Tensor decomposition and tensor rank from the point of view of Classical Algebraic Geometry

RTG Workshop

Tensors and their Geometry in High Dimensions

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Content of the three talks

- **Wednesday** Rank and symmetric rank. Tensor decomposition. Classical apolarity and Sylvester algorithm. Secant varieties. Clebsch quartics. Sum of squares, sum of k-th powers.
- **Thursday** Cases where classical apolarity fails. Vector bundles and non abelian apolarity. Equations for secant varieties, infinitesimal criterion for smoothness. Scorza map and Lüroth quartics.
- **Friday** Actions of $SL(2)$. The complexity of Matrix Multiplication Algorithm.

The rank of a matrix

Let A be a $m \times n$ matrix with entries in a field K.

Basic Fact

A has rank one $\iff$ there exist nonzero $x \in K^m$, $y \in K^n$ such that $A = x \cdot y^t$, that is $a_{ij} = x_i y_j$

Proposition

A has rank $\leq r \iff$ there exist A_i such that rank $A_i = 1$ and $A = A_1 + \ldots + A_r$.

Proof $\Longleftarrow$ trivial

$\Longrightarrow$ There are $G \in GL(m)$, $H \in GL(n)$ such that

$$GAH = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & 1 & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} =$$

end of the proof

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \ldots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \ldots & 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & 1 & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} =$$

$= GA_1H + GA_2H + \ldots + GA_rH$

Then $A = A_1 + \ldots + A_r$.

Gaussian elimination

Operation $A \mapsto GAH$ where $G \in GL(m)$, $H \in GL(n)$ is essentially Gaussian elimination (on both rows and columns).
It reduces every matrix to its canonical form where there are r entries equal to 1 on the diagonal, otherwise zero.

Not uniqueness

The expression $A = \sum_{i=1}^{r} A_i$ where $\mathrm{rank}\ A_i = 1$ is far to be unique.

The reason is that there are infinitely many G, H such that

$$GAH = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & 1 & \vdots \\ 0 & & \cdots & 0 \end{bmatrix}$$

The only invariant is the number of summands, the individual summands are not uniquely determined.

Symmetric matrices

Let A be a $n \times n$ symmetric matrix with entries in the field $K = \mathbb{R}$ or $\mathbb{C}$.

Basic Fact

A has rank one $\iff$ there exist nonzero $x \in K^n$, such that $A = \pm x \cdot x^t$, that is $a_{ij} = \pm x_i x_j$

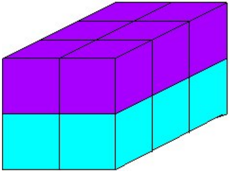
Proposition

A has rank $\leq r$ $\iff$ there exist symmetric A_i such that rank $A_i = 1$ and $A = A_1 + \ldots + A_r$.

Proof is the same, with symmetric gaussian elimination $A \mapsto G^t AG$. Works in every field where any element is a square or the opposite of a square.

Tensors

Let V_i be complex (or real) vector spaces. A tensor is an element $f \in V_1 \otimes \ldots \otimes V_k$, that is a multilinear map $V_1^\vee \times \ldots \times V_k^\vee \to K$
A tensor can be visualized as a multidimensional matrix.



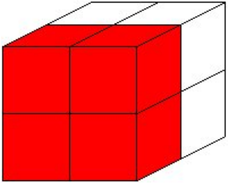
Entries of f are labelled by k indices, as $a_{i_1 \ldots i_k}$

Definition

A tensor is *decomposable* if there exist $x^i \in V_i$ for $i = 1, \ldots, k$ such that $a_{i_1 \ldots i_k} = x^1_{i_1} x^2_{i_2} \ldots x^k_{i_k}$

For a nonzero usual matrix, decomposable $\iff$ rank one.

