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Local boundedness of solutions to quasilinear elliptic systems

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Abstract. The mathematical analysis to achieve everywhere regularity in the interior of weak solutions to nonlinear elliptic systems usually starts from their local boundedness. Having in mind De Giorgi's counterexamples, some structure conditions must be imposed to treat systems of partial differential equations. On the contrary, in the scalar case of a general elliptic single equation a well established theory of regularity exists. In this paper we propose a unified approach to local boundedness of weak solutions to a class of quasilinear elliptic systems, with a structure condition inspired by Ladyzhenskaya–Ural'tseva's work for linear systems, as well as valid for the general scalar case. Our growth assumptions on the nonlinear quantities involved are new and general enough to include anisotropic systems with sharp exponents and the p, q-growth case.

1. Introduction

The study of regularity for generalized solutions of second order *quasilinear* (i.e., linear with respect to second derivatives) *elliptic systems* has been strongly motivated and at the same time conditioned by the De Giorgi's example of existence of the nonsmooth weak solution

$$u(x) = \frac{x}{|x|^{\gamma}}, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \tag{1.1}$$

to the *linear elliptic system*

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{\beta=1}^{n} a_{ij}^{\alpha\beta}(x)\; u_{x_j}^{\beta} \right) = 0, \quad \forall\, \alpha = 1, 2, \ldots, n.$$

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Here $n \geq 3$ and the measurable coefficients $a_{ij}^{\alpha\beta}$ are bounded *discontinuous* at $x = 0$. The exponent γ in (1.1) is given by

$$\gamma = \frac{n}{2}\left(1 - \frac{1}{\sqrt{(2n-2)^2 + 1}}\right)$$

and, being greater than one, the solution u in (1.1) is unbounded around the origin. De Giorgi's example was published in [8], while a modification of it, due by Giusti and Miranda [17], deals with *continuous* (in fact analytic) coefficients $a_{ij}^{\alpha\beta}(u)$ depending on u instead than of x. We also mention the extensions due to Frehse [10,11], Nečas [34], Hildebrant–Widman [18], up to the recent contribution by Šverák Yan [38]. For a description of this lack of regularity and related questions we also refer the reader to Giaquinta [14] and Giusti [16].

Motivated by these examples we find in the mathematical literature at least two directions of research about regularity of generalized solutions of elliptic systems: (i) *partial regularity*, i.e., smoothness of solutions up to a set of zero measure, or up to a better mathematically characterized set, see Mingione [32] for a detailed discussion; (ii) *everywhere regularity* in the interior of the given domain Ω of $\mathbb{R}^n$, starting—as usual in this context—from the *local boundedness* of the solution. In the last case, having in mind the above counterexamples, some *structure assumptions* must be considered to treat *systems* of partial differential equations, in contrast with the scalar case of a single equation, where a well established theory of regularity exists since the work of De Giorgi, Moser, Morrey, Nash, Serrin and many others.

Ladyzhenskaya and Ural'tseva [21, Chap. 7] first proposed the local boundedness of solutions $u = \left(u^1, u^2, \ldots, u^m\right)$ to the *linear* elliptic system

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(x)\; u_{x_j}^{\alpha} + \sum_{\beta=1}^{m} b_i^{\alpha\beta}(x)\; u^{\beta} + f_i^{\alpha}(x)\right) \\ + \sum_{i=1}^{n}\sum_{\beta=1}^{m} c_i^{\alpha\beta}(x)\; u_{x_i}^{\beta} + \sum_{\beta=1}^{m} d^{\alpha\beta}(x)\; u^{\beta} = f^{\alpha}(x), \quad \forall\, \alpha = 1, 2, \ldots, m,\end{aligned} \tag{1.2}$$

with bounded measurable coefficients a_{ij}, $b_i^{\alpha\beta}$, $c_i^{\alpha\beta}$, $d^{\alpha\beta}$ and given functions f_i^{α}, f^{α}. Here the *structure condition* is stated in terms of the positive definite $n \times n$ matrix $\left(a_{ij}\right)$, which does not depend on α, β.

Meier [31] extended these results to a class of *quasilinear elliptic systems*, introducing a structure condition based on a so called *indicator function* and assuming natural growth conditions on the quantities involved; i.e., assuming polynomial p-growth on the nonlinear coefficients (instead of $p = 2$). Quasilinear elliptic equations have been previously studied by Serrin [35,36]. Meier's motivations were based on some related researches by the Bonn school in pde's, mainly by Hildebrandt–Widman [18,19] and Frehse [11]. More recently sufficient conditions for boundedness of weak solutions have been given by Landes [22,23] and by Krömer [20]. In the nonlinear case one is led to consider $W^{1,p} \cap L^{\infty}$ as the natural

Sobolev class where to start to get regularity of weak solutions; see for example Hildebrandt [19] (see also [2,24,25]).

In this paper we consider a generalization of the *linear* case by Ladyzhenskaya and Ural'tseva in (1.2) to *quasilinear* elliptic systems of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}\,(x, u, Du)\ u^{\alpha}_{x_j} + b^{\alpha}_i\,(x, u, Du) \right) \\ = f^{\alpha}\,(x, u, Du)\,, \quad \forall\, \alpha = 1, 2, \ldots, m. \tag{1.3}$$

It is worth remarking that systems of this type, even with a linear principal part as in (1.2), arise in many problems in differential geometry such as harmonic mappings between manifolds or surfaces of prescribed mean curvature; see for instance [14]. Contrary to many papers in the mathematical literature about regularity for systems, the elliptic scalar case $m = 1$ is included in full generality in our context. A relevant example for $m > 1$ which enters in our analysis is given by the Euler's first variation of integrals of the calculus of variations such as, for instance, below in (1.10).

Let us enter in more details about our assumptions. Here we allow some *general growth conditions*, which we list in this introduction in a simplified version, for the sake of simplicity. Precisely, we assume that there exist exponents $p_1, p_2, \ldots, p_n \in (1, +\infty)$ and positive constants M_1, M_2 such that, for almost every $x \in \Omega \subset \mathbb{R}^n$ and for every $u \in \mathbb{R}^m, \xi \in \mathbb{R}^{m\times n}, \xi = (\xi_i)_{i=1,\ldots,n} = (\xi^{\alpha}_i)_{\substack{i=1,\ldots,n\\ \alpha=1,\ldots,m}}$, and $\lambda = (\lambda_i)_{i=1,\ldots,n} \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x, u, \xi)\lambda_i\lambda_j \geq M_1 \sum_{i=1}^{n} \lambda_i^2 \left(\sum_{\alpha=1}^{m} \left(\xi^{\alpha}_i\right)^2 \right)^{\frac{p_i-2}{2}}, \tag{1.4}$$

$$\left| \sum_{j=1}^{n} a_{ij}\,(x, u, \xi)\ \xi^{\alpha}_j \right| \leq M_2 \left\{ \sum_{j=1}^{n} \left|\xi_j\right|^{p_j} + |u|^{\gamma} + 1 \right\}^{1-\frac{1}{p_i}}, \quad \forall\, i,\ \alpha,$$

$$\left| b^{\alpha}_i\,(x, u, \xi) \right| \leq M_2 \left\{ \sum_{j=1}^{n} \left|\xi_j\right|^{p_j(1-\epsilon)} + |u|^{\gamma} + 1 \right\}^{1-\frac{1}{p_i}}, \quad \forall\, i,\ \alpha,$$

$$\left| f^{\alpha}\,(x, u, \xi) \right| \leq M_2 \left\{ \sum_{j=1}^{n} |\xi_j|^{p_j\,(1-\delta)} + |u|^{\gamma-1} + 1 \right\}, \quad \forall\, \alpha,$$

for suitable γ, ϵ and δ. Note that (1.4) is a weaker assumption with respect to the usual ellipticity and it reduces to the ordinary ellipticity condition only if $p_1 = p_2 = \cdots = p_n = 2$.

Our analysis unifies the scalar case (one single equation) and the vector valued one (system of pde's) with special structure. In fact, as we already said, the elliptic scalar case $m = 1$ is a special case which enters in the above assumptions.

More precisely this means that we can consider a *general quasilinear elliptic equation* of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a_i \left(x, u, Du \right) \right) = f \left(x, u, Du \right), \tag{1.5}$$

with a_i of class C^1 in the gradient variable. In fact we have

$$a_i \left(x, u, Du \right) - a_i \left(x, u, 0 \right) = \int_0^1 \frac{d}{dt} a_i \left(x, u, t\ Du \right)\ dt$$

$$= \int_0^1 \left\{ \sum_{j=1}^{n} \frac{\partial a_i}{\partial \xi_j} \left(x, u, t\ Du \right)\ u_{x_j} \right\}\ dt = \sum_{j=1}^{n} u_{x_j} \int_0^1 \frac{\partial a_i}{\partial \xi_j} \left(x, u, t\ Du \right)\ dt.$$

Therefore, if we pose

$$b_i(x, u) = a_i \left(x, u, 0 \right), \quad a_{ij}(x, u, \xi) = \int_0^1 \frac{\partial a_i}{\partial \xi_j} \left(x, u, t\ \xi \right)\ dt,$$

then the pde in (1.5) becomes a particular case of the system in (1.3) and the *ellipticity assumption* (1.4) on a_{ij}, in terms of the vector field $(a_i)_{i=1,2,\ldots,n}$, is satisfied, with constant $M_1 \cdot \min\left\{ \frac{1}{p_i - 1} : i = 1, \ldots, n \right\}$, when

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \xi_j}(x, u, \xi)\lambda_i \lambda_j \geq M_1 \sum_{i=1}^{n} |\xi_i|^{p_i - 2}\, \lambda_i^2. \tag{1.6}$$

Corollary 2.4 below gives specific conditions in order to get local boundedness of weak solutions to the Eq. (1.5) with anisotropic growth.

Let us go back to the general *system* (1.3). We need a restriction on the exponents $\{p_i\}$ to achieve the local boundedness of the solutions. Let us denote by $\overline{p}$ the harmonic average of the $\{p_i\}$ and by $\overline{p}^*$ the Sobolev exponent of $\overline{p}$; i.e.,

$$\frac{1}{\overline{p}} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}, \quad \overline{p}^* := \begin{cases} \frac{n\overline{p}}{n-\overline{p}} & \text{if } \overline{p} < n \\ \text{any } \mu > \overline{p} & \text{if } \overline{p} \geq n. \end{cases} \tag{1.7}$$

Theorem 1.1. *Under the previous assumptions, if*

$$\max\{p_1, p_2, \ldots, p_n\} < \overline{p}^*, \quad 1 < \gamma < \overline{p}^*, \quad 0 < \epsilon < 1, \quad \frac{1}{\overline{p}^*} < \delta < 1, \tag{1.8}$$

then every weak solution u to the quasilinear elliptic system (1.3) is locally bounded and for every R such that $B_R(x_0) \subset \Omega$ there exist constants c and $\theta \geq 0$ such that

$$\sup_{B_{R/2}(x_0)} |u| \leq c \left\{ \int_{B_R(x_0)} (|u| + 1)^{\overline{p}^*}\, dx \right\}^{\frac{1+\theta}{\overline{p}^*}}. \tag{1.9}$$

Assumption (1.8) is sharp, in the sense that even in the scalar case $m = 1$ (and n large) is possible to produce examples of unbounded generalized solutions when the reverse inequality $\max\{p_i : i = 1, 2, \ldots, n\} > \overline{p}^*$ is satisfied; see Giaquinta–Marcellini [15,26–28]. In the case of a single equation ($m = 1$) the local boundedness of weak solutions has been widely investigated; see for instance [3,5,12,13, 27,28,37,39] and, more recently, [6]. About partial regularity for systems ($m > 1$) see for instance [1,4,9,32,33]. In the last years there has been a large amount of papers dealing with the regularity under p, q-growth and we refer the interested reader to the survey by Mingione [32].

