Lectures on the Geometry of Tensors

Informal Notes for the Nordfjordeid Summer School 2010

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1 The Sylvester algorithm for complex binary forms

In these lectures we will be particularly interested in the two cases $K = \mathbb{R}, \mathbb{C}$. If it is not specified otherwise, we will tacitly assume that $K = \mathbb{C}$.

The first source about tensor decomposition is probably the Babylonian technique to "completing the square" in order to solve the equation of second degree, like

$x^2 + 10xy + 21y^2 = (x + 5y)^2 - 4y^2$

In undergraduate mathematics this leads to decompose every homogeneous polynomial of degree 2 as a sum of squares of linear forms (on the real numbers some minus signs can be needed, obtaining the concept of signature). This decomposition is possible as well for many variables, and it is well known that a homogeneous polynomial of rank 2 corresponds to a symmetric matrix, and the minimum number of summands in the decomposition as a sum of squares is exactly the *rank* of the matrix.

In the above example

$$x^2 + 10xy + 21y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 5 & 21 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and $\begin{pmatrix} 1 & 5 \\ 5 & 21 \end{pmatrix}$ has rank 2. The decomposition as a sum of squares is never unique, unless the original polynomial is a square, which means that the matrix has rank 1.

Sylvester studied the analogous problem to decompose a homogeneous polynomial $F(x, y)$ of degree d as a sum of d-th powers of linear forms. He solved completely the problem, obtaining the nice result that the decomposition is unique for general polynomials of odd degree. Sylvester algorithm became the prototype of more general tensor decompositions. In the case of polynomials (symmetric tensors), the decomposition as sum of powers is known as the Waring problem for polynomials. In this section we discuss the Sylvester algorithm.

Let $F(x, y)$ be a complex homogeneous polynomial of degree d.

The Sylvester algorithm finds a decomposition

$$F = \sum_{i=1}^{r} l_i^d$$

where l_i are homogeneous polynomials of degree 1 and where r is minimal. Such minimal r is called the *rank* of F.

The elegant solution of this problem uses the dual ring of differential operators, that we introduce in more generality.

Let U be a vector space over K. We consider the polynomial ring $S = \oplus_i Sym^i(U)$. If $x_0, \ldots, x_n$ denote a basis of U, then S is the ring of polynomials in $x_0, \ldots, x_n$ with indeterminates in K. The dual basis of $U^\vee$ can be denoted $\partial_0 = \frac{\partial}{\partial x_0}, \ldots, \partial_n = \frac{\partial}{\partial x_n}$.

Its dual ring is $T = \oplus_i Sym^i(U^\vee)$. S and T are dual of each other, so that ∂_i is an operator acting on x_j, as well x_i is an operator acting on ∂_j.

There are linear maps

$$Sym^p(U) \otimes Sym^q(U^\vee) \to Sym^{p-q}(U) \qquad \text{for } p \geq q$$

or

$$Sym^p(U) \otimes Sym^q(U^\vee) \to Sym^{q-p}(U^\vee) \qquad \text{for } p \leq q$$

Both maps can be defined for any p, q, with the convention that Sym^i is zero for negative i.

From now on we set $\dim U = 2$. For any $l = \alpha x_0 + \beta x_1 \in U$ we denote $l^\perp = -\beta\partial_0 + \alpha\partial_1$.

Note that

$$l^\perp(l^d) = 0 \tag{1}$$

so that $l^\perp$ is well defined (without referring to coordinates) up to scalar multiples.

Let e be an integer Any $f \in S^dU$ defines $A(f)_{e,d-e} \colon Sym^e(U^\vee) \to Sym^{d-e}U$

Also

$(l_1^\perp \circ \ldots \circ l_e^\perp) \colon S^dU \to S^{d-e}U$

Proposition 1.1 *Let l_i be distinct for $i = 1, \ldots, e$. There are $c_i \in K$ such that $f = \sum_{i=1}^e c_i(l_i)^d$ if and only if $(l_1^\perp \circ \ldots \circ l_e^\perp)f = 0$*

Proof: The implication $\Longrightarrow$ is immediate from (1). It can be summarized by the inclusion $< (l_1)^d, \ldots, (l_e)^d > \subseteq \ker(l_1^\perp \circ \ldots \circ l_e^\perp)$. The other inclusion follows by dimensional reasons, because both spaces have dimension e. This proves the implication $\Longleftarrow$. $\square$

The following corollary is a dual reformulation. Both formulations are useful.