Decomposition in $2 \times 2 \times 2$ case



Theorem (Segre)

A general tensor t of format $2 \times 2 \times 2$ has a unique decomposition as a sum of two decomposable tensors

Sketch of proof

Assume we have a decomposition (with obvious notations)
$t = x_1 \otimes y_1 \otimes z_1 + x_2 \otimes y_2 \otimes z_2$
Consider t as a linear map $A_t \colon {\mathbb{C}^2}^\vee \otimes {\mathbb{C}^2}^\vee \to \mathbb{C}^2$
Let (x_1', x_2'), (y_1', y_2') be dual basis.
$\ker A_t$ is a two dimensional subspace of the source, which contains $x_1' \otimes y_2'$ and $x_2' \otimes y_1'$, hence it is equal to their linear span.
In the source space there is the quadratic cone of decomposable elements given by $\det = 0$. Cutting with the kernel get just the two lines spanned by $x_1' \otimes y_2'$ and $x_2' \otimes y_1'$. These are two linear functions with common zero locus (on decomposable elements in $\mathbb{C}^2 \otimes \mathbb{C}^2$) given by $(x_1 \otimes y_1)$, $(x_2 \otimes y_2)$ (and their scalar multiples), that can be found uniquely from t.

Why geometry?

Corrado Segre in XIX century understood the previous decomposition in terms of projective geometry.
The tensor t is a point of the space $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$.
The decomposable tensors make the "Segre variety"

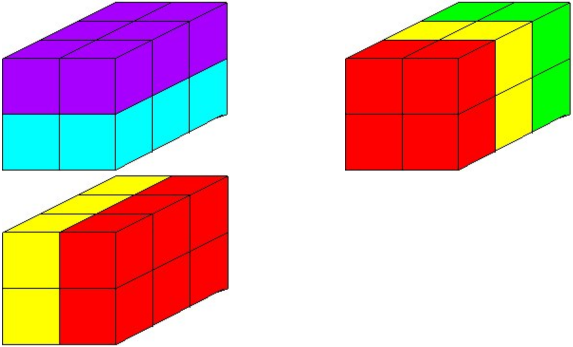
$$\begin{array}{rcc} X = \mathbb{P}(\mathbb{C}^2) \otimes \mathbb{P}(\mathbb{C}^2) \otimes \mathbb{P}(\mathbb{C}^2) & \to & \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \\ ((a_0, a_1), (b_0, b_1), (c_0, c_1)) & \mapsto & (a_0 b_0 c_0, a_0 b_0 c_1, \ldots, a_1 b_1 c_1) \end{array}$$

From t there is a unique secant line meeting X in two points.
This point of view is extremely useful also today.

J.M. Landsberg, Tensors: Geometry and Applications, AMS 2012

Slices

Just as matrices can be cutted in rows or in columns, higher dimensional tensors can be cut in slices



The three ways to cut a $3 \times 2 \times 2$ matrix into parallel slices

For a tensor of format $a_1 \times \ldots \times a_d$, there are a_1 slices of format $a_2 \times \ldots \times a_d$.

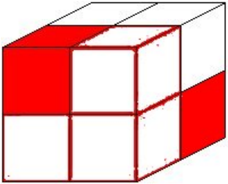
Multidimensional Gauss elimination

We can operate adding linear combinations of a slice to another slice, just in the case of rows and columns.
This amounts to multiply A of format $n_1 \times \ldots \times n_k$ for $G_1 \in GL(n_1)$, then for $G_i \in GL(n_i)$.
The group acting is quite big $G = GL(n_1) \times \ldots \times GL(n_k)$.

Canonical form in the format $2 \times 2 \times 2$

Theorem

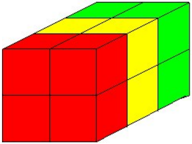
For a tensor A of format $2 \times 2 \times 2$ such that $Det(A) \neq 0$, (hyperdeterminant) then there exist $H_0, H_1, H_2 \in GL(2)$ such that (with obvious notations) $H_0 * A * H_1 * H_2$ has entries 1 in the red opposite corners and 0 otherwise.