We emphasize that systems under consideration include the first variation of integrals of the calculus of variations of the form

$$\int_\Omega g\left(x, u, |Du|\right)\, dx \tag{1.10}$$

and the local boundedness result of Theorem 1.1 can be applied to the minimizers. In fact they are weak solutions to the system (1.3) when we define

$$a_{ij}\left(x, u, \xi\right) = \frac{1}{|\xi|}\,\frac{\partial g\left(x, u, |\xi|\right)}{\partial\,|\xi|}\,\delta_{ij}, \quad \forall\, i, j = 1, 2, \ldots, n,$$

and as usual for the lower order terms. Under a nonstandard growth condition the local boundedness of minimizers of vectorial integral functionals as in (1.10) has been studied by Dall'Aglio–Mascolo [7] when $g = g(x, |Du|)$ is a N-function in the Δ_2-class. For Lipschitz and higher regularity see Marcellini [29] and Marcellini Papi [30], who extended to general growth condition the regularity results obtained by Uhlenbeck [41] for the p-Laplacian.

Finally in the last section we deal with systems satisfying a p, q-growth condition. We assume ellipticity and growth conditions of p, q-type, see (4.5), (4.6) for precise assumptions. We prove that weak solutions $u \in W^{1,q}$ to (1.3) satisfy an a priori estimate as in (1.9).

2. The anisotropic growth

Let us consider the nonlinear system of pde's

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}\left(x, u, Du\right)\, u^\alpha_{x_j} + b^\alpha_i\left(x, u, Du\right)\Big) = f^\alpha\left(x, u, Du\right), \quad \forall\, \alpha = 1, 2, \ldots, m \tag{2.1}$$

on an open set Ω of $\mathbb{R}^n$, $n \geq 2$, $m \geq 1$. We assume that $a_{ij} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ and $b_i, f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}^m$ are Carathé odory functions, $i, j = 1, \ldots, n$.

We need some notations. If $\xi \in \mathbb{R}^{m\times n}$ we write $\xi = (\xi_1, \ldots, \xi_n)$, where $\xi_i = (\xi^1_i, \ldots, \xi^m_i) \in \mathbb{R}^m$ for $i = 1, \ldots, n$. In particular, $Du = (u_{x_1}, \ldots, u_{x_n})$ and $u_{x_i} = (u^1_{x_i}, \ldots, u^m_{x_i})$. Analogously, $b_i = (b^1_i, \ldots, b^m_i)$ and similarly for f. Given $p_1, \ldots, p_n$ exponents greater than 1, we define

$$p := \min\{p_1, \ldots, p_n\} \quad \text{and} \quad q := \max\{p_1, \ldots, p_n\}.$$

As usual p' is the conjugate exponent of p; i.e., $1/p + 1/p' = 1$. Moreover, $\overline{p}$ stands for the harmonic average of $\{p_i\}$ and $\overline{p}^*$ is the Sobolev exponent of $\overline{p}$ as defined in (1.7).

We assume the following conditions for almost every $x \in \Omega$ and for every $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{m\times n}$ and $\lambda = (\lambda_i)_{i=1,\dots,n} \in \mathbb{R}^n$,

(H1) (ellipticity condition)

$$\sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\lambda_i\lambda_j \geq M_1 \sum_{i=1}^{n} \lambda_i^2 |\xi_i|^{p_i-2}, \tag{2.2}$$

(H2) (growth conditions)

$$\left|\sum_{j=1}^{n} a_{ij}(x,u,\xi)\xi_j\right| \leq M_2 \left\{\sum_{j=1}^{n} |\xi_j|^{p_j} + b_1(x)|u|^{\gamma} + a_1(x)\right\}^{1-\frac{1}{p_i}}, \quad \forall\, i \tag{2.3}$$

$$|b_i(x,u,\xi)| \leq M_2 \left\{\sum_{j=1}^{n} |\xi_j|^{p_j(1-\epsilon)} + b_2(x)|u|^{\gamma} + a_2(x)\right\}^{1-\frac{1}{p_i}}, \quad \forall\, i \tag{2.4}$$

$$|f(x,u,\xi)| \leq M_2 \sum_{j=1}^{n} |\xi_j|^{p_j(1-\delta)} + b_3(x)|u|^{\gamma-1} + a_3(x), \tag{2.5}$$

where

$$M_1, M_2 > 0, \quad 1 < \gamma < \overline{p}^*, \quad 0 < \epsilon < 1, \quad \frac{1}{\overline{p}^*} < \delta < 1 \tag{2.6}$$

and, for $i = 1, 2, 3$,

$$b_i \in L^s_{\mathrm{loc}}(\Omega) \ \text{ with} \left(\frac{\overline{p}^*}{\gamma}\right)' < s \leq +\infty \ \text{ and } \ a_i \in L^t_{\mathrm{loc}}(\Omega) \ \text{ with} \left(\frac{\overline{p}^*}{q}\right)' < t \leq +\infty. \tag{2.7}$$

Our aim is to prove the local boundedness of weak solutions to (2.1). We consider the following *anisotropic Sobolev space*

$$W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m) := \left\{u \in W^{1,1}(\Omega;\mathbb{R}^m) : u_{x_i} \in L^{p_i}(\Omega;\mathbb{R}^m), \text{ for all } i{=}1,\dots,n\right\},$$

endowed with the norm

$$\|u\|_{W^{1,(p_1,\dots,p_n)}(\Omega)} := \|u\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

We write $W_0^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$ in place of $W_0^{1,1}(\Omega;\mathbb{R}^m) \cap W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$. For some properties of these spaces we refer to [40]; in particular the following embedding result holds.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider $u \in W_0^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$, $p_i \geq 1$ for all $i = 1,\dots,n$. Let $\max\{p_i\} < \overline{p}^*$, with $\overline{p}^*$ as in (1.7). Then $u \in L^{\overline{p}^*}(\Omega;\mathbb{R}^m)$. Moreover, there exists c, depending on $n, p_1,\dots,p_n$, such that*

$$\|u\|^n_{L^{\overline{p}^*}(\Omega)} \leq c \prod_{i=1}^n \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

Thanks to the imbedding theorem above and to Hölder inequality, the growth conditions (H2) allow a meaningful definition of weak solutions.

Definition 2.2. A function $u \in W^{1,(p_1,\dots,p_n)}_{\text{loc}}(\Omega;\mathbb{R}^m)$ is a weak solution to (2.1) if

$$\int_\Omega \left\{ \sum_{i,j=1}^n \Big(a_{ij}(x,u,Du)u^\alpha_{x_j} + b^\alpha_i(x,u,Du)\Big)\varphi^\alpha_{x_i} + f^\alpha(x,u,Du)\varphi^\alpha \right\} dx = 0 \tag{2.8}$$

for all $\alpha = 1,\dots,m$ and all $\varphi \in C^1_c(\Omega;\mathbb{R}^m)$ (or equivalently $\varphi \in W_0^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$).

Theorem 2.3. *Assume (H1) and (H2) and let $1 < p \leq q < \overline{p}^*$. Then every weak solution $u \in W^{1,(p_1,\dots,p_n)}_{\text{loc}}(\Omega;\mathbb{R}^m)$ to (2.1) is locally bounded. Moreover, for every $B_R(x_0) \subset \Omega$ there exists a positive constant c such that*

$$\sup_{B_{R/2}(x_0)} |u| \leq c \left\{ \int_{B_R(x_0)} (|u|+1)^{\overline{p}^*}\, dx \right\}^{\frac{1+\theta}{\overline{p}^*}}, \tag{2.9}$$

where $\theta = \frac{\tilde{q}}{p}\frac{\tilde{q}-p}{\overline{p}^*-\tilde{q}}$ with $\tilde{q} = \max\left\{\frac{1}{\delta}, \gamma s', qt'\right\}$.

The above theorem also gives the local boundedness of weak solutions to the general *quasilinear equation* $(m = 1)$

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(a_i\left(x,u,Du\right)\right) = f\left(x,u,Du\right), \tag{2.10}$$

where $a_i(x,u,\xi)$, $\frac{\partial a_i}{\partial \xi_j}(x,u,\xi)$ are Carathéodory functions. Let us assume

$$\sum_{i,j=1}^n \frac{\partial a_i}{\partial \xi_j}(x,u,\xi)\lambda_i\lambda_j \geq M_1 \sum_{i=1}^n \lambda_i^2 |\xi_i|^{p_i-2}, \quad \forall\, \lambda \in \mathbb{R}^n \tag{2.11}$$

$$|a_i(x,u,\xi)| \leq M_2 \left\{ \sum_{j=1}^n |\xi_j|^{p_j} + b_1(x)|u|^\gamma + a_1(x) \right\}^{1-\frac{1}{p_i}}, \quad \forall\, i \tag{2.12}$$

and

$$|f(x,u,\xi)| \le M_2 \sum_{j=1}^{n} |\xi_j|^{p_j(1-\delta)} + b_3(x)|u|^{\gamma-1} + a_3(x), \tag{2.13}$$

where $M_1, M_2, \gamma, \delta, a_i, b_i$ $(i = 1, 3)$ satisfy (2.6) and (2.7).

Corollary 2.4. *Under assumptions (2.11)–(2.13), if* $1 < p \le q < \overline{p}^*$, *then every weak solution* $u \in W^{1,(p_1,\dots,p_n)}_{\mathrm{loc}}(\Omega)$ *to the pde (2.10) is locally bounded and the estimate (2.9) holds.*

3. Proof of Theorem 2.3

First we give two preliminary results.

Lemma 3.1. *Under the ellipticity condition (2.2), for almost every* $x \in \Omega \subset \mathbb{R}^n$, *for every* $u \in \mathbb{R}^m$ *and* $\xi \in \mathbb{R}^{m\times n}$, *we have*

$$\sum_{\alpha=1}^{m} \sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\xi_i^\alpha \xi_j^\alpha \ge M_3 \sum_{i=1}^{n} \left(\sum_{\alpha=1}^{m} \left(\xi_i^\alpha\right)^2 \right)^{\frac{p_i}{2}}, \tag{3.1}$$

with $M_3 = M_1 m^{1-q}$.