Corollary 1.2 *Let l_i be distinct for $i = 1, \ldots, e$. Let $r \leq d - e$. There are $c_i \in K$ such that $f = \sum_{i=1}^e c_i(l_i)^d$ if and only if Im $A(f)_{r,d-r} \subseteq < (l_1)^{d-r}, \ldots (l_e)^{d-r} >$.*

Proof: Considering $A(f)_{e,d-e} \colon Sym^e(U^\vee) \to Sym^{d-e}(U)$, the proposition Prop. 1.1 says that there are $c_i \in K$ such that $f = \sum_{i=1}^e c_i(l_i)^d$ if and only if $(l_1^\perp \circ \ldots \circ l_e^\perp) \in \ker A(f)_{e,d-e}$.

The transpose of $A(f)_{e,d-e}$ is$A(f)_{d-e,e}$ and the thesis follows from the two equalities $(\text{Im } A(f)_{e,d-e})^\perp = \ker A(f)_{d-e,e}$ and $< l_1^\perp \circ \ldots \circ l_e^\perp \circ S^{d-r-e} > = < (l_1)^{d-r}, \ldots, (l_e)^{d-r} >^\perp$. For the last one, the inclusion $\subseteq$ is trivial, and the equality holds because both spaces have the same dimension $d - r - e + 1$. $\square$

The Prop. 1.1 is the core of the Sylvester algorithm, because the differential operators killing f allow to define the decomposition of f, as we see in the next algorithm.

Theorem 1.3 *Let $k \leq e \leq \frac{d}{2}$*

$$f \in \sigma_k(C_d) \Leftrightarrow \textit{rank } A(f)_{e,d-e} \leq k$$

The $(k+1)$-minors of $A(f)_{e,d-e}$ define scheme theoretically $\sigma_k(C_d)$ (i.e. the minors cut $\sigma_k(C_d)$ transversally).

Proof: If $f = v^d \in C_d$ then Im $A(f)_{e,d-e} =< v^{d-e} >$, hence rk $A(f)_{e,d-e} = 1$. It follows that if $f = \sum_{i=1}^k v_i^d$ has rank k, then

$$\text{rk } A(f)_{e,d-e} = \text{rk } \sum_{i=1}^k A(v_i^d)_{e,d-e} \leq \sum_{i=1}^k \text{rk } A(v_i^d)_{e,d-e} = \sum_{i=1}^k 1 = k$$

This proves the implication $\Rightarrow$. Conversely, let assume $e = k$ and let the general case to the exercises. Iif rank $A(f)_{k,d-k} \leq k$ consider the kernel of $A(f)_{k,d-k}$. It is generated by a polynomial of degree k in the dual ring. If it factors with d distinct linear forms, then rk $f = k$ by the Prop. 1.1. Otherwise it is easy to check that there is a sequence $f_n \to \phi$ where the kernel of $A(f_n)_{k,d-k}$ contains a polynomial consisting of k distinct linear forms. Indeed let $< l_1^\perp \circ \ldots \circ l_k^\perp >= \ker A(f)_{k,d-k}$. Consider that $\phi \in \ker(l_1^\perp \circ \ldots \circ l_k^\perp)$ For any n we may choose *distinct* $l_{n,i}$ such that $l_{n,i} \to l_i$. Note that $\ker(l_{n,1}^\perp \circ \ldots \circ l_{n,k}^\perp)$ has always the same affine dimension k for every n. Hence we may choose $f_n \to f$ such that $\phi_n \in \ker(l_{n,1}^\perp \circ \ldots \circ l_{n,k}^\perp)$ and we have done.

This proves the set-theoretic version. We will prove that the minors define $\sigma_k(C_d)$ scheme-theoretically, as a consequence of a more general result about degeneracy loci.

□

Remark The $(k+1)$-minors of A_ϕ even generate the ideal of $\sigma_k(C_d)$. A reference is [Iarrobino-Kanev, LNM 1721].

Theorem 1.4 *(i) For odd $d = 2k+1$, the general $f \in Sym^d(\mathbb{C}^2)$ has a unique decomposition as a sum of $k+1$ d-th powers of linear forms.*

(ii) For even $d = 2k$, the general $f \in Sym^d(\mathbb{C}^2)$ has infinitely many decompositions as a sum of $k+1$ d-th powers of linear forms.

