aligned} \tag{2.8}$$

where v is the unit outward normal to ∂B_δ and ds the $(n-1)$-dimensional area element on ∂B_δ. Since $u(x) = \chi(x)$ on ∂B_δ and $u(x) < \chi(x)$ for $x \in A$, (2.5) and (2.8) imply $J^{**}(u) > J^{**}(\chi)$. This contradicts that $u(x)$ is a solution of (2.2), and so $Du(x_0)$ cannot belong to $K(x_0)$.

Now, if $\sum_i D_i m_i(x_0) < 0$, with the same argument as above, by using ψ_+ instead of ψ_-, we can prove that $Du(x_0) \notin K(x_0)$.

Remark. Let us make a comparison between the above assumptions and the ones made in [16]. In that paper we assume $f(\cdot, p) \in C^{0,1}(G)$ for each $p \in \mathbb{R}^N$ (we mean uniformly with respect to p), which implies $f^{**}(\cdot, p) \in C^{0,1}(G)$, in order to obtain that the functions $m_i(x)$ in (2.3) are a.e. differentiable in G. Here $f(x, p)$ is only in $C^0(\bar{G} \times \mathbb{R}^N)$ but to prove Theorem 2.1 we need a stronger condition on $f^{**}(x, p)$ for p in $K(x)$, i.e., on the functions $m_i(x)$.

Remark. The idea of the above proof is also present in Lemma 2.2 of [16] where we prove that, for a particular solution w of the relaxed problem (see definition (2.5) in [16]), $Dw(x) \in R^N - K(x)$ for a.e. $x \in G$. In that proof we consider $\psi_- \in C^{0,1}(B_\delta)$ and $\varphi \in C_0^\infty(B_\delta)$ satisfying (2.4) and (2.7) and we claim the existence of $\delta' < \delta$ such that $\psi_- + \varepsilon\varphi = w$ on $\partial B_{\delta'}$. This is not acceptable in general.[3] However, Lemma 2.2 continues to be true: the arguments of Theorem 2.1 can be also used with the assumptions made in [16]. In fact we can only say that there exists an open subset A of B_δ such that $\psi_- + \varepsilon\varphi > w$ in A and $\psi_- + \varepsilon\varphi = w$ on ∂A. Then we define

$$\bar{\chi}(x) = \begin{cases} \psi_-(x) + \varepsilon\varphi(x), & x \in A \\ w(x), & x \in G - A \end{cases}$$

and, by proceeding as in the above formula (2.8) we get

$$\int_G f^{**}(x, D\bar{\chi}(x))\, dx \leq \int_G f^{**}(x, Dw(x))\, dx,$$

which contradicts the definition of w.

Now we state existence theorems for problem (2.1).

THEOREM 2.2. *Assume that $f(x, p)$ verifies* (1.1) *and* (i) *in some S_{t_0}. If* (iv) *and* (v) *hold, every solution of* (2.2) *is also a $C^{0,1}_{\text{loc}}$-solution of* (2.1).

Proof. From theorem 1.2, every solution u of (2.2) is in $C^{0,1}_{\text{loc}}(G)$, then, from theorem 2.1, u is a solution of (2.1).

Proceeding as before, if we apply Theorem 1.3 instead of Theorem 1.2, we get

THEOREM 2.3. *Assume that $f(x, p)$ verifies* (1.1) *and* (i) *in some S_{t_0} and that G and u_0 verify* (ii) *and* (iii). *If* (iv) *and* (v) *hold, every solution of* (2.2) *is a $C^{0,1}(\bar{G})$-solution of* (2.1).

The following existence theorem is proved under different assumptions on the functions $m_i(x)$. Suppose

(vi) For each $i = 1, \ldots, N$, $m_i \in C^{0,1}_{\text{loc}}(G)$ and $\sum_i D_i m_i(x) = 0$ a.e. in G.

THEOREM 2.4. *Assume that $f(x, p)$ verifies* (1.1) and (i) *in some S_{t_0}. Moreover G and u_0 verify* (ii) *and* (iii). *If* (iv) *and* (vi) *hold,* 2.1 *has at least one solution in $C^{0,1}(\bar{G})$.*

[3] We wish to thank the referee of the present paper for pointing this out.

Proof. Consider, for $L>0$, the set

$$M_L=\{v\in C^{0,1}(G),\, v \text{ solution of } (2.2),\ |Dv(x)|\leqslant L \text{ a.e. in } G\}.$$

From Theorem 1.3, for a suitable L, $M_L\neq\emptyset$; moreover (see [15]) the function $u(x)=\sup\{v(x),\, v\in M_L\}\in M_L$. Define χ as in Theorem 2.1, by proceeding as in (2.8) from (vi) we get

$$J^{**}(u)-J^{**}(\chi)\geqslant\int_{\partial B_\delta}\sum_i m_i(x)(u-\chi)\,v_i\,ds=0,$$

i.e., $J^{**}(u)\geqslant J^{**}(\chi)$. Therefore $\chi\in M_L$. Since $\chi(x_0)\geqslant u(x_0)$ by construction, we get a contradiction with the definition of u.

Theorem 2.4 can be considered an extension of Theorem 1.4 in [16] to more general boundary data.

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