Proof. For fixed $\alpha \in \{1, 2, \dots, m\}$ we pose $\lambda = \left(\xi_i^\alpha\right)_{i=1,\dots,n} \in \mathbb{R}^n$ and we get

$$\sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\xi_i^\alpha \xi_j^\alpha \ge M_1 \sum_{i=1}^{n} \left(\xi_i^\alpha\right)^2 \left(\sum_{\beta=1}^{m} \left(\xi_i^\beta\right)^2 \right)^{\frac{p_i-2}{2}} \ge M_1 \sum_{i=1}^{n} \left|\xi_i^\alpha\right|^{p_i}. \tag{3.2}$$

Fixed $i \in \{1, \dots, n\}$, by the convexity of the function $t \in \mathbb{R}_+ \to t^{p_i}$ we have the inequality

$$\left(\frac{1}{m} \sum_{\alpha=1}^{m} \left|\xi_i^\alpha\right| \right)^{p_i} \le \frac{1}{m} \sum_{\alpha=1}^{m} \left|\xi_i^\alpha\right|^{p_i}.$$

If we sum up both sides of (3.2) with respect to $\alpha = 1, 2, \dots, m$ we obtain

$$\sum_{\alpha=1}^{m} \sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\xi_i^\alpha \xi_j^\alpha \ge M_1 \sum_{i=1}^{n} \sum_{\alpha=1}^{m} \left|\xi_i^\alpha\right|^{p_i} \ge M_1 \sum_{i=1}^{n} m^{1-p_i} \left(\sum_{\alpha=1}^{m} \left|\xi_i^\alpha\right| \right)^{p_i}.$$

The conclusion (3.1) follows from the fact that $\sum_{\alpha=1}^{m} \left|\xi_i^\alpha\right| \ge \left(\sum_{\alpha=1}^{m} \left(\xi_i^\alpha\right)^2 \right)^{1/2}$. □

Lemma 3.2. *Let $\nu, \gamma, \delta, \sigma$ be positive numbers, and assume that there exists $\tau \in (1, +\infty]$ such that $\gamma\tau', \delta\tau' \leq \sigma$. Let $v \in L^{\sigma(\nu+1)}(\Omega)$, $v \geq 1$, and let $a \in L^{\tau}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a measurable set with finite Lebesgue measure. Then*

$$\int_\Omega |a(x)|[v(x)]^{\gamma+\delta\nu}\,dx \leq \|a\|_{L^\tau}\|v\|_{L^\sigma}^{\frac{\sigma-\delta\tau'}{\tau'}}\left(\int_\Omega [v(x)]^{\sigma(\nu+1)}\,dx\right)^{\frac{\delta}{\sigma}}.$$

Proof. By Hölder inequality

$$\int_\Omega |a|\,v^{\gamma+\delta\nu}\,dx \leq \|a\|_{L^\tau}\left(\int_\Omega v^{\gamma\tau'+\delta\tau'\nu}\,dx\right)^{\frac{1}{\tau'}}.$$

Let us consider the two cases separately: $\gamma \leq \delta$ and $\gamma > \delta$. In the first case, since $\gamma\tau' \leq \delta\tau'$ and $v \geq 1$ we have $\int v^{\gamma\tau'+\delta\tau'\nu}\,dx \leq \int v^{\delta\tau'(\nu+1)}\,dx$.

If $\delta\tau' = \sigma$ we conclude; otherwise, we proceed with the chain of inequalities:

$$\int_\Omega v^{\delta\tau'(\nu+1)}\,dx \leq \left(\int_\Omega v^{\sigma(\nu+1)}\,dx\right)^{\frac{\delta\tau'}{\sigma}}|\Omega|^{\frac{\sigma-\delta\tau'}{\sigma}} \leq \left(\int_\Omega v^{\sigma(\nu+1)}\,dx\right)^{\frac{\delta\tau'}{\sigma}}\left(\int_\Omega v^{\sigma}\,dx\right)^{\frac{\sigma-\delta\tau'}{\sigma}}.$$

Let us now deal with the case $\gamma > \delta$. We have $\int v^{\gamma\tau'+\delta\tau'\beta}\,dx \leq \int v^{\sigma-\delta\tau'}v^{(\beta+1)\delta\tau'}\,dx$ and if $\delta\tau' = \sigma$ we have done; otherwise, by Hölder inequality

$$\int_\Omega v^{\sigma-\delta\tau'}v^{(\beta+1)\delta\tau'}\,dx \leq \left(\int_\Omega v^{\sigma}\,dx\right)^{\frac{\sigma-\delta\tau'}{\sigma}}\left(\int_\Omega v^{(\nu+1)\sigma}\,dx\right)^{\frac{\delta\tau'}{\sigma}}$$

and we get the thesis. □

Proof of Theorem 2.3 We split the proof into steps. Without loss of generality we assume that *the functions a_i, b_i, $i = 1, 2, 3$, in (H2) are a.e. greater than or equal to* 1.

Step 1. We define a sequence of test functions $(\varphi_k)_k$ to insert in (2.8), with $\varphi_k \in W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$ and supp $\varphi_k \Subset \Omega$. Fix a ball $B_{R_0}(x_0) \Subset \Omega$. Notice that when it is obvious by the context, we write B_r and $W^{1,(p_1,\dots,p_n)}(B_r)$ in place of $B_r(x_0)$ and $W^{1,(p_1,\dots,p_n)}(B_r(x_0);\mathbb{R}^m)$. Let us assume $0 < \rho < R \leq R_0$ and let $\eta \in C_c^\infty(\Omega)$ be a cut-off function, satisfying the following assumptions:

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_\rho, \quad \operatorname{supp}\eta \Subset B_R, \quad |D\eta| \leq \frac{2}{R-\rho}. \tag{3.3}$$

Let us approximate the identity function id : $\mathbb{R}_+ \to \mathbb{R}_+$ with an *increasing* sequence of C^1 functions $g_k : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$g_k(t) = \begin{cases} 0 & \text{for all } t \in [0, \frac{1}{k+1}] \\ k & \text{for all } t \geq k, \end{cases} \quad 0 \leq g_k'(t) \leq 2 \quad \text{and} \quad g_k'(t)t \leq g_k(t) + \frac{2}{k} \quad \text{in } \mathbb{R}_+. \tag{3.4}$$

Notice that the last inequality can be assumed since the restriction of g_k to the interval $\left[\frac{1}{k+1}, k\right]$ can be seen as a smooth approximation of the linear function $G_k(t) = \frac{k(k+1)}{k(k+1)-1}\left(t - \frac{1}{k+1}\right)$, whose graph is the line of the plane connecting $(\frac{1}{k+1}, 0)$ and (k, k) and G_k satisfies $G_k'(t)t \leq G_k(t) + \frac{1}{k}$. Fixed $h = 1, \dots, n$, $k \in \mathbb{N}$ and $\nu > 0$, let $\Phi^{(h)}_{k,\nu} : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function defined as

$$\Phi^{(h)}_{k,\nu}(t) := g_k(t^{p_h \nu}).$$

By (3.4) we obtain

$$(\Phi^{(h)}_{k,\nu})'(t)t \leq p_h\nu \left\{\Phi^{(h)}_{k,\nu}(t) + \frac{2}{k}\right\} \leq q\nu \left\{\Phi^{(h)}_{k,\nu}(t) + \frac{2}{k}\right\}. \tag{3.5}$$

Finally, define $\varphi^{(h)}_{k,\nu} : B_{R_0} \to \mathbb{R}^m$,

$$\varphi^{(h)}_{k,\nu}(x) := \Phi^{(h)}_{k,\nu}(|u(x)|)u(x)[\eta(x)]^q \quad \text{for every } x \in B_{R_0}. \tag{3.6}$$

From now on, we write $\varphi^{(h)}_k$ and $\Phi^{(h)}_k$ instead of $\varphi^{(h)}_{k,\nu}$ and $\Phi^{(h)}_{k,\nu}$. We claim that

$$\varphi^{(h)}_k \in W^{1,(p_1,\dots,p_n)}(B_{R_0}; \mathbb{R}^m), \quad \operatorname{supp}\, \varphi^{(h)}_k \Subset B_R.$$

Indeed, $\Phi^{(h)}_k$ is in $C^1(\mathbb{R}_+)$, bounded, because $\|\Phi^{(h)}_k\|_{L^\infty(\mathbb{R}_+)} \leq k$, and with bounded derivative. Precisely, if $a^{(h)}_k = (k+1)^{-\frac{1}{p_h\nu}}$ and $b^{(h)}_k = k^{\frac{1}{p_h\nu}}$, then

$$(\Phi^{(h)}_k)'(s) = \begin{cases} 0 & \text{if } s \in \mathbb{R}_+ \setminus [a^{(h)}_k, b^{(h)}_k] \\ p_h\nu g_k'(s^{p_h\nu})s^{p_h\nu-1} & \text{if } s \in [a^{(h)}_k, b^{(h)}_k] \end{cases}$$

and

$$\|(\Phi^{(h)}_k)'\|_{L^\infty(\mathbb{R}_+)} \leq 2p_h\nu\, \|s^{p_h\nu-1}\|_{L^\infty(a^{(h)}_k, b^{(h)}_k)} = 2p_h\nu\, \max\left\{\left[a^{(h)}_k\right]^{p_h\nu-1}, \left[b^{(h)}_k\right]^{p_h\nu-1}\right\} < \infty.$$

As a consequence, taking into account that $u \in W^{1,(p_1,\dots,p_n)}(B_{R_0})$ we have that $\Phi^{(h)}_k(|u|)u$ is in $W^{1,(p_1,\dots,p_n)}(B_{R_0})$ and the claim follows. By density arguments, we can use $\varphi^{(h)}_k$ in (3.6) as a test function in (2.8).

Step 2. Assume that η is the cut-off function in Step 1. We aim to prove that for every $h = 1, \dots, n$ and every $\nu > 0$

$$\begin{aligned} &\frac{M_3}{4} \int_{B_R} \sum_{i=1}^n |u_{x_i}|^{p_i} |u|^{p_h\nu}\, \eta^q\, dx \\ &\leq \frac{c \max\{\nu, 1\}^{\frac{p'}{\epsilon}}}{(R-\rho)^q} \int_{B_R} \left\{|u|^q + |u|^{1/\delta} + \sum_{j=1}^3 b_j\, |u|^\gamma + a_3|u| + \sum_{j=1}^2 a_j\right\} |u|^{p_h\nu}\, dx, \end{aligned} \tag{3.7}$$

where $M_3 = M_1 m^{1-q}$ and c is a positive constant depending on the data and R_0, but is independent of ν, R and ρ.
Insert $\varphi_k^{(h)}$ in (2.8) as test function. Notice that

$$\left(\varphi_{k,\nu}^{(h)}\right)_{x_i}(x) = \sum_{\beta=1}^{m} (\Phi_k^{(h)})'(|u|) \frac{u^\beta}{|u|} u_{x_i}^\beta u \eta^q + \Phi_k^{(h)}(|u|) u_{x_i} \eta^q + q \Phi_k^{(h)}(|u|) u \eta^{q-1} \eta_{x_i}.$$

We recall that $(\Phi_k^{(h)})'(s) = 0$ in $[0, a_k^{(h)}]$. Then (2.8) implies

$$\begin{aligned} I_1 + I_2 &:= \int_{B_R} \langle \tilde{a}(x, u, Du), Du \rangle \Phi_k^{(h)}(|u|)\, \eta^q\, dx \\ &\quad + \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha,\beta=1}^{m} \tilde{a}_i^\alpha(x, u, Du)\, u^\alpha \frac{u^\beta}{|u|}\, u_{x_i}^\beta\, (\Phi_k^{(h)})'(|u|)\, \eta^q\, dx \\ &= q \int_{B_R} \langle \tilde{a}(x, u, Du), -u \otimes D\eta \rangle \Phi_k^{(h)}(|u|)\, \eta^{q-1}\, dx \\ &\quad - \int_{B_R} \langle f(x, u, Du), u \rangle \Phi_k^{(h)}(|u|)\, \eta^q\, dx =: I_3 + I_4, \end{aligned} \tag{3.8}$$

where $\tilde{a} = (\tilde{a}_i^\alpha)_{\substack{i=1,\dots,n \\ \alpha=1,\dots,m}}$ is the matrix with entries

$$\tilde{a}_i^\alpha(x, u, \xi) = \sum_{j=1}^{n} a_{ij}(x, u, \xi)\, \xi_j^\alpha + b_i^\alpha(x, u, \xi) \tag{3.9}$$

and we used the following notation: $u \otimes D\eta := \left(u^\alpha \eta_{x_i}\right)_{\substack{i=1,\dots,n \\ \alpha=1,\dots,m}}$. Separately we consider and estimate I_i, $i = 1, \dots, 4$.