Proof: In the case $d = 2k+1$, for general f the kernel of $A(f)_{k+1,k}$ is one dimensional and it is generated by a polynomial with distinct factors. In the case $d = 2k$ the kernel of $A(f)_{k+1,k-1}$ has dimension two. □

It is interesting that for several values of m we get different contraction operators, and their minors define the same variety. Let us visualize the situation in the case $d = 6$.

For $m = 1$ the matrix of A_ϕ, where $\phi = \sum_{i=0}^6 \binom{6}{i} a_i x^i y^{6-i}$, is

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}$$

and its 2-minors define $\sigma_1(C_6) = C_6$.

For $m = 2$ the matrix of A_ϕ is

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \end{bmatrix}$$

and again its 2-minors define C_6 but also its 3-minors define $\sigma_2(C_6)$.

For $m = 3$ the matrix of A_ϕ is

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}$$

and again its 2-minors define $\sigma_1(C_5) = C_5$, its 3-minors define $\sigma_2(C_5)$, but also its determinant define $\sigma_4(C_5)$.

For $m = 4$ the matrix of A_ϕ is the transpose of the one constructed for $m = 2$ and so on.

Sylvester algorithm for general f Compute the decomposition of a general $f \in S^d U$

Pick a generator g of $\ker A_f$. Decompose g as product of linear factors, $g = (l_1^\perp \circ \ldots \circ l_r^\perp)$ Solve the system $f = \sum_{i=1}^r c_i (l_i)^d$ in the unknowns c_i.

Sylvester algorithm to compute the rank Comas and Seiguer prove that if the border rank of ϕ is r ($r \geq 2$), then there are only two possibilities, the rank of ϕ is r or the rank of ϕ is $d - r + 2$. The first case corresponds to the case when the generator of $A(\phi)_{r,d-r}$ has distinct roots, the second case when there are multiple roots.

Remark In higher dimension the possibilities for the rank are much more complicated, see recent work by Landsberg, Teitler, Bernardi, Ida', Gimigliano.

Remark In space of linear maps, the rank is a lower semicontinuous function, that is the locus where the rank is $\leq k$ is closed. In general, the rank is neither lower nor upper semicontinuous.

Exercise Prove that in $S^3\mathbb{C}^3$ the rank is neither lower nor upper semicontinuous, by finding a sequence of tensors of rank 2 which has limit a tensor of rank 3, and another sequence of tensors of rank 2 which has limit a tensor of rank 1.

Geometric solution of the equation of third degree (with complex coefficients) If $f = a_0x^3 + 3a_1x^2y + 3a_2a_2xy^2 + a_3y^3$ consider

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{bmatrix}$$

The kernel is given by

$$\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \partial_0 - \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \partial_1 + \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} \partial_2$$

Decompose it as

$$(-\beta_1\partial_x+\alpha_1\partial_y)(-\beta_2\partial_x+\alpha_2\partial_y)$$

The condition that this decomposition has two distinct factors is

$$\Delta\neq 0$$

where

$$\Delta := \begin{bmatrix} a_0 & a_2 \\ a_1 & a_3 \end{bmatrix}^2 - 4\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}\cdot\begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}$$

Then (by putting c_i inside the linear forms we get $f=(\alpha_1x+\beta_1y)^3+(\alpha_2x+\beta_2y)^3$ so that the solution of the equation $f=0$ are given by $\left[\frac{\alpha_1x+\beta_1y}{\alpha_2x+\beta_2y)^3}\right]^3=-1$ Let $\epsilon_j=\exp{(1+2j)\pi\sqrt{-1}}$ for $j=0,1,2$ be the three cubic roots of -1, then the solutions are given by the solution of the three linear equations

$$(\alpha_1x+\beta_1y)+\epsilon_j(\alpha_2x+\beta_2y)=0 \text{ for } j=0,1,2$$

Geometrically, the line through $(\alpha_1x+\beta_1y)^3$ and $(\alpha_2x+\beta_2y)^3$ is the unique line through f which is secant to the twisted cubic.

Exercises

1. Decompose x^3+3xy^2 as a sum of two cubes
2. Decompose x^3-3xy^2 as a sum of two cubes
3. Modify the solution of the equation of third degree, in order to cover the case $\Delta=0$
4. Decompose (with the help of a computer), a general homogeneous polynomial $F(x,y)$of degree 5.
5. What is the dual variety of the hypersurface of forms of rank $\leq d$ in $\mathbf{P}(S^{2d}U)$?

2 Determinantal Varieties in Spaces of Linear Maps

Let V, W be K-vector spaces, with $\dim V = n+1$, $\dim W = m+1$.