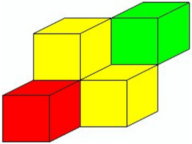


The group acting

Basic computation of dimensions. Let $\dim V_i = n_i$
$\dim V_1 \otimes \ldots \otimes V_k = \prod_{i=1}^{k} n_i$
$\dim GL(n_1) \times \ldots \times GL(n_k) = \sum_{i=1}^{k} n_i^2$
For $k \geq 3$, the dimension of the group is in general much less that the dimension of the space where it acts.
This makes a strong difference between the classical case $k = 2$ and the case $k \geq 3$.

case $3 \times 2 \times 2$ has canonical form



For a tensor A of format $3 \times 2 \times 2$ such that $Det(A) \neq 0$, (hyperdeterminant) then there exist $G \in GL(3)$ $H_1, H_2 \in GL(2)$ such that (with obvious notations) $G * A * H_1 * H_2$ is equal to the "identity matrix".
The "identity matrix" corresponds to polynomial multiplication $\mathbb{C}^2 \otimes \mathbb{C}^2 \to S^2(\mathbb{C}^2)$ represented by $3 \times 2 \times 2$ matrix which in convenient basis is the "identity". 

Few cases where multidimensional Gaussian elimination works

There are finitely many orbits for the action of $GL(k_1) \times GL(k_2) \times GL(k_3)$ over $\mathbb{C}^{k_1} \otimes \mathbb{C}^{k_2} \otimes \mathbb{C}^{k_3}$ just in the following cases ([Parfenov])

(k_1, k_2, k_3)	# orbits
$(2, 2, 2)$	7
$(3, 2, 2)$	9
$(n \geq 4, 2, 2)$	10
$(3, 3, 2)$	18
$(4, 3, 2)$	24
$(5, 3, 2)$	26
$(n \geq 6, 3, 2)$	27

Tensor Decomposition and Rank

Let $V_1, \ldots, V_k$ be complex vector spaces. A *decomposition* of $f \in V_1 \otimes \ldots \otimes V_k$ is

$$f = \sum_{i=1}^{r} c_i v_{i,1} \otimes \ldots \otimes v_{i,k} \qquad \text{with } c_i \in \mathbb{C}, \quad v_{i,j} \in V_j$$

Definition

$\mathrm{rk}(f)$ is the minimum number of summands in a decomposition of f. A minimal decomposition has $\mathrm{rk}(f)$ summands and it is called CANDECOMP or PARAFAC.

Note that for usual matrices, this definition of rank agrees with the classical one.
We may assume $c_i = 1$, although in practice it is more convenient to determine $v_{i,k}$ up to scalars, and then solve for c_i.

Rank from slices, in three dimensional case

Let $A = [A_1, \dots A_m]$, where A_i are its two-dimsnional slices. Note that

$$A = \sum_{i=1}^{r} x_i \otimes y_i \otimes z_i$$

if and only if

$$< A_1, \dots, A_m > \subseteq < x_1 y_1, \dots, x_r y_r >$$

We get

Rank from slices

The rank of $A = [A_1, \dots A_m]$ is the minimum r such that there exists a span of r matrices of rank one containing $< A_1, \dots, A_m >$.

Symmetric tensors = homogeneous polynomials

In the case $V_1 = \ldots = V_k = V$ we may consider symmetric tensors $f \in S^d V$.
Elements of $S^d V$ can be considered as homogeneous polynomials of degree d in $x_0, \ldots x_n$, basis of V.
So polynomials have rank (as all tensors) and also symmetric rank (next slides).