Estimate of I_1

By (2.2) and Lemma 3.1 we easily get

$$\begin{aligned} \langle \tilde{a}(x, u, Du), Du \rangle &= \sum_{i,j=1}^{n} \sum_{\alpha=1}^{m} a_{ij}(x, u, Du) u_{x_j}^\alpha u_{x_i}^\alpha + \sum_{i=1}^{n} \sum_{\alpha=1}^{m} b_i^\alpha(x, u, Du) u_{x_i}^\alpha \\ &\geq M_3 \sum_{i=1}^{n} |u_{x_i}|^{p_i} - \sum_{i=1}^{n} |b_i(x, u, Du)||u_{x_i}|. \end{aligned}$$

By (2.4) and the Young inequality (applied first with exponent p_i and then with exponent $\frac{1}{1-\epsilon}$) we obtain

$$\begin{aligned} |b_i(x, u, Du)||u_{x_i}| &\leq \frac{M_3}{8n} |u_{x_i}|^{p_i} + c_1 \left\{ \sum_{j=1}^{n} |u_{x_j}|^{p_j(1-\epsilon)} + b_2(x)|u|^\gamma + a_2(x) \right\} \\ &\leq \frac{M_3}{4n} \sum_{j=1}^{n} |u_{x_j}|^{p_j} + c_2 \left\{ b_2(x)|u|^\gamma + a_2(x) + 1 \right\} \end{aligned} \tag{3.10}$$

with c_2 depending also on ϵ. Therefore, defining $c_3 = 2nc_2$ we get (recall that $a_i, b_i \geq 1$)

$$I_1 \geq \frac{3M_3}{4} \int_{B_R} \sum_{i=1}^{n} |u_{x_i}|^{p_i} \Phi_k^{(h)}(|u|)\eta^q \, dx - c_3 \int_{B_R} \left\{ b_2|u|^\gamma + a_2 \right\} \Phi_k^{(h)}(|u|) \, dx. \tag{3.11}$$

Estimate of I_2

For a.e. $x \in \{|u| > 0\}$

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{\alpha,\beta=1}^{m} \tilde{a}_i^\alpha(x, u, Du)\, u^\alpha \frac{u^\beta}{|u|}\, u_{x_i}^\beta \\
&= \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} a_{ij}(x, u, Du) u_{x_j}^\alpha u^\alpha \frac{u^\beta}{|u|}\, u_{x_i}^\beta + \sum_{i=1}^{n} \sum_{\alpha,\beta=1}^{m} b_i^\alpha(x, u, Du)\, u^\alpha \frac{u^\beta}{|u|}\, u_{x_i}^\beta.
\end{aligned}$$

By (2.2), with $\lambda_i = \sum_{\alpha=1}^{m} u^\alpha\, u_{x_i}^\alpha$, we have that

$$\begin{aligned}
\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} a_{ij}(x, u, Du) u_{x_j}^\alpha u^\alpha u^\beta\, u_{x_i}^\beta &= \sum_{i,j=1}^{n} a_{ij}(x, u, Du) \left\{ \sum_{\alpha=1}^{m} u^\alpha\, u_{x_j}^\alpha \right\} \left\{ \sum_{\alpha=1}^{m} u^\alpha\, u_{x_i}^\alpha \right\} \\
&\geq 0.
\end{aligned} \tag{3.12}$$

Thus, by $(\Phi_k^{(h)})' \geq 0$ we have

$$\int_{B_R} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} a_{ij}(x, u, Du) u_{x_j}^\alpha u^\alpha \frac{u^\beta}{|u|}\, u_{x_i}^\beta (\Phi_k^{(h)})'(|u|)\, \eta^q \, dx \geq 0. \tag{3.13}$$

The above inequality and (3.5) imply

$$\begin{aligned}
I_2 &\geq \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha,\beta=1}^{m} b_i^\alpha(x, u, Du)\, u^\alpha \frac{u^\beta}{|u|}\, u_{x_i}^\beta\, (\Phi_k^{(h)})'(|u|)\, \eta^q \, dx \\
&\geq - \int_{B_R} \sum_{i=1}^{n} |b_i(x, u, Du)||u_{x_i}|\, (\Phi_k^{(h)})'(|u|)|u|\, \eta^q \, dx \\
&\geq -q\nu \int_{B_R} \sum_{i=1}^{n} |b_i(x, u, Du)||u_{x_i}| \left\{ \Phi_k^{(h)}(|u|) + \frac{2}{k} \right\} \eta^q \, dx.
\end{aligned}$$

Reasoning as done in (3.10), since $a_2 \geq 1$

$$\begin{aligned}
q\nu |b_i(x, u, Du)||u_{x_i}| &\leq \frac{M_3}{8n} |u_{x_i}|^{p_i} + c_4 M_2 \max\{\nu, 1\}^{p'} \\
&\quad \times \left\{ \sum_{j=1}^{n} |u_{x_j}|^{p_j(1-\epsilon)} + b_2(x)|u|^\gamma + a_2(x) \right\} \\
&\leq \frac{M_3}{4n} \sum_{j=1}^{n} |u_{x_j}|^{p_j} + c_5 \max\{\nu, 1\}^{\frac{p'}{\epsilon}} \left\{ b_2(x)|u|^\gamma + a_2(x) \right\}.
\end{aligned} \tag{3.14}$$

Thus, we obtain

$$I_2 \geq -\frac{M_3}{4} \int_{B_R} \sum_{j=1}^{n} |u_{x_j}|^{p_j} \left\{ \Phi_k^{(h)}(|u|) + \frac{2}{k} \right\} \eta^q \, dx$$
$$-c_5 \, n \max\{\nu, 1\}^{\frac{p'}{\epsilon}} \int_{B_R} \left\{ b_2|u|^\gamma + a_2 \right\} \left\{ \Phi_k^{(h)}(|u|) + \frac{2}{k} \right\} dx. \quad (3.15)$$

Estimate of I_3

For a.e. $x \in B_{R_0}$ $|u_{x_j}|^{p_j(1-\epsilon)} \leq \max\{|u_{x_j}|, 1\}^{p_j(1-\epsilon)} \leq |u_{x_j}|^{p_j} + 1 \leq |u_{x_j}|^{p_j} + a_2(x)$; therefore (2.4) implies

$$|b_i(x, u, Du)| \leq 2M_2 \left\{ \sum_{j=1}^{n} |u_{x_j}|^{p_j} + b_2(x)|u|^\gamma + a_2(x) \right\}^{1-\frac{1}{p_i}} \quad \forall i.$$

Thus, the above inequality and (2.3) imply that for a.e. x

$$|\tilde{a}_i(x, u, Du)| \leq 3M_2 \left\{ \sum_{j=1}^{n} |u_{x_j}|^{p_j} + (b_1(x) + b_2(x))|u|^\gamma + a_1(x) + a_2(x) \right\}^{1-\frac{1}{p_i}}.$$

By (2.3) and the properties of η in (3.3) we obtain

$$q\langle \tilde{a}(x, u, Du), -u \otimes D\eta \rangle \eta^{q-1} \leq \sum_{i=1}^{n} \frac{2q\eta^{q-1}}{R-\rho} |u||\tilde{a}_i(x, u, Du)|$$
$$\leq \sum_{i=1}^{n} \frac{6M_2 q \eta^{q-1}}{R-\rho} |u| \left\{ \sum_{j=1}^{n} |u_{x_j}|^{p_j} + (b_1(x)+b_2(x))|u|^\gamma + a_1(x) + a_2(x) \right\}^{1-\frac{1}{p_i}}.$$

Using the Young inequality and $\eta \leq 1$ we get

$$\frac{6M_2 q \eta^{q-1}}{R-\rho} |u| \left\{ \sum_{j=1}^{n} |u_{x_j}|^{p_j} + (b_1 + b_2)|u|^\gamma + a_1 + a_2 \right\}^{1-\frac{1}{p_i}}$$
$$= \frac{6M_2 q \eta^{\frac{q-p_i}{p_i}}}{R-\rho} |u| \left\{ \eta^q \left(\sum_{j=1}^{n} |u_{x_j}|^{p_j} + (b_1 + b_2)|u|^\gamma + a_1 + a_2 \right) \right\}^{1-\frac{1}{p_i}}$$
$$\leq \frac{M_3}{8n} \eta^q \left\{ \sum_{j=1}^{n} |u_{x_j}|^{p_j} + (b_1 + b_2)|u|^\gamma + a_1 + a_2 \right\} + \left\{ \frac{c_6}{R-\rho} |u| \right\}^{p_i},$$

with c_6 depending on n, m, M_1, M_2, q. Since

$$\sum_{i=1}^{n}\left\{\frac{c_6}{R-\rho}|u|\right\}^{p_i} \le \sum_{i=1}^{n}\left(\max\left\{\frac{c_6}{R-\rho}|u|, 1\right\}\right)^{p_i} \le n\left(\left\{\frac{c_6}{R-\rho}|u|\right\}^{q}+1\right)$$
$$\le n\left(\left\{\frac{c_6}{R-\rho}|u|\right\}^{q}+a_2\right)$$

we conclude that there exists $c_7 > 0$, possibly depending on R_0, such that

$$q\langle\tilde{a}(x,u,Du), -u\otimes D\eta\rangle\eta^{q-1} \le$$
$$\le \frac{M_3}{8}\eta^q\sum_{i=1}^{n}|u_{x_i}|^{p_i} + \left\{\frac{c_7}{R-\rho}\right\}^{q}\left\{|u|^q+(b_1+b_2)|u|^\gamma+a_1+a_2\right\}$$

and the following estimate of I_3 follows:

$$I_3 \le \frac{M_3}{8}\int_{B_R}\sum_{i=1}^{n}|u_{x_i}|^{p_i}\,\Phi_k^{(h)}(|u|)\,\eta^q\,dx$$
$$+\left\{\frac{c_7}{R-\rho}\right\}^{q}\int_{B_R}\left\{|u|^q+(b_1+b_2)|u|^\gamma+a_1+a_2\right\}\Phi_k^{(h)}(|u|)\,dx. \quad (3.16)$$

Estimate of I_4

Let us now deal with I_4. Using (2.5) we obtain

$$I_4 \le \int_{B_R}\left\{M_2\sum_{i=1}^{n}|u_{x_i}|^{p_i(1-\delta)}|u|+b_3|u|^\gamma+a_3|u|\right\}\Phi_k^{(h)}(|u|)\,\eta^q\,dx$$
$$\le \int_{B_R}\left\{M_2\eta^q\sum_{i=1}^{n}|u_{x_i}|^{p_i(1-\delta)}|u|+b_3|u|^\gamma+a_3|u|\right\}\Phi_k^{(h)}(|u|)\,dx.$$

By the Young inequality

$$M_2\sum_{i=1}^{n}|u_{x_i}|^{p_i(1-\delta)}|u| \le \frac{M_3}{8}\sum_{i=1}^{n}|u_{x_i}|^{p_i}+c_8|u|^{\frac{1}{\delta}},$$

with $c_8 \ge 1$ depending on M_1, M_2, m, q, δ, we get

$$I_4 \le \frac{M_3}{8}\int_{B_R}\sum_{i=1}^{n}|u_{x_i}|^{p_i}\,\Phi_k^{(h)}(|u|)\,\eta^q\,dx + c_8\int_{B_R}\left\{|u|^{\frac{1}{\delta}}+b_3|u|^\gamma+a_3|u|\right\}\Phi_k^{(h)}(|u|)\,dx. \quad (3.17)$$

Conclusion of Step 2.