Consider the space of linear maps $V \otimes W = Hom(V^\vee, W)$.

The *rank* of $f \in Hom(V^\vee, W)$ is by definition $\dim \mathrm{Im}\, f$. The linear maps of rank $\leq r$ are defined by minors of size $r+1$. Let

$$D_r = \{f \in V \otimes W | \mathrm{rk}\ f \leq r\}$$

Proposition 2.1 *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. The singular locus of D_r coincides with D_{r-1}, that is $D_r \setminus D_{r-1}$ is a smooth variety. The codimension of D_r is $(n+1-r)(m+1-r)$.*

Proof: $D_r \setminus D_{r-1}$ is an orbit under the action of $GL(V) \times GL(W)$. After a choice of basis, a representant for this orbit is given by the block matrix

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

where I_r is the identity matrix of size r. In order to compute the codimension, consider that, in a neighborhood of the previous matrix, the local equations of D_r are given by the vanishing of all minors obtained adding one row and column to the North-West $r \times r$ block. In this way the entries of the $(n+1-r) \times (m+1-r)$ South-East block are determined (on D_r) by the other ones. $\square$

Parametric description of D_r For any pairs $(v_i, w_i) \in V \times W$ consider $\sum_{i=1}^{r} v_i \otimes w_i \in D_r$. In this way we get all of D_r. Note that the parameters involved are more than the dimension of D_r, so that they are superabundant. The description is better understood in terms of projective geometry, because all D_r are cones. The projectivization of D_1 in $\mathbf{P}(V \otimes W)$ is isomorphic to the Segre variety $\mathbf{P}(V) \times \mathbf{P}(W)$, which is defined.

In other terms, the Segre variety $\mathbf{P}(V) \times \mathbf{P}(W)$ consists of all the *decomposable* tensors.

This allows to define the rank in a second way. If $X \subseteq \mathbf{P}^N$ is a projective variety, the rank of $p \in \mathbf{P}^N$ with respect to X is the minimum r such that there exists $x_1 \ldots x_r \in X$ such that $p \in < x_1, \ldots, x_r >$. By abuse of notations, we identify the points in the projective space with their suitable affine representative and we write $p = \sum_{i=1}^{r} x_i$. So the rank can be considered as the minimum length of an additive decomposition of p in terms of X. Note that the rank is 1 if and only if $p \in X$.

Exercise The rank of $f \in \mathbf{P}(V \otimes W)$ as linear map and the rank of f with respect to $\mathbf{P}(V) \times \mathbf{P}(W)$ coincide.

Cartesian description of D_r D_r is the variety given by the vanishing of all $(r+1)$-minors. This is the reason why D_r is called a determinantal variety. These are homogeneous equations, then define also the projectivizations. Note that $D_1 \simeq \mathbf{P}(V) \times \mathbf{P}(W)$ is given by the vanishing of all 2-minors, which are quadratic equations.

It is known even more, that the ideal of D_r is generated by all the $(r+1)$-minors, they are homogeneous polynomials of degree $r+1$.

2.1 Symmetric case

The symmetric product S^2V consists of all symmetric matrices of size $(n+1)\times(n+1)$. It can be identified with homogeneous quadratic polynomials in $n+1$ variables.

Let

$$DS_r = \{f \in S^2V | \text{rk } f \leq r\}$$

Again the singular locus of DS_r coincides with DS_{r-1}, that is $DS_r \setminus DS_{r-1}$ is a smooth variety.

If $\dim V = n+1$, the codimension of DS_r is $\binom{n+2-r}{2}$.

Parametric description of DS_r For any $v_i \in V$ consider $\sum_{i=1}^r v_i^2 \in SD_r$. In this way we get all of DS_r. Again the variables involved are more than the dimension of DS_r, with the exception $r=1$.

Cartesian description of DS_r DS_r is the variety given by the vanishing of all $(r+1)$-minors.

It is known even more, that the ideal of DS_r is generated by all the $(r+1)$-minors, they are homogeneous polynomials of degree $r+1$.

Exercise In the real case, note that the quadratic Veronese embedding gives just one component of the affine variety DS_1 of symmetric maps of rank 1, namely the positive semidefinite one. Modify the parametrization of DS_r, which has now several connected components, according to the signature. For example the projectivization of DS_2consists of two semialgebraic varieties, in one the zero locus is a pair of real hyperplanes, in the other one is a pait of imaginary conjugate hyperplanes.