Symmetric Tensor Decomposition (Waring)

A *Waring decomposition* of $f \in S^d V$ is

$$f = \sum_{i=1}^{r} c_i (l_i)^d \qquad \text{with } l_i \in V$$

with minimal r

Example: $7x^3 - 30x^2y + 42xy^2 - 19y^3 = (-x+2y)^3 + (2x-3y)^3$
$\mathrm{rk}\left(7x^3 - 30x^2y + 42xy^2 - 19y^3\right) = 2$

Symmetric Rank and Comon Conjecture

The minimum number of summands in a Waring decomposition is called the symmetric rank

Comon Conjecture

Let t be a symmetric tensor. Are the rank and the symmetric rank of t equal ? Comon conjecture gives affirmative answer.

Known to be true when $t \in S^d\mathbb{C}^{n+1}$, $n = 1$ or $d = 2$ and few other cases.

Apolarity and Waring decomposition, I

For any $l = \alpha x_0 + \beta x_1 \in \mathbb{C}^2$ we denote $l^\perp = -\beta\partial_0 + \alpha\partial_1 \in {\mathbb{C}^2}^\vee$.
Note that

$$l^\perp(l^d) = 0 \qquad (1)$$

so that $l^\perp$ is well defined (without referring to coordinates) up to scalar multiples. Let e be an integer. Any $f \in S^d\mathbb{C}^2$ defines $C_f^e \colon S^e({\mathbb{C}^2}^\vee) \to S^{d-e}\mathbb{C}^2$
Elements in $S^e({\mathbb{C}^2}^\vee)$ can be decomposed as $(l_1^\perp \circ \ldots \circ l_e^\perp)$ for some $l_i \in \mathbb{C}^2$.

Apolarity and Waring decomposition, II

Proposition

Let l_i be distinct for $i = 1, \ldots, e$. There are $c_i \in K$ such that $f = \sum_{i=1}^{e} c_i (l_i)^d$ if and only if $(l_1^{\perp} \circ \ldots \circ l_e^{\perp}) f = 0$

Proof: The implication $\Longrightarrow$ is immediate from (1). It can be summarized by the inclusion $< (l_1)^d, \ldots, (l_e)^d > \subseteq \ker(l_1^{\perp} \circ \ldots \circ l_e^{\perp})$. The other inclusion follows by dimensional reasons, because both spaces have dimension e. $\square$

The previous Proposition is the core of the Sylvester algorithm, because the differential operators killing f allow to define the decomposition of f, as we see in the next slide.

Sylvester algorithm for Waring decomposition

Sylvester algorithm for general f Compute the decomposition of a general $f \in S^d U$

- Pick a generator g of $\ker C_f^a$ with $a = \lfloor \frac{d+1}{2} \rfloor$.
- Decompose g as product of linear factors, $g = (l_1^\perp \circ \ldots \circ l_r^\perp)$
- Solve the system $f = \sum_{i=1}^r c_i (l_i)^d$ in the unknowns c_i.

Remark When d is odd the kernel is one-dimensional and the decomposition is unique. When d is even the kernel is two-dimensional and there are infinitely many decompositions.

The catalecticant matrices for two variables

If $f(x,y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$ then

$$C_f^1 = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

and

$$C_f^2 = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix}$$

The catalecticant algorithm at work

The catalecticant matrix associated to
$f = 7x^3 - 30x^2 + 42x - 19 = 0$ is

$$A_f = \begin{bmatrix} 7 & -10 & 14 \\ -10 & 14 & -19 \end{bmatrix}$$

$\ker A_f$ is spanned by $\begin{bmatrix} 6 \\ 7 \\ 2 \end{bmatrix}$ which corresponds to
$6\partial_x^2 + 7\partial_x\partial_y + 2\partial_y^2 = (2\partial_x + \partial_y)(3\partial_x + 2\partial_y)$