Collecting (3.11), (3.15), (3.16) and (3.17), the equality (3.8) gives

$$\frac{M_3}{4}\int_{B_R}\sum_{i=1}^{n}|u_{x_i}|^{p_i}\left\{\Phi_k^{(h)}(|u|)-\frac{2}{k}\right\}\eta^q\,dx \le c_3\int_{B_R}\left\{b_2|u|^\gamma+a_2\right\}\Phi_k^{(h)}(|u|)\,dx$$
$$+c_5\max\{\nu,1\}^{\frac{p'}{\epsilon}}\int_{B_R}\left\{b_2|u|^\gamma+a_2\right\}\left\{\Phi_k^{(h)}(|u|)+\frac{2}{k}\right\}dx$$
$$+\left\{\frac{c_7}{R-\rho}\right\}^q\int_{B_R}\left\{|u|^q+(b_1+b_2)|u|^\gamma+a_1+a_2\right\}\Phi_k^{(h)}(|u|)\,dx$$
$$+c_8\int_{B_R}\left\{|u|^{\frac{1}{\delta}}+b_3|u|^\gamma+a_3|u|\right\}\Phi_k^{(h)}(|u|)\,dx.$$

Since the sequence Φ_k is increasing and, by Theorem 2.1, $|u|\in L^{\overline{p}^*}$, we get (3.7) when k goes to ∞.

Step 3. In this step we prove that

$$\left\{\int_{B_\rho} v^{\overline{p}^*(\nu+1)}\,dx\right\}^{\frac{1}{\overline{p}^*}} \le \left\{\frac{c[\nu+1]^{q+\frac{p'}{\epsilon}}}{[R-\rho]^q}\right\}^{\frac{1}{p}}\left\{1+\int_{B_{R_0}} v^{\overline{p}^*}\right\}^{\frac{\tilde{q}-p}{\overline{p}^* p}}\left\{\int_{B_R} v^{\tilde{q}(\nu+1)}\right\}^{\frac{1}{\tilde{q}}}, \tag{3.18}$$

where

$$v:=\max\{|u|,1\},\quad q\le\tilde{q}:=\max\left\{\frac{1}{\delta},\gamma s',qt'\right\}<\overline{p}^*$$

and c is a positive constant depending on the data, R_0 and on the Lebesgue norms of b_i and a_i, $i=1,2,3$. We point out that c is independent of ν, R and ρ. We begin noticing that

$$\int_{B_R}\left|\left[\eta^q(|u|^{\nu+1}+1)\right]_{x_h}\right|^{p_h}dx \le 2^{q-1}[\nu+1]^q\int_{B_R}\left\{|u|^\nu|u_{x_h}|\,\eta^q\right\}^{p_h}dx$$
$$+2^{q-1}\int_{B_R}\left\{\left|\left[\eta^q\right]_{x_h}\right|(|u|^{\nu+1}+1)\right\}^{p_h}dx = J_1+J_2. \tag{3.19}$$

By (3.7) we can estimate J_1 as follows:

$$J_1\le 2^{q-1}[\nu+1]^q\int_{B_R}\left\{|u|^\nu|u_{x_h}|\right\}^{p_h}\eta^q\,dx \le 2^{q-1}[\nu+1]^q\int_{B_R}\sum_{i=1}^{n}|u_{x_i}|^{p_i}|u|^{p_h\nu}\eta^q\,dx$$
$$\le\frac{c_9[\nu+1]^{q+\frac{p'}{\epsilon}}}{[R-\rho]^q}\int_{B_R}\left\{v^{\tilde{q}}+\sum_{j=1}^{3}b_j v^\gamma+a_3 v+a_1+a_2\right\}|u|^{p_h\nu}\,dx \tag{3.20}$$

where we used that $\eta^{qp_h} \le \eta^q$. Moreover, by the assumptions on η, see (3.3), $p_h \le q$ and the Hölder inequality we have

$$J_2 \le \frac{c_{10}}{[R-\rho]^q} \left\{ \int_{B_R} v^{\tilde{q}(\nu+1)}\,dx \right\}^{\frac{p_h}{\tilde{q}}}. \tag{3.21}$$

By (3.19)–(3.21) we obtain

$$\int_{B_R} \left| \left[\eta^q (|u|^{\nu+1}+1) \right]_{x_h} \right|^{p_h} dx \le \frac{c_{10}}{(R-\rho)^q} \left\{ \int_{B_R} v^{\tilde{q}(\nu+1)}\,dx \right\}^{\frac{p_h}{\tilde{q}}}$$
$$+\frac{c_9[\nu+1]^{q+\frac{p'}{\epsilon}}}{[R-\rho]^q} \int_{B_R} \left\{ v^{\tilde{q}} + \sum_{j=1}^{3} b_j v^{\gamma} + a_3 v + a_1 + a_2 \right\} v^{p_h \nu}\,dx. \tag{3.22}$$

We remark that

$$b_1, b_2, b_3 \in L^s(B_{R_0}) \quad \text{with} \quad \left(\frac{\overline{p}^*}{\gamma}\right)' < s \le +\infty \quad \text{and} \quad \gamma s' \le \tilde{q},$$
$$a_1, a_2, a_3 \in L^t(B_{R_0}) \quad \text{with} \quad \left(\frac{\overline{p}^*}{q}\right)' < t \le +\infty \quad \text{and} \quad q t' \le \tilde{q}.$$

Thus, since $p_h \le q \le \tilde{q}$, we can repeatedly use Lemma 3.2 with $\delta = p_h, \sigma = \tilde{q}$ and suitable exponents γ and τ:

$$\int_{B_R} \sum_{j=1}^{3} b_j v^{\gamma+p_h\nu}\,dx$$
$$\le c(\|b_1\|_s, \|b_2\|_s, \|b_3\|_s) \left\{ \int_{B_{R_0}} v^{\tilde{q}} \right\}^{\frac{1}{s'}-\frac{p_h}{\tilde{q}}} \left\{ \int_{B_R} v^{\tilde{q}(\nu+1)} \right\}^{\frac{p_h}{\tilde{q}}} \qquad (\gamma=\gamma,\ \tau=s)$$

$$\int_{B_R} (a_1+a_2) v^{p_h\nu}\,dx$$
$$\le c(\|a_1\|_t, \|a_2\|_t) \left\{ \int_{B_{R_0}} v^{\tilde{q}} \right\}^{\frac{1}{t'}-\frac{p_h}{\tilde{q}}} \left\{ \int_{B_R} v^{\tilde{q}(\nu+1)} \right\}^{\frac{p_h}{\tilde{q}}} \qquad (\gamma=0,\ \tau=t)$$

$$\int_{B_R} a_3 v^{1+p_h\nu}\,dx$$
$$\le c(\|a_3\|_t) \left\{ \int_{B_{R_0}} v^{\tilde{q}} \right\}^{\frac{1}{t'}-\frac{p_h}{\tilde{q}}} \left\{ \int_{B_R} v^{\tilde{q}(\nu+1)} \right\}^{\frac{p_h}{\tilde{q}}} \qquad (\gamma=1,\ \tau=t)$$

and, by Hölder inequality,

$$\int_{B_R} v^{\tilde{q}+p_h\nu}\,dx = \int_{B_R} v^{\tilde{q}-p_h} v^{p_h(\nu+1)}\,dx \le \left\{\int_{B_{R_0}} v^{\tilde{q}}\right\}^{1-\frac{p_h}{\tilde{q}}} \left\{\int_{B_R} v^{\tilde{q}(\nu+1)}\right\}^{\frac{p_h}{\tilde{q}}}.$$

Collecting these inequalities, by $\max\{\frac{1}{t'}, \frac{1}{s'}, 1\} = 1$ there exists c_{11} depending on R_0 and the Lebesgue norms of $a_i, b_i, i = 1, 2, 3$, such that

$$\int_{B_R} \left|\left[\eta^q(|u|^{\nu+1}+1)\right]_{x_h}\right|^{p_h} dx \le \frac{c_{11}[\nu+1]^{q+\frac{p'}{\epsilon}}}{[R-\rho]^q} \left\{1+\int_{B_{R_0}} v^{\tilde{q}}\right\}^{1-\frac{p_h}{\tilde{q}}} \left\{\int_{B_R} v^{\tilde{q}(\nu+1)}\right\}^{\frac{p_h}{\tilde{q}}}.$$

If we choose $c_{12} \ge c_{11}$ large so that $\frac{c_{12}[\nu+1]^{q+\frac{p'}{\epsilon}}}{R_0^q} \ge 1$ the above inequality, together with $p_h \ge p$, implies

$$\left\{\int_{B_R} \left|\left[\eta^q(|u|^{\nu+1}+1)\right]_{x_h}\right|^{p_h} dx\right\}^{\frac{1}{p_h}} \le \left\{\frac{c_{12}[\nu+1]^{q+\frac{p'}{\epsilon}}}{[R-\rho]^q}\right\}^{\frac{1}{p}} \left\{1+\int_{B_{R_0}} v^{\tilde{q}}\right\}^{\frac{\tilde{q}-p}{p\tilde{q}}} \left\{\int_{B_R} v^{\tilde{q}(\nu+1)}\right\}^{\frac{1}{\tilde{q}}}.$$

This inequality holds for every $h = 1, \ldots, n$; therefore, we get

$$\prod_{h=1}^{n} \left\{\int_{B_R} \left|\left[\eta^q(|u|^{\nu+1}+1)\right]_{x_h}\right|^{p_h} dx\right\}^{\frac{1}{p_h}} \le \left\{\frac{c_{12}[\nu+1]^{q+\frac{p'}{\epsilon}}}{[R-\rho]^q}\right\}^{\frac{n}{p}} \left\{1+\int_{B_{R_0}} v^{\tilde{q}}\right\}^{n\frac{\tilde{q}-p}{p\tilde{q}}} \left\{\int_{B_R} v^{\tilde{q}(\nu+1)}\right\}^{\frac{n}{\tilde{q}}}.$$

By Theorem 2.1 we get

$$\left\{\int_{B_\rho} v^{\overline{p}^*(\nu+1)}\,dx\right\}^{\frac{1}{\overline{p}^*}} \le \left\{\int_{B_\rho}\left\{|u|^{\nu+1}+1\right\}^{\overline{p}^*}\,dx\right\}^{\frac{1}{\overline{p}^*}}$$
$$\le \left\{\int_{B_R}\left\{\eta^q\left(|u|^{\nu+1}+1\right)\right\}^{\overline{p}^*}\,dx\right\}^{\frac{1}{\overline{p}^*}}$$
$$\le \left\{\frac{c_{13}[\nu+1]^{q+\frac{p'}{\epsilon}}}{[R-\rho]^q}\right\}^{\frac{1}{p}}\left\{1+\int_{B_{R_0}} v^{\tilde{q}}\right\}^{\frac{\tilde{q}-p}{p\tilde{q}}}\left\{\int_{B_R} v^{\tilde{q}(\nu+1)}\right\}^{\frac{1}{\tilde{q}}} \tag{3.23}$$

and by the Hölder inequality we get the conclusion (3.18).