2.2 Skew-symmetric case

The alternating product $\wedge^2V$ consists of all skew-symmetric matrices of size $(n+1)\times(n+1)$. It can be identified with homogeneous quadratic polynomials in $n+1$ variables.

Let

$$DA_r = \{f \in \wedge^2V | \text{rk } f \leq r\}$$

Since the rank of skew-symmetric matrices is always even, we may assume that r is even. Again the singular locus of DA_r coincides with DA_{r-2}, that is $DA_r \setminus DA_{r-2}$ is a smooth variety.

If $\dim V = n+1$, the codimension of DA_r is $\binom{n+1-r}{2}$.

Parametric description of DA_r For any pairs $(v_i, v_i') \in V \times V$ consider $\sum_{i=1}^r v_i \wedge v_i' \in DA_r$. In this way we get all of DA_r. Again the variables involved are more than the dimension of DA_r.

Cartesian description of DA_r DA_r is the variety given by the vanishing of all $(r+2)$-subpfaffians.

It is known even more, that the ideal of DA_r is generated by all the $(r+2)$-subfaffians, they are homogeneous polynomials of degree $\frac{r+2}{2}$.

2.3 k-secant varieties

Let $X \subseteq \mathbf{P}^N$. We remind that the *r-secant variety* $\sigma_r(X)$ is defined as the Zariski closure of all the points of rank r. It contains all points of rank $\leq r$, but it may contain also points of higher rank (the easiest example is when X is the twisted cubic).

Over $\mathbb{R}$ it is better defined as the Euclidean closure

The expected dimension of $\sigma_r(X)$ is $r \dim X + (r-1)$, unless it fills the ambient space, so it is better written as $\min\{r \dim X + (r-1), N\}$.

The three basic varieties X we are interested in these lectures are

1.) the Veronese variety $v_d(\mathbf{P}(V))$, which corresponds to the decomposable tensors in $S^d(V)$.

2.) the Segre variety $\mathbf{P}(V_1) \times \ldots \times \mathbf{P}(V_k)$, which corresponds to the decomposable tensors in $V_1 \otimes \ldots \otimes V_k$ (called the space of k-way tensors). The "hypercubic" case when all V_i are equal to V is particularly important. The Veronese case is the symmetrization of this one.

3.) the Grassmann variety $Gr(\mathbf{P}^k, \mathbf{P}(V))$, which corresponds to the decomposable tensors in $\wedge^{k+1}(V)$. The Grasmannian parametrizes all vector subspaces of dimension $k+1$ of V.

In terms of secant varieties, $D_r = \sigma_r(D_1)$. In particular D_r can be found by vector space operations by knowing just D_1. Note that $D_1 = \mathbf{P}^n \times \mathbf{P}^m$ corresponds to the subcase $k = 2$ of the 2.) above. Since that case has been understood with minors of matrices, this explains why the research for equations of $\sigma_k(\mathbf{P}(V_1) \times \ldots \times \mathbf{P}(V_k))$ is understood as "generalized minors" (of hypermatrices).

In the same vein, $DS_1 = v_2(\mathbf{P}(V)$ corresponds to the subcase $d = 2$ of 1.) above. Hence the research for equations of $\sigma_k(v_d(\mathbf{P}(V))$ is understood as "generalized minors" of symmetric hypermatrices.

Finally, $DA_1 = Gr(\mathbf{P}^1, \mathbf{P}(V))$ corresponds to the subcase $k = 1$ of 3.) above. Hence the research for equations of $\sigma_s(Gr(\mathbf{P}^k, \mathbf{P}(V)))$ is understood as a look for "hyperpfaffians".

In all the cases 1.), 2.), 3.), a parametric description is natural, but a cartesian description is still missing, and the main topic of these lectures is how to overcome this problem.