Hence the decomposition

$$7x^3 - 30x^2y + 42xy^2 - 19y^3 = c_1(-x + 2y)^3 + c_2(2x - 3y)^3$$

Solving the linear system, we get $c_1 = c_2 = 1$

Application to the solution of the cubic equation

$$7x^3 - 30x^2 + 42x - 19 = 0$$

$$7x^3 - 30x^2 + 42x - 19 = (-x+2)^3 + (2x-3)^3$$

$$\left(\frac{-x+2}{-2x+3}\right)^3 = 1$$

three linear equations

$-x + 2 = (-2x+3)\omega^j \quad \text{for } j = 0,1,2 \qquad \omega = \exp\frac{2\pi i}{3}$

$$x = \frac{3\omega^j - 2}{2\omega^j - 1}$$

Secant varieties

Secant varieties give basic interpretation of rank of tensors in Geometry.
Let $X \subset \mathbb{P}V$ be irreducible variety.

$$\sigma_k(X) := \overline{\bigcup_{x_1,\ldots,x_k \in X} < x_1, \ldots, x_k >}$$

where $< x_1, \ldots, x_k >$ is the projective span.
There is a filtration $X = \sigma_1(X) \subset \sigma_2(X) \subset \ldots$
This ascending chain stabilizes when it fills the ambient space.
So $\min\{k | \sigma_k(X) = \mathbb{P}V\}$ is called the generic X-rank.

Examples of secant varieties

$X = \mathbb{P}V \otimes \mathbb{P}W$
Then $\sigma_k(X)$ parametrizes linear maps $V^\vee \to W$ of rank $\leq k$.
In this case the Zariski closure is not necessary, the union is already closed.

The symmetric case

$X = v_2\mathbb{P}V$ quadratic Veronese embedding of $\mathbb{P}V$.
Then $\sigma_k(X)$ parametrizes symmetric linear maps $V^\vee \to V$ of rank $\leq k$.
Also in this case the Zariski closure is not necessary, the union is already closed.

The skew-symmetric case is parametrized by secants of a Grassmannian.

Rank of tensors has wild behaviour

$\mathrm{rk}(x^3) = 1$

$\mathrm{rk}(x^3 + y^3) = 2$

$\mathrm{rk}(x^2 y) = 3$ because $x^2 y = \frac{1}{6}\left[(x+y)^3 - (x-y)^3 - 2y^3\right]$, but.....

$x^2 y = \lim_{t \to 0} \frac{(x+ty)^3 - x^3}{3t}$
so that a polynomial of rank 3 can be approximated by polynomials of rank 2. In this case we say that the *border rank* of $x^2 y$ is 2.
$t \in \sigma_r(X) \iff \text{border rank } (t) \leq r$
Similar phenomena happen in the nonsymmetric case.

Sylvester algorithm for rank of binary forms

Sylvester algorithm to compute the rank Comas and Seiguer prove that if the border rank of $f \in S^d\mathbb{C}^2$ is r ($r \geq 2$), then there are only two possibilities, the rank of f is r or the rank of f is $d - r + 2$. The first case corresponds to the case when the generator of C_f^r has distinct roots, the second case when there are multiple roots.

Veronese variety parametrizes symmetric tensors of (symmetric) rank one

Let V be a (complex) vector space of dimension $n+1$. We denote by S^dV the d-th symmetric power of V. The d-Veronese embedding of $\mathbb{P}^n$ is the variety image of the map

$$\begin{array}{ccc} \mathbb{P}V & \rightarrow & \mathbb{P}S^dV \\ v & \mapsto & v^d \end{array}$$

We denote it by $v_d(\mathbb{P}V)$.

Theorem

A linear function $F\colon S^dV \rightarrow K$ is defined if and only if it is known on the Veronese variety. So knowing $F(x^d)$ for every x linear allows to define $F(f)$ for every $f \in S^dV$.