Step 4. We prove the boundedness of u and the estimate (2.9), using Moser's iteration technique. For all $h\in\mathbb{N}$ define $\nu_h=-1+\left(\frac{\overline{p}^*}{\tilde{q}}\right)^h$, $\rho_h=R_0/2+R_0/2^{h+1}$ and $R_h=R_0/2+R_0/2^h$. Notice that $\rho_h=R_{h+1}$ and $\overline{p}^*(\nu_h+1)=\tilde{q}(\nu_{h+1}+1)$; therefore, by (3.18), replacing ν, R and ρ with ν_h, R_h and ρ_h, respectively, we have that $v\in L^{\tilde{q}(\nu_h+1)}(B_{R_h})$ implies $v\in L^{\tilde{q}(\nu_{h+1}+1)}(B_{R_{h+1}})$. Precisely,

$$\|v\|^{\nu_h+1}_{L^{\tilde{q}(\nu_{h+1}+1)}(B_{R_{h+1}})}=\left\{\int_{B_{R_{h+1}}} v^{\tilde{q}(\nu_{h+1}+1)}\,dx\right\}^{\frac{1}{\overline{p}^*}}=\left\{\int_{B_{R_{h+1}}} v^{\overline{p}^*(\nu_h+1)}\,dx\right\}^{\frac{1}{\overline{p}^*}}$$
$$\le\left\{\frac{c_{14}2^{(h+1)q}\left[\frac{\overline{p}^*}{\tilde{q}}\right]^{h\left(q+\frac{p'}{\epsilon}\right)}}{R_0^q}\right\}^{\frac{1}{p}}\left\{1+\int_{B_{R_0}} v^{\overline{p}^*}\right\}^{\frac{\tilde{q}-p}{\overline{p}^*p}}\left\{\int_{B_{R_h}} v^{\tilde{q}(\nu_h+1)}\,dx\right\}^{\frac{1}{\tilde{q}}}$$
$$\le [c_{15}]^h\left\{1+\int_{B_{R_0}} v^{\overline{p}^*}\right\}^{\frac{\tilde{q}-p}{\overline{p}^*p}}\|v\|^{\nu_h+1}_{L^{\tilde{q}(\nu_h+1)}(B_{R_h})}. \tag{3.24}$$

Thus,

$$\|v\|_{L^{\tilde{q}(\nu_{h+1}+1)}(B_{R_{h+1}})}\le[c_{15}]^{h\left(\frac{\tilde{q}}{\overline{p}^*}\right)^h}\left\{1+\int_{B_{R_0}} v^{\overline{p}^*}\right\}^{\frac{\tilde{q}-p}{\overline{p}^*p}\left(\frac{\tilde{q}}{\overline{p}^*}\right)^h}\|v\|_{L^{\tilde{q}(\nu_h+1)}(B_{R_h})}.$$

Taking into account that $\tilde{q}(\nu_1+1)=\overline{p}^*$ and that $\frac{\tilde{q}-p}{\overline{p}^* p}\sum_{h=1}^{\infty}\left(\frac{\tilde{q}}{\overline{p}^*}\right)^h=\frac{\tilde{q}}{\overline{p}^* p}\frac{\tilde{q}-p}{\overline{p}^*-\tilde{q}}$, an iterated use of (3.24) implies

$$\|v\|_{L^\infty(B_{R_0/2})}\le c_{16}\left\{1+\int_{B_{R_0}} v^{\overline{p}^*}\right\}^{\frac{\tilde{q}}{\overline{p}^* p}\frac{\tilde{q}-p}{\overline{p}^*-\tilde{q}}}\|v\|_{L^{\overline{p}^*}(B_{R_0})}.$$

Therefore, since $v=\max\{|u|,1\}$ then

$$\sup_{B_{R_0/2}(x_0)}|u|\le c_{17}\left\{\int_{B_{R_0}}(|u|+1)^{\overline{p}^*}\,dx\right\}^{\frac{1+\theta}{\overline{p}^*}}$$

with $\theta=\frac{\tilde{q}}{p}\frac{\tilde{q}-p}{\overline{p}^*-\tilde{q}}$ and we get the thesis. □

4. Boundedness under p,q-growth

In this section we deal with the system (2.1), assuming a suitable p,q-growth with $1<p\le q$. For the sake of simplicity we use the following notations

$$a=(a_i^\alpha)_{\substack{i=1,\dots,n\\ \alpha=1,\dots,m}},\quad a_i^\alpha(x,u,\xi):=\sum_{j=1}^n a_{ij}(x,u,\xi)\xi_j^\alpha\quad \forall i=1,\dots,n,\quad \alpha=1,\dots,m$$

and similarly for $b=(b_i^\alpha)$. We assume that the following inequalities hold for a.e. $x\in\Omega$ and for every $u\in\mathbb{R}^m$, $\xi\in\mathbb{R}^{m\times n}$, $\lambda=(\lambda_i)_{i=1,\dots,n}\in\mathbb{R}^n$

(A1) (ellipticity condition)

$$\sum_{i,j=1}^n a_{ij}(x,u,\xi)\lambda_i\lambda_j\ge M_1\sum_{i=1}^n\lambda_i^2|\xi_i|^{p-2}, \tag{4.1}$$

(A2) (growth conditions)

$$|a(x,u,\xi)|\le M_2\left\{|\xi|^{q-1}+b_1(x)|u|^{\frac{\gamma}{p'}}+a_1(x)\right\}, \tag{4.2}$$

$$|b(x,u,\xi)|\le M_2\left\{|\xi|^{(p-1)(1-\epsilon)}+b_2(x)|u|^{\frac{\gamma}{p'}}+a_2(x)\right\}, \tag{4.3}$$

$$|f(x,u,\xi)|\le M_2\left\{|\xi|^{p(1-\delta)}+b_3(x)|u|^{\gamma-1}+a_3(x)\right\}, \tag{4.4}$$

for some positive constants M_1, M_2, $1<\gamma<p^*$, $0<\epsilon<1$, $\frac{1}{p^*}<\delta<1$ and, for $i=1,2,3$,

$$b_1^{p'},b_2^{p'},b_3\in L^s_{\rm loc}(\Omega)\text{ with }\left(\frac{p^*}{\gamma}\right)'<s\le+\infty,$$

$$a_1^{p'},a_2^{p'},a_3\in L^t_{\rm loc}(\Omega)\quad\text{with }\left(\frac{p^*}{p}\right)'<t\le+\infty.$$

Theorem 4.1. *Let (A1) and (A2) hold. Assume also that either*

$$\langle a(x,u,\xi)-a(x,u,\eta),\xi-\eta\rangle \geq 0 \quad \forall \xi,\eta \in \mathbb{R}^{n\times m} \text{ and } q < p\frac{n-1}{n-p} \text{ if } p<n, \tag{4.5}$$

or there exists a Carathéodory function $A : \Omega \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}_+$, $t \to A(x,u,t)t$ increasing, such that

$$a_{ij}(x,u,\xi) = A(x,u,|\xi|)\delta_{ij} \quad \forall i,j=1,\dots,n, \quad \forall \xi \in \mathbb{R}^{n\times m} \text{ and } q<p^* \text{ if } p<n. \tag{4.6}$$

Then any weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ to (2.1) is locally bounded. Moreover, for every $B_R(x_0) \Subset \Omega$ there exists a constant $c>0$ such that

$$\sup_{B_{R/2}(x_0)} |u| \leq c \left\{ \int_{B_R(x_0)} (|u|+1)^{p^*}\, dx \right\}^{\frac{1+\theta}{p^*}}, \tag{4.7}$$

where $\theta = \frac{\tilde q}{p}\frac{\tilde q - p}{p^*-\tilde q}$ with $\tilde q = \max\left\{q\frac{p'}{q'}, \frac{1}{\delta}, \gamma s', pt'\right\}$ if (4.5) holds, otherwise $\tilde q = \max\left\{q, \frac{1}{\delta}, \gamma s', pt'\right\}$ if (4.6) holds.

Remark 4.2. Inequality (4.1) implies

$$\langle a(x,u,\xi),\xi\rangle = \sum_{i,j=1}^{n}\sum_{\alpha=1}^{m} a_{ij}(x,u,\xi)\xi_j^\alpha \xi_i^\alpha \geq M_1 n^{1-p}|\xi|^p \quad \forall \xi \in \mathbb{R}^{m\times n}. \tag{4.8}$$

Notice that if (4.6) holds then

$$\langle a(x,u,\xi),\eta\rangle = A(x,u,|\xi|)\langle \xi,\eta\rangle \quad \forall \xi,\eta \in \mathbb{R}^{m\times n}, \tag{4.9}$$

so the inequality (4.8) is equivalent to

$$A(x,u,|\xi|)|\xi|^2 \geq M_1 n^{1-p}|\xi|^p. \tag{4.10}$$

Moreover, under the structure assumption (4.6) we have that the growth condition (4.2) is equivalent to

$$A(x,u,|\xi|)|\xi| \leq M_2\left\{|\xi|^{q-1} + b_1(x)|u|^{\frac{\gamma}{p'}} + a_1(x)\right\}. \tag{4.11}$$

By the monotonicity assumption on A it is easy to prove that

$$A(x,u,|\xi|)\langle\xi,\eta\rangle \leq A(x,u,|\xi|)|\xi||\eta| \leq A(x,u,|\xi|)|\xi|^2 + A(x,u,|\eta|)|\eta|^2 \tag{4.12}$$

or equivalently

$$\sum_{i,j=1}^{n}\sum_{\alpha=1}^{m} a_{ij}(x,u,\xi)\xi_j^\alpha\eta_i^\alpha \leq \sum_{i,j=1}^{n}\sum_{\alpha=1}^{m} a_{ij}(x,u,\xi)\xi_i^\alpha\xi_j^\alpha + \sum_{i,j=1}^{n}\sum_{\alpha=1}^{m} a_{ij}(x,u,\eta)\eta_i^\alpha\eta_j^\alpha \quad \forall \xi,\eta \in \mathbb{R}^{m\times n}.$$

Proof of Theorem 4.1 The scheme of the proof is analogous to the proof of Theorem 2.3. Also in this case, without loss of generality we assume that *the functions* $a_i, b_i, i = 1, 2, 3,$ *in* (4.2)–(4.4) *are a.e. greater than or equal to* 1. We split the proof into steps.

Step 1. We define a sequence of test functions $(\varphi_k)_k$. Consider $B_{R_0}(x_0) \Subset \Omega$, $0 < r \leq R \leq R_0$, $\eta \in C_c^\infty(B_R)$ and the increasing sequence of C^1 functions $g_k : \mathbb{R}_+ \to \mathbb{R}_+$ as in the proof of Theorem 2.3. Fixed $k \in \mathbb{N}$ and $\nu \geq 0$, let $\Phi_{k,\nu} : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function $\Phi_{k,\nu}(t) := g_k(t^{p\nu})$. Notice that as in (3.5) we have

$$(\Phi_{k,\nu})'(t)t \leq p\nu \left\{ \Phi_{k,\nu}(t) + \frac{2}{k} \right\}. \tag{4.13}$$

Finally, define $\varphi_{k,\nu}(x) := \Phi_{k,\nu}(|u(x)|)u(x)[\eta(x)]^\mu$ for every $x \in B_{R_0}$, with $\mu = q\frac{p'}{q'} \geq q$. Notice that $\varphi_{k,\nu}$ is in $W^{1,q}(B_{R_0}; \mathbb{R}^m)$, supp $\varphi \Subset B_R$.