The degrees of the determinantal varieties in space of linear maps were computed in the XIX century (Schubert, Segre, Giambelli,...) and they are interesting

$$\deg D_r = \deg\left(\sigma_r(\mathbf{P}^n \times \mathbf{P}^m)\right) = \prod_{i=0}^{n-k} \frac{(m+i+1)!i!}{(r+i)!(m-k+i+1)!}$$

In particular $\deg D_1 = \deg\left(\mathbf{P}^n \times \mathbf{P}^m\right) = \binom{m+n}{n}$ and in the case $n \leq m$ $\deg D_n = \deg\left(\sigma_n(\mathbf{P}^n \times \mathbf{P}^m)\right) = \binom{m+1}{n}$.

$$\deg DS_r = \deg\left(\sigma_r(v_2\mathbf{P}^n)\right) = \prod_{i=0}^{n-r} \frac{\binom{n+i+1}{n-r-i+1}}{\binom{2i+1}{i}}$$

In particular $\deg DS_1 = \deg(v_2\mathbf{P}^n) = 2^n$ and $\deg DS_n = n{+}1$ (it is a determinant).

$$\deg DA_r = \deg\left(\sigma_r(Gr(\mathbf{P}^1,\mathbf{P}^n)\right) = \frac{1}{2^{n-r}}\prod_{i=0}^{n-r-1}\frac{\binom{n+i+1}{n-r-i}}{\binom{2i+1}{i}}$$

In particular $\deg DA_1 = \deg(Gr(\mathbf{P}^1,\mathbf{P}^n) = \frac{1}{n}\binom{2n-2}{n-1}$ (Catalan number) and in the case n odd $\deg DA_{n-1} = \frac{n+1}{2}$ (it is a pfaffian) and in the case n even $\deg DA_{n-2} = \frac{1}{4}\binom{n+2}{3}$.

2.4 The tangent and the normal spaces

To understand better a variety, it is very useful to know its tangent (and normal) space. They are linear objects that describe the infinitesimal behaviour.

Parametric description of tangent space at D_r

Proposition 2.2 *The tangent space of D_1 at $v\otimes w$ is given by*

$$v\otimes W + V\otimes w = \{v\otimes w' + v'\otimes w, \forall v'\in V, w'\in W\}$$

Proof: Consider any curve $v(t)\otimes w(t)\in D_1$ such that $v(0)=v$, $w(0)=w$.

The derivative for $t=0$ is given by $v'(0)\otimes w + v\otimes w'(0)$. As $v'(0)$ and $w'(0)$ are arbitrary vectors, the thesis follows. □

The previous proposition is the case $r=1$ of the following more general

Proposition 2.3 *The tangent space of D_r at $\sum_{i=1}^r v_i\otimes w_i\in D_r$ is given by*

$$\sum_{i=1}^r v_i\otimes W + V\otimes w_i$$

The proof is exactly the same. This can be seen also as the first display of the basic *Terracini lemma*.

If $f=\sum_{i=1}^r v_i\otimes w_i$ with minimal r, we get that both v_i and w_i are independent, otherwise we can express f as a sum of fewer r. For higher way tensors this is no more possible.

Cartesian description of tangent space at D_r

Theorem 2.4 *Let $f\in D_r\subseteq Hom(V^\vee,W)$. The tangent space to D_r at f is given by $\{g\in Hom(V^\vee,W)|g(\ker f)\subseteq \mathrm{Im}\ f\}$*

There are several proofs of this theorem, we propose the following one which is natural in the setting of tensor decomposition.

Proof: By assumption there are $v_i\in V$ and $w_i\in W$ such that $f=\sum_{i=1}^r v_i\otimes w_i$ Note that $\ker f = <v_1,\ldots,v_r>^\perp$ and $\mathrm{Im}\ f = <w_1,\ldots w_r>$

If $g\in\sum_{i=1}^r v_i\otimes W + V\otimes w_i$ then $g(\ker f)\subseteq \mathrm{Im}\ f$. This proves one inclusion. The second inclusion follows by a dimensional count. □

Corollary 2.5 *The normal space at $f \in D_r$ is given by*

$$Hom\ (\ker f, W/Im\ f)$$

The conormal space (it is the dual of the normal space) at $f \in D_r$ is given by

$$(\ker f) \otimes (Im\ f)^{\perp} \subseteq V^{\vee} \otimes W^{\vee}$$

The conormal space is quite useful because it coincides with $T_f^{\perp}$, the orthogonal of the tangent space.