Geometric interpretation of border rank

A tensor t has border rank $\leq r \iff t \in \sigma_r(\mathrm{Segre\ variety})$

A symmetric tensor t has symmetric border rank $\leq r \iff t \in \sigma_r(\mathrm{Veronese\ variety})$

Terracini Lemma describes the tangent space at a secant variety

Lemma

Terracini Let $z \in < x_1, \ldots, x_k >$ *be general. Then* $T_z\sigma_k(X) =< T_{x_1}X, \ldots, T_{x_k}X >$

Dual varieties

If $X \subset \mathbb{P}V$ then

$$X^\vee := \overline{\{H \in \mathbb{P}V^\vee | \exists \text{ smooth point } x \in X \text{ s.t. } T_x X \subset H\}}$$

is called the *dual variety* of X. So $X^\vee$ consists of hyperplanes tangent at some smooth point of X.
By Terracini Lemma

$$\sigma_k(X)^\vee = \{H \in PV^\vee | H \supset T_{x_1}X, \ldots, T_{x_k}X \text{ for smooth points } x_1, \ldots, x_k\}$$

namely, $\sigma_k(X)^\vee$ consists of hyperplanes tangent at $\geq k$ smooth points of X.

Examples of dual to secant varieties

$\sigma_1(v_3(\mathbb{P}^2))$	l^3
$\sigma_2(v_3(\mathbb{P}^2))$	three concurrent lines
$\sigma_3(v_3(\mathbb{P}^2))$	Aronhold hypersurface, orbit of Fermat cubic
$\sigma_1(v_3(\mathbb{P}^2))^\vee$	discriminant (singular cubics)
$\sigma_2(v_3(\mathbb{P}^2))^\vee$	reducible cubics
$\sigma_3(v_3(\mathbb{P}^2))^\vee$	triangles (split variety)

Sum of k-th powers

Problem Write a homogeneous polynomial of degree dk as a sum of k-th powers of degree d homogeneous polynomials.
$f = \sum_{i=1}^{r} (f_i)^k$, $\deg f_i = d$

$k = 2$ is sum of squares.
$d = 1$ is Waring decomposition.

Theorem (Fröberg - O - Shapiro)

Let $k \geq 2$. Any generic form f of degree kd in $n+1$ variables is the sum of at most k^n k-th powers. Moreover, for a fixed n, this number is sharp for $d \gg 0$.

Indeed

$$\frac{\dim S^{kl}\mathbb{C}^{n+1}}{\dim S^{l}\mathbb{C}^{n+1}} < k^n \quad \text{and} \quad \lim_{l\to\infty} \frac{\dim S^{kl}\mathbb{C}^{n+1}}{\dim S^{l}\mathbb{C}^{n+1}} = k^n.$$

Configuration of points with roots of unity

Let $\xi_i = e^{2\pi i/k}$ for $i = 0, \ldots k-1$ be the k-th roots of unity.
In the proof it is crucial to consider the grid of points

$$(1, \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n})$$

Let
$\mathrm{SOS}^{n,d} = \{p \in \mathbb{R}[x_1, \ldots, x_n]_{2d} | p = \sum_{i=1}^{k} l_i^2\}$
$C_+^{n,d}\{p \in \mathbb{R}[x_1, \ldots, x_n]_{2d} | p \geq 0\}$
SOS=Sum Of Squares
There is an inclusion of convex cones

$$\mathrm{SOS}^{n,d} \subseteq C_+^{n,d}$$

Theorem (Hilbert)

The inclusion is an equality if and only if $n \leq 2$, $d = 1$ or $(n, d) = (3, 2)$

Clebsch quartics

A plane quartic $f \in S^4V$ is called *Clebsch* if it has an apolar conic, that is if there exists a nonzero $q \in S^2V^\vee$ such that $q \cdot f = 0$.
One defines, for any $f \in S^4V$, the catalecticant map $C_f \colon S^2V^\vee \to S^2V$ which is the contraction by f. If

$$f = a_{00}x^4 + 4a_{10}x^3y + 4a_{01}x^3z + 6a_{20}x^2y^2 + 12a_{11}x^2yz + 6a_{02}x^2z^2 + 4a_{30}xy^3$$