Step 2. We aim to prove that for every $\nu \geq 0$, $k \in \mathbb{N}$,

$$\begin{aligned}
&\frac{5}{8} \int_{B_R} \langle a(x, u, Du), Du\rangle \left\{ \Phi_k(|u|) - \frac{1}{5k} \right\} \eta^\mu \, dx \\
&\quad \leq \mu \int_{B_R} \langle \tilde{a}(x, u, Du), -u \otimes D\eta\rangle \Phi_k(|u|)\, \eta^{\mu-1} \, dx \\
&\qquad + c \max\{\nu, 1\}^{\frac{p'}{\epsilon}} \int_{B_R} \left\{ |u|^{\frac{1}{\delta}} + \left(b_2^{p'} + b_3 \right) |u|^\gamma + a_3|u| + a_2^{p'} \right\} \left\{ \Phi_k(|u|) + \frac{2}{k} \right\} dx,
\end{aligned} \tag{4.14}$$

where we used the notations in Step 2 of the proof of Theorem 2.3, see in particular (3.9). Using φ_k as a test function in (2.8) we get

$$\begin{aligned}
I_1 + I_2 &:= \int_{B_R} \langle \tilde{a}(x, u, Du), Du\rangle \Phi_k(|u|)\, \eta^\mu \, dx \\
&\quad + \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha,\beta=1}^{m} \tilde{a}_i^\alpha(x, u, Du)\, u^\alpha \frac{u^\beta}{|u|} u_{x_i}^\beta \, (\Phi_k)'(|u|)\, \eta^\mu \, dx \\
&= \mu \int_{B_R} \langle \tilde{a}(x, u, Du), -u \otimes D\eta\rangle \Phi_k(|u|)\, \eta^{\mu-1} \, dx \\
&\quad - \int_{B_R} \langle f(x, u, Du), u\rangle \Phi_k(|u|)\, \eta^\mu \, dx =: I_3 + I_4.
\end{aligned} \tag{4.15}$$

Now, we separately consider and estimate I_i, $i = 1, 2, 4$.

Estimate of I_1

By (4.3) the Young inequality (applied first with exponent p and then with exponent $\frac{1}{1-\epsilon}$) and (4.8) we have that for some positive $\tau \ll 1$

$$\begin{aligned}|\langle b(x,u,Du),Du\rangle| &\le M_2\left\{|Du|^{(p-1)(1-\epsilon)}+b_2|u|^{\frac{\gamma}{p'}}+a_2(x)\right\}|Du|\\ &\le \tau|Du|^p+c_\tau\left\{|Du|^{p(1-\epsilon)}+b_2^{p'}(x)|u|^\gamma+a_2^{p'}(x)\right\}\\ &\le \frac{M_1}{4n^{p-1}}|Du|^p+c_1\left\{b_2^{p'}(x)|u|^\gamma+a_2^{p'}(x)+1\right\}\\ &\le \frac{1}{4}\langle a(x,u,Du),Du\rangle+c_2\left\{b_2^{p'}(x)|u|^\gamma+a_2^{p'}(x)\right\}\end{aligned} \tag{4.16}$$

with c_2 depending also on ϵ. Thus,

$$I_1 \ge \frac{3}{4}\int_{B_R}\langle a(x,u,Du),Du\rangle\Phi_k(|u|)\eta^\mu\,dx - c_2\int_{B_R}\left\{b_2^{p'}|u|^\gamma+a_2^{p'}\right\}\Phi_k(|u|)\,dx. \tag{4.17}$$

Estimate of I_2

As in the proof of Theorem 2.3, using (3.12) and (4.13) we have

$$\begin{aligned}I_2 &\ge \int_{B_R}\sum_{i=1}^{n}\sum_{\alpha,\beta=1}^{m} b_i^\alpha(x,u,Du)\,u^\alpha\frac{u^\beta}{|u|}\,u^\beta_{x_i}\,(\Phi_k)'(|u|)\,\eta^\mu\,dx\\ &\ge -\int_{B_R} p\nu|b(x,u,Du)||Du|\left\{\Phi_k(|u|)+\frac{2}{k}\right\}\eta^\mu\,dx.\end{aligned}$$

Reasoning as in (3.14) and in (4.16), by (4.3) and the Young inequality it follows that

$$p\nu|b(x,u,Du)||Du| \le \frac{1}{16}\langle a(x,u,Du),Du\rangle + c_3\max\{\nu,1\}^{\frac{p'}{\epsilon}}\left\{b_2^{p'}(x)|u|^\gamma+a_2^{p'}(x)\right\}.$$

Thus, we obtain

$$\begin{aligned}I_2 \ge &-\frac{1}{16}\int_{B_R}\langle a(x,u,Du),Du\rangle\left\{\Phi_k(|u|)+\frac{2}{k}\right\}\eta^\mu\,dx\\ &-c_3\max\{\nu,1\}^{\frac{p'}{\epsilon}}\int_{B_R}\left\{b_2^{p'}(x)|u|^\gamma+a_2^{p'}(x)\right\}\left\{\Phi_k(|u|)+\frac{2}{k}\right\}dx.\end{aligned} \tag{4.18}$$

Estimate of I_4

Let us now deal with I_4. Using (4.4) we obtain

$$I_4 \le M_2\int_{B_R}\left\{\eta^\mu|Du|^{p(1-\delta)}|u|+b_3|u|^\gamma+a_3|u|\right\}\Phi_k(|u|)\,dx.$$

Now, let us estimate the right-hand side using the Young inequality and (4.8). We have that there exists c_4, depending on M_1, M_2, n, p, δ, such that for a.e. x

$$M_2|Du|^{p(1-\delta)}|u| \le \frac{M_1}{16n^{p-1}}|Du|^p + c_4|u|^{\frac{1}{\delta}} \le \frac{1}{16}\langle a(x,u,Du), Du\rangle + c_4|u|^{\frac{1}{\delta}}.$$

Therefore,

$$I_4 \le \frac{1}{16}\int_{B_R}\langle a(x,u,Du), Du\rangle \Phi_k(|u|)\eta^\mu\,dx + c_4\int_{B_R}\left\{|u|^{\frac{1}{\delta}} + b_3|u|^\gamma + a_3|u|\right\}\Phi_k(|u|)\,dx. \tag{4.19}$$

Collecting (4.15), (4.17), (4.18) and (4.19) we get

$$\begin{aligned}&\frac{5}{8}\int_{B_R}\langle a(x,u,Du), Du\rangle\left\{\Phi_k(|u|) - \frac{1}{5k}\right\}\eta^\mu\,dx\\ &\le I_3 + c_5\max\{\nu,1\}^{\frac{p'}{\epsilon}}\int_{B_R}\left\{|u|^{\frac{1}{\delta}} + \left(b_2^{p'} + b_3\right)|u|^\gamma + a_3|u| + a_2^{p'}\right\}\left\{\Phi_k(|u|) + \frac{2}{k}\right\}dx\end{aligned} \tag{4.20}$$

and the claim follows.

Step 3. In this step we provide two different estimates of I_3 depending on whether (4.5) or (4.6) holds true. We recall that

$$I_3 = \mu\int_{B_R}\langle a(x,u,Du) + b(x,u,Du), -u\otimes D\eta\rangle\Phi_k(|u|)\,\eta^{\mu-1}\,dx.$$

Estimate of I_3 under assumption (4.5) For a.e. $x \in B_{R_0}\cap\{\eta \ne 0\}$ by (4.5) with $\xi = Du(x)$ and $\eta = -8\mu u(x)\otimes\frac{D\eta(x)}{\eta(x)}$, we obtain

$$\begin{aligned}\mu\,\langle a(x,u,Du), -u\otimes D\eta\rangle\,\eta^{\mu-1} &= \frac{\eta^\mu}{8}\left\langle a(x,u,Du), -8\mu u\otimes\frac{D\eta}{\eta}\right\rangle\\ &\le \frac{\eta^\mu}{8}\langle a(x,u,Du), Du\rangle + \frac{\eta^\mu}{8}\left\langle a\left(x,u,-8\mu u\otimes\frac{D\eta}{\eta}\right), -8\mu u\otimes\frac{D\eta}{\eta}\right\rangle\\ &\quad-\frac{\eta^\mu}{8}\left\langle a\left(x,u,-8\mu u\otimes\frac{D\eta}{\eta}\right), Du\right\rangle.\end{aligned} \tag{4.21}$$

By (4.2) and the assumptions on η, see (3.3),

$$\begin{aligned}&\frac{\eta^\mu}{8}\left\langle a\left(x,u,-8\mu u\otimes\frac{D\eta}{\eta}\right), -8\mu u\otimes\frac{D\eta}{\eta}\right\rangle\\ &\le 8^{q-1}\mu^q\eta^\mu M_2\left\{\left|u\otimes\frac{D\eta}{\eta}\right|^q + \left(b_1|u|^{\frac{\gamma}{p'}} + a_1\right)\left|u\otimes\frac{D\eta}{\eta}\right|\right\}\\ &\le \frac{c_6}{(R-\rho)^q}\left\{|u|^q + b_1|u|^{\frac{\gamma}{p'}+1} + a_1|u|\right\} \le \frac{c_7}{(R-\rho)^q}\left\{|u|^q + |u|^p + b_1^{p'}|u|^\gamma + a_1^{p'}\right\}\end{aligned} \tag{4.22}$$

with c_7 depending on M_2, p, q and R_0. Notice that we used that $\mu \geq q$ and $\frac{1}{(R-\rho)^a} \leq \frac{R_0^{b-a}}{(R-\rho)^b}$ if $0 < a < b$. Let us now estimate the last term in (4.21) using (4.2) once more. We get

$$-\frac{\eta^\mu}{8}\left\langle a\left(x, u, -8\mu u \otimes \frac{D\eta}{\eta}\right), Du\right\rangle \leq \frac{c_8}{(R-\rho)^{q-1}}\eta^{\mu-q+1}|Du|\left\{|u|^{q-1} + b_1|u|^{\frac{\gamma}{p'}} + a_1\right\} \tag{4.23}$$

with c_8 depending on M_2, p, q. To estimate the term at the right-hand side of (4.23) we use the Young inequality and (4.8) (notice that $\eta^{\mu-q+1} = \eta^{\frac{\mu}{p}}\eta^{\mu\frac{p-1}{p}-q+1}$ and that $\mu\frac{p-1}{p} - q + 1 \geq 0$). Thus, for a.e. x

$$\begin{aligned}
&\left\{\eta^{\frac{\mu}{p}}|Du|\right\}\left\{\frac{c_8}{(R-\rho)^{q-1}}\eta^{\mu\frac{p-1}{p}-q+1}\left(|u|^{q-1} + b_1|u|^{\frac{\gamma}{p'}} + a_1\right)\right\} \\
&\leq \frac{M_1}{16n^{p-1}}\eta^\mu|Du|^p + \frac{c_9}{(R-\rho)^{(q-1)p'}}\left\{|u|^{(q-1)p'} + b_1^{p'}|u|^\gamma + a_1^{p'}\right\} \\
&\leq \frac{1}{16}\eta^\mu\langle a(x, u, Du), Du\rangle + \frac{c_9}{(R-\rho)^{(q-1)p'}}\left\{|u|^{(q-1)p'} + b_1^{p'}|u|^\gamma + a_1^{p'}\right\}.
\end{aligned} \tag{4.24}$$