Exercise For $f \in \mathrm{Hom}\ (V^{\vee}, W)$ denote by $f^t \in \mathrm{Hom}\ (W^{\vee}, V)$ the *transpose* of f, defined by $f^t(w)(v) = f(v)(w)$ for any $w \in W^{\vee}$, $v \in V^{\vee}$. Prove that $(\ker f)^{\perp} = \mathrm{Im}\ f^t$. Let $V = W$,prove that f is symmetric if and only if $f = f^t$, f is skew-symmetric if and only if $f = -f^t$

Corollary 2.6 *Symmetric case In the symmetric case ($V = W$ and $f \in S^2V$) we have that $(\ker f)^{\perp} = \mathrm{Im}\ f$ and the conormal space to DS_r at f is given by*

$$S^2(\ker f) \subseteq S^2(V^{\vee})$$

Corollary 2.7 *Skew-ymmetric case In the skew-symmetric case ($V = W$ and $f \in \wedge^2 V$) we have again that $(\ker f)^{\perp} = \mathrm{Im}\ f$ and the conormal space to DA_r at f is given by*

$$\wedge^2(\ker f) \subseteq \wedge^2(V^{\vee})$$

In practice, the rank of any $f \in V \otimes W$ can be computed efficiently by Gaussian elimination. Theoretically, Gaussian elimination consists in rewriting $\sum_{i=1}^{r} v_i \otimes w_i$ in such a way that both v_i and w_j are independent. Then the rank is the minimum number of summands. This method does not work for higher format tensor.

Exercises Prove that a matrix of rank r is sum of r matrices of rank one. Is the decomposition unique ? Prove that a symmetric matrix of rank r is sum of r symmetric matrices of rank one. Is the decomposition unique ?

Exercises on the pfaffian

Let $\omega = [\omega_{ij}]$ be a skew-symmetric matrix $2n \times 2n$ with entries in $K = \mathbb{R}$ or $\mathbb{C}$.

Let $\{e_1, \ldots, e_{2n}\}$ be the standard basis of $V = K^{2n}$. The matrix ω corresponds to the 2-form

$$\tilde{\omega} = \sum_{i<j} \omega_{ij} e_i \wedge e_j \in \wedge^2 V$$

We set

$$\tilde{\omega}^{\wedge n} = (n!)\mathrm{Pf}(\omega) e_1 \wedge \ldots \wedge e_{2n}$$

where $\mathrm{Pf}(\omega) \in K$ is called the *pfaffian* of ω.

The pfaffian is defined up to the choice of an isomorphism $\wedge^{2n} V \simeq K$.

For example

$$\mathrm{Pf} \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a$$

$$\mathrm{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + cd$$

- 1) Prove that if $g \in GL(2n)$ we have $\mathrm{Pf}(g^t \omega g) = \det g \cdot \mathrm{Pf}(\omega)$.
- 2) Prove that $\mathrm{rk}\,\omega \geq 2k$ if and only if $\tilde{\omega}^{\wedge k} \neq 0$.
- 3) Deduce that $\mathrm{rk}\,\omega \geq 2k$ if and only if there exists a nonsingular principal submatrix of size $2k \times 2k$. Is the analogous statement for symmetric matrices true ?
- 4) Let J_r be the skew-symmetric matrix of size $2n \times 2n$ which has the first r diagonal blocks equal to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Prove that $\mathrm{Pf}(J_n) = 1$ and that there exists $g \in GL(2n)$ such that $g^t \omega g = J_r$ for some r. Deduce that $\mathrm{rk}\,\omega$ is even. *(Hint: we may assume that $\omega_{12} \neq 0$. There exists $g \in GL(2n)$ such that $(g^t \omega g)_{1i} = 0$ for $i \geq 3 \ldots$)*
- 5) Deduce by 1) and 4) that $\mathrm{Pf}^2(\omega) = \det(\omega)$.

Let ω be a skew-symmetric matrix of size $(2n+1) \times (2n+1)$.

- 6) Let $\mathrm{ad}\omega$ be the adjoint matrix of ω. Prove that $\mathrm{rk}\,(\mathrm{ad}\omega) \leq 1$ and moreover $\mathrm{rk}\,(\mathrm{ad}\omega) = 1$ if and only if $\mathrm{rk}\,(\omega) = 2n$.
- 7) Let C_i be the i-th principal subpfaffian obtained by deleting by ω the i-th row and the i-th column. If M_{ij} is the submatrix of ω obtained by deleting by ω the i-th row and the j-th column, prove that $\det(M_{ij}) = C_i C_j$.

Gaussian elimination rephrased Let A_i be the rows (resp. A^j be the columns of a $n \times m$ matrix. Then $A = \sum_{i=1}^n e_i \otimes A_i = \sum_{j=1}^m A^j \otimes e_j$. Prove that if $B_1, \ldots B_r$ is a basis of the row space, then there exist $v_i \in K^n$ such that $A = \sum_{i=1}^r v_i \otimes B_i$. Prove that if $B^1, \ldots B^r$ is a basis of the column space, then there exist $w_i \in K^m$ such that $A = \sum_{i=1}^r B^i \otimes w_i$.