$$4a_{03}xz^3 + a_{40}y^4 + 4a_{31}y^3z + 6a_{22}y^2z^2 + 4a_{13}yz^3 + a_{04}z^4$$

then the matrix of C_f is

$$C_f = \begin{bmatrix} a_{00} & a_{10} & a_{01} & a_{20} & a_{11} & a_{02} \\ a_{10} & a_{20} & a_{11} & a_{30} & a_{21} & a_{12} \\ a_{01} & a_{11} & a_{02} & a_{21} & a_{12} & a_{03} \\ a_{20} & a_{30} & a_{21} & a_{40} & a_{31} & a_{22} \\ a_{11} & a_{21} & a_{12} & a_{31} & a_{22} & a_{13} \\ a_{02} & a_{12} & a_{03} & a_{22} & a_{13} & a_{04} \end{bmatrix}$$

Computation of the catalecticant for plane quartics

This matrix has been computed acting with the following differential operators
To any quartic we can associate the catalecticant matrix constructed in the following way

$$\begin{array}{c|cccccc} & \partial_{00} & \partial_{01} & \partial_{02} & \partial_{11} & \partial_{12} & \partial_{22} \\ \hline \partial_{00} & & & & & & \\ \partial_{01} & & & & & & \\ \partial_{02} & & & & & & \\ \partial_{11} & & & & & & \\ \partial_{12} & & & & & & \\ \partial_{22} & & & & & & \end{array}$$

$rank(f) = rank(C_f)$ it relates the rank of a tensor with the rank of a usual matrix.

Clebsch quartics have border rank five

We get that a plane quartic f is Clebsch if and only if $\det C_f = 0$. The basic property is that if $f = l^4$ is the 4-th power of a linear form, then C_f has rank 1. It follows that if $f = \sum_{i=1}^5 l_i^4$ is the sum of five 4-th powers of linear forms, then

$$\mathrm{rk}\ C_f = \mathrm{rk}\ \sum_{i=1}^5 C_{l_i^4} \leq \sum_{i=1}^5 \mathrm{rk}\ C_{l_i^4} = \sum_{i=1}^5 1 = 5$$

Theorem (Clebsch)

A plane quartic f is Clebsch if and only if there is an expression $f = \sum_{i=0}^4 l_i^4$ (or a limit of such an expression)

In conclusion, $\det C_f = 0$ is the equation (of degree six) of $\sigma_5(v_4(\mathbb{P}^2))$, which is called the Clebsch hypersurface.

Expected dimension for secant varieties

Let $X \subset \mathbb{P}^N$ be an irreducible variety.
The naive dimensional count says that

$$\dim \sigma_k(X) + 1 \leq k(\dim X + 1)$$

When $\dim \sigma_k(X) = \min\{N, k(\dim X + 1) - 1\}$ then we say that $\sigma_k(X)$ has the expected dimension. Otherwise we say that X is k-defective.
Correspondingly, the expected value for the general X-rank is

$$\lceil \frac{N+1}{\dim X + 1} \rceil$$

In defective cases, the general X-rank can be bigger than the expected one.

Defectivity of plane quartics

It is expected by naive dimensional count that the general rank for a plane quartic is five.
On the contrary, the general rank is six. Five summands are not sufficient, and describe Clebsch quartics.
A general Clebsch quartic f can be expressed as a sum of five 4-th powers in ∞^1 many ways. Precisely the 5 lines l_i belong to a unique smooth conic Q in the dual plane, which is apolar to f and it is found as the generator of $\ker\ C_f$.

Dual of Clebsch hypersurface

Question What is the dual of the Clebsch hypersurface in $\mathbb{P}^{14} = \mathbb{P}(S^4\mathbb{C}^3)$?
It consists of quartics that are singular in five points.
$\sigma_5(v_4(\mathbb{P}^2))^\vee = CC$, variety of squares
Every general $f \in S^4\mathbb{C}^3$ can be expressed as a sum $f = q_1^2 + q_2^2 + q_3^2$ in ∞^3 ways. These different ways describe a variety with exactly 63 components.