As far as the integral

$$\mu\int_{B_R}\langle b(x, u, Du), -u \otimes D\eta\rangle\Phi_k(|u|)\,\eta^{\mu-1}\,dx$$

is concerned, reasoning as in (4.16) and using (4.3) and (4.8) we get

$$\begin{aligned}
&\mu\langle b(x, u, Du), -u \otimes D\eta\rangle\eta^{\mu-1} \\
&\leq \frac{M_1}{16n^{p-1}}\eta^\mu|Du|^p + \frac{c_{10}}{(R-\rho)^p}\left\{|u|^p + b_2^{p'}|u|^\gamma + a_2^{p'}\right\} \\
&\leq \frac{1}{16}\eta^\mu\langle a(x, u, Du), Du\rangle + \frac{c_{10}}{(R-\rho)^p}\left\{|u|^p + b_2^{p'}|u|^\gamma + a_2^{p'}\right\}.
\end{aligned} \tag{4.25}$$

Collecting (4.21)–(4.25) we get the following estimate of I_3 ($(q-1)p' \geq q \geq p$)

$$\begin{aligned}
I_3 &\leq \frac{1}{4}\int_{B_R}\langle a(x, u, Du), Du\rangle\Phi_k(|u|)\,\eta^\mu\,dx \\
&\quad+\frac{c_{11}}{(R-\rho)^{(q-1)p'}}\int_{B_R}\left\{|u|^{(q-1)p'} + (b_1^{p'} + b_2^{p'})|u|^\gamma + a_1^{p'} + a_2^{p'}\right\}\Phi_k(|u|)\,dx.
\end{aligned} \tag{4.26}$$

Estimate of I_3 under assumption (4.6) By definition of $\tilde{a}(x,u,Du)$ in (3.9) and (4.9)

$$\mu\langle\tilde{a}(x,u,Du),-u\otimes D\eta\rangle\eta^{\mu-1}\le\mu A(x,u,|Du|)\langle Du,-u\otimes D\eta\rangle\eta^{\mu-1}$$
$$+\mu|b(x,u,Du)||u\otimes D\eta|\eta^{\mu-1}. \quad (4.27)$$

By applying (4.12), with $\xi=Du(x)$ and $\eta=-8\mu u(x)\otimes\frac{D\eta(x)}{\eta(x)}$,

$$\mu A(x,u,|Du|)\langle Du,-u\otimes D\eta\rangle\eta^{\mu-1}=\frac{\eta^\mu}{8}A(x,u,|Du|)\left\langle Du,-8\mu u\otimes\frac{D\eta(x)}{\eta(x)}\right\rangle$$
$$\le\frac{\eta^\mu}{8}\left\{A(x,u,|Du|)|Du|^2+A\left(x,u,\left|8\mu u\otimes\frac{D\eta}{\eta}\right|\right)\left|8\mu u\otimes\frac{D\eta}{\eta}\right|^2\right\}. \quad (4.28)$$

Now, by (4.11) and the Young inequality

$$\frac{\eta^\mu}{8}A\left(x,u,\left|8\mu u\otimes\frac{D\eta}{\eta}\right|\right)\left|8\mu u\otimes\frac{D\eta}{\eta}\right|^2$$
$$\le\mu\eta^\mu M_2\left\{(8\mu)^{q-1}\left|u\otimes\frac{D\eta}{\eta}\right|^{q-1}+a_1(x)|u|^{\frac{\gamma}{p'}}+a_1(x)\right\}\left|u\otimes\frac{D\eta}{\eta}\right|$$
$$\le\frac{c_{12}}{(R-\rho)^q}|u|^q+\frac{c_{12}}{R-\rho}\left\{b_1(x)|u|^{\frac{\gamma}{p'}+1}+a_1(x)|u|\right\}$$
$$\le\frac{c_{13}}{(R-\rho)^q}\left\{|u|^q+|u|^p+b_1^{p'}(x)|u|^\gamma+a_1^{p'}(x)\right\} \quad (4.29)$$

with c_{13} depending on M_2, p,q and R_0. Taking into account (4.3) and reasoning as in (4.16) we get

$$\mu|b(x,u,Du)||u\otimes D\eta|\eta^{\mu-1}\le\frac{M_1\eta^\mu}{8n^{p-1}}|Du|^p+\frac{c_{14}}{(R-\rho)^p}\left\{|u|^p+b_2^{p'}|u|^\gamma+a_2^{p'}\right\}$$

which implies, by using (4.9) and (4.10),

$$\mu|b(x,u,Du)||u\otimes D\eta|\eta^{\mu-1}\le\frac{\eta^\mu}{8}\langle a(x,u,Du),Du\rangle$$
$$+\frac{c_{15}}{(R-\rho)^{(q-1)p'}}\left\{|u|^p+b_2^{p'}|u|^\gamma+a_2^{p'}\right\}. \quad (4.30)$$

Collecting (4.27)–(4.30) we get the following estimate of I_3

$$I_3\le\frac{1}{4}\int_{B_R}\langle a(x,u,Du),Du\rangle\Phi_k(|u|)\,\eta^\mu\,dx$$
$$+\frac{c_{16}}{(R-\rho)^{(q-1)p'}}\int_{B_R}\left\{|u|^q+|u|^p+\sum_{i=1}^2 b_i^{p'}|u|^\gamma+\sum_{i=1}^2 a_i^{p'}\right\}\Phi_k(|u|)\,dx,$$
$$(4.31)$$

which implies an inequality analogous to (4.26):

$$I_3 \le \frac{1}{4}\int_{B_R} \langle a(x,u,Du), Du\rangle \Phi_k(|u|)\,\eta^\mu\,dx + \frac{c_{17}}{(R-\rho)^{(q-1)p'}}\int_{B_R}\left\{|u|^q + (b_1^{p'}+b_2^{p'})|u|^\gamma + a_1^{p'} + a_2^{p'}\right\}\Phi_k(|u|)\,dx. \tag{4.32}$$

Eventually, by (4.14), (4.26) and (4.32) we have

$$\frac{3}{8}\int_{B_R}\langle a(x,u,Du), Du\rangle\left\{\Phi_k(|u|) - \frac{1}{3k}\right\}\eta^\mu\,dx \le \frac{c_{18}[\nu+1]^{\frac{p'}{\epsilon}}}{(R-\rho)^{(q-1)p'}}\int_{B_R}\left\{|u|^\theta + |u|^{\frac{1}{\delta}} + \left(\sum_{i=1}^2 b_i^{p'} + b_3\right)|u|^\gamma + a_3|u| + \sum_{i=1}^2 a_i^{p'}\right\}\left\{\Phi_k(|u|) + \frac{2}{k}\right\}dx,$$

where

$$\theta := \begin{cases} (q-1)p' = q\frac{p'}{q'} & \text{if (4.5) holds} \\ q & \text{if (4.6) holds.}\end{cases}$$

Since $\Phi_k(|u|) \to |u|^{p\nu}$ as k go to $+\infty$, passing to the limit and using (4.8) we obtain

$$\int_{B_R}|Du|^p|u|^{p\nu}\eta^\mu\,dx \le \frac{c_{19}[\nu+1]^{\frac{p'}{\epsilon}}}{(R-\rho)^{(q-1)p'}}\int_{B_R}\left\{|u|^\theta + |u|^{\frac{1}{\delta}} + \left(\sum_{i=1}^2 b_i^{p'} + b_3\right)|u|^\gamma + a_3|u| + \sum_{i=1}^2 a_i^{p'}\right\}|u|^{p\nu}\,dx \tag{4.33}$$

where c is a suitable positive constant depending on the data and R_0, but not on ν.

Step 4. In this step we conclude. We follow the scheme of Steps 3 and 4 of the proof of Theorem 2.3, taking into account that now $p_h = p$ and that a_i and b_i are now replaced by $a_i^{p'}$ and $b_i^{p'}$, $i = 1, 2$, respectively. We limit ourselves to outline the main first inequalities. First we estimate the left-hand side in (4.33) proceeding as in (3.19):

$$\begin{aligned}
&\int_{B_R} \left| D\left[\eta^{\frac{\mu}{p}}\left(|u|^{\nu+1}+1\right)\right]\right|^p dx \\
&\quad \le c_{20}\int_{B_R} \eta^{\mu-p}|D\eta|^p[|u|^{\nu+1}+1]^p\,dx + c_{20}[\nu+1]^p\int_{B_R}|Du|^p|u|^{p\nu}\eta^\mu\,dx \\
&\quad \le \frac{c_{21}}{(R-\rho)^p}\int_{B_R}[\max\{|u|,1\}]^{p+p\nu}\,dx + c_{20}[\nu+1]^p\int_{B_R}|Du|^p|u|^{p\nu}\eta^\mu\,dx \\
&\quad \le \frac{c_{22}}{(R-\rho)^{(q-1)p'}}\int_{B_R}[\max\{|u|,1\}]^{p+p\nu}\,dx + c_{20}[\nu+1]^p\int_{B_R}|Du|^p|u|^{p\nu}\eta^\mu\,dx
\end{aligned}$$

with c_{20} depending only on p and c_{22} depending also on q and R_0. Then, defining $v := \max\{|u|, 1\}$ and using the classical Sobolev imbedding theorem and (4.33) we get

$$\begin{aligned}
&\left(\int_{B_\rho} v^{p^*(\nu+1)}\,dx\right)^{\frac{p}{p^*}} \le \left(\int_{B_R}\left|\eta^{\frac{\mu}{p}}\left(|u|^{\nu+1}+1\right)\right|^{p^*}dx\right)^{\frac{p}{p^*}} \\
&\quad \le c_{23}\int_{B_R}\left|D\left[\eta^{\frac{\mu}{p}}\left(|u|^{\nu+1}+1\right)\right]\right|^p dx \le \frac{c_{24}[\nu+1]^{p+\frac{p'}{\epsilon}}}{(R-\rho)^{(q-1)p'}} \\
&\quad \int_{B_R}\left\{|v|^{\theta+p\nu}+|v|^{\frac{1}{\delta}+p\nu}+\left(\sum_{i=1}^2 b_i^{p'}+b_3\right)v^{\gamma+p\nu}+a_3v^{1+p\nu}+\sum_{i=1}^2 a_i^{p'}v^{p\nu}\right\}dx.
\end{aligned}$$

By Lemma 3.2 with $\delta = p, \sigma = \tilde{q}$ and a suitable choice of γ and τ, taking into account that $\max\{\frac{1}{t'}, \frac{1}{s'}, 1\} = 1$, we obtain

$$\left(\int_{B_\rho} v^{p^*(\nu+1)}\,dx\right)^{\frac{1}{p^*}} \le \frac{c_{25}\,[\nu+1]^{1+\frac{1}{(p-1)\epsilon}}}{(R-\rho)^{\frac{q-1}{p-1}}}\left\{1+\int_{B_{R_0}} v^{\tilde{q}}\,dx\right\}^{\frac{\tilde{q}-p}{p\tilde{q}}}\left\{\int_{B_R} v^{\tilde{q}(\nu+1)}\right\}^{\frac{1}{\tilde{q}}}$$

the analogue of (3.23). Taking into account that $\tilde{q} < p^*$, from now on the proof goes as in the previous section. □

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