Generalization to sextics

Proposition

Let Y_{10} be the determinantal hypersurface in the space $\mathbb{P}S^6\mathbb{C}^3$ of sextics having a apolar cubic.
(i) $Y_{10} = \sigma_9(v_6(\mathbb{P}^2))$
(ii) The dual variety $Y_{10}^{\vee}$ is the variety of sextics whic are square of a cubic(double cubics).

Proposition[Blekherman-Hauenstein-Ottem-Ranestad-Sturmfels]

The variety of 3-secant to $Y_{10}^{\vee}$ consists in sextics which are sum of three squares. It is an hypersurface of degree 83200.

Such a hypersurface coincides with the locus of sextic curves which admit an effective theta-characteristic (theta locus).
The question of computing the degree of the theta locus is interesting and open for all the even plane curves.

Theorem (Campbell, [1891] Terracini, [1916] Alexander-Hirschowitz [1995])

The general $f \in S^d\mathbb{C}^{n+1}$ $(d \geq 3)$ has rank

$$\lceil \frac{\binom{n+d}{d}}{n+1} \rceil$$

which is called the generic rank, with the only exceptions

- $S^4\mathbb{C}^{n+1}$, $2 \leq n \leq 4$, *where the generic rank is* $\binom{n+2}{2}$
- $S^3\mathbb{C}^5$, *where the generic rank is* 8, *sporadic case*

Toward an Alexander-Hirschowitz Theorem in the non symmetric case

Defective examples

$\dim V_i = n_i + 1$, $n_1 \leq \ldots \leq n_k$
Only known examples where the general $f \in V_1 \otimes \ldots \otimes V_k$ $(k \geq 3)$ has rank different from the *generic rank*

$$\lceil \frac{\prod (n_i + 1)}{\sum n_i + 1} \rceil$$

are

- unbalanced case, where $n_k \geq \prod_{i=1}^{k-1}(n_i + 1) - \left(\sum_{i=1}^{k-1} n_i\right) + 1$, note that for $k = 3$ it is simply $n_3 \geq n_1 n_2 + 2$
- $k = 3$, $(n_1, n_2, n_3) = (2, m, m)$ with m even [Strassen],
- $k = 3$, $(n_1, n_2, n_3) = (2, 3, 3)$, sporadic case [Abo-O-Peterson]
- $k = 4$, $(n_1, n_2, n_3, n_4) = (1, 1, n, n)$

Theorem (Strassen-Lickteig)

there are no exceptions (no defective cases) $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ *beyond the variety* $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$

Theorem

- *The unbalanced case is completely understood [Catalisano-Geramita-Gimigliano].*
- *The exceptions listed in the previous slide are the only ones in the cases:*
  (i) $k = 3$ *and* $n_i \leq 9$
  (ii) $s \leq 6$ *[Abo-O-Peterson]*
  (iii) $\forall k,\ n_i = 1$ *(deep result, [Catalisano-Geramita-Gimigliano])*

Proof uses an inductive technique, developed first for $k = 3$ in [Bürgisser-Claussen-Shokrollai].

Asymptotical behaviour

[Abo-O-Peterson]

Asymptotically ($n \to \infty$), the general rank for tensors in $\mathbb{C}^{n+1} \otimes \ldots \otimes \mathbb{C}^{n+1}$ (k times) tends to

$$\frac{(n+1)^k}{nk+1}$$

as expected.

Gesmundo result

For any $n_1, \dots, n_k$ there is Θ_k such that for $s \le \Theta_k \frac{\prod n_i}{1+\sum_i (n_i - 1)}$ then σ_s has the expected dimension.

In case $n_i = 2^{d_i}$ then $\Theta_k \to 1$ for $k \to \infty$