Exercises Find the tangent space of $D_r \subseteq \mathrm{Hom}\ (K^{n+1}, K^{m+1})$ at the following matrices of rank r

1. i)

$$\begin{bmatrix} & 1 & \\ & & \\ & & \end{bmatrix} \text{ where 1 is at place } (i,j)$$

2. i)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Answer $a_{i,j} = x_i + y_j$

3. i)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

4. i)

$$\begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & \end{bmatrix}$$

Find the tangent space of DS_r (as subspace of symmetric matrices) at the following symmetric matrices of rank r

1. i)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Answer $a_{i,j} = x_i + x_j$

2. i)

$$\begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & \end{bmatrix}$$

Find the tangent space of D_r (as subspace of skew-symmetric matrices) at the following skew-symmetric matrices of rank r

1. i)

$$\begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & \\ & & & \end{bmatrix}$$

2. i)

$$\begin{bmatrix} 0 & 2 & & \\ -2 & & & \\ & & & 3 \\ & & -3 & \end{bmatrix}$$

Compute the intersection of the two tangent spaces of D_r (as subspace of symmetric matrices) at the following two symmetric matrices of rank r

$$\begin{bmatrix} 2 & & \\ & & \\ & & \end{bmatrix} \quad \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}$$

Let A be the tensor of format $a \times b \times c$ such that $A_{i,j,k} = t^{i+j+k}$ for a fixed $t \in K$. Compute the rank of A and the decomposition of A.

Answer If $t = 0$ the rank is zero. If $t \neq 0$ the rank is one, indeed $A = (t, \ldots, t^a) \otimes (t, \ldots, t^b) \otimes (t, \ldots, t^c)$.

It is well known that Vandermonde matrices appear in polynomial interpolation in one variable. The generalized Vandermonde matrices appear in polynomial interpolation in more variable.

2.5 The Terracini Lemma

Lemma 2.8 (First Terracini lemma) *Let $p_1, \ldots p_k \in Y$ be general points and $z \in< p_1, \ldots, p_k >$ a general point. Then*

$$T_z\sigma_k(Y) =< T_{p_1}Y, \ldots, T_{p_k}Y > .$$

Proof. Let $Y(\tau) = Y(\tau_1, \ldots, \tau_n)$ be a local parametrization of Y. We denote by $Y_j(\tau)$ the partial derivative with respect to τ_j. Let p_i be the point corresponding to $\tau^i = (\tau^i_1, \ldots, \tau^i_n)$. The space $< T_{p_1}Y, \ldots, T_{p_k}Y >$ is spanned by the $k(n+1)$ rows of the following matrix

$$\begin{matrix} \vdots \\ Y(\tau^i) \\ Y_1(\tau^i) \\ \vdots \\ Y_n(\tau^i) \\ \vdots \end{matrix}$$

(here we write only the i-th block of rows, $i = 1, \ldots, k$).

We write also the local parametrization of $\sigma_k(Y)$ given by

$$\Phi(\tau^1, \ldots, \tau^k, \lambda_1, \ldots, \lambda_{k-1}) = \sum_{i=1}^{k-1} \lambda_i Y(\tau^i) + Y(\tau^k)$$

depending on kn parameters τ_j^i and $k-1$ parameters λ_i. The matrix whose rows are given by Φ and its $kn+k-1$ partial derivatives computed at z is

$$\begin{matrix} \sum_{i=1}^{k-1} \lambda_i Y(\tau^i) + Y(\tau^k) \\ \vdots \\ \lambda_i Y_1(\tau^i) \\ \vdots \\ \lambda_i Y_n(\tau^i) \\ \vdots \\ Y_1(\tau^k) \\ \vdots \\ Y_n(\tau^k) \\ Y(\tau^1) \\ \vdots \\ Y(\tau^{k-1}) \end{matrix}$$

and its rows span $T_z\sigma_k(Y)$. It is elementary to check that the two above matrices are obtained one from the other by performing elementary operations on rows, hence they have the same row space and the same rank. $\square$

3 The symmetric flattening (catalecticant)

Clebsch noticed in the XIX century that $\sigma_5(v_4(\mathbf{P}^2))$ has not the expected dimension and it gives an interesting defective example.

Clebsch main insight was to write a quartics as a"quadric of quadrics", in the following way

$$f = a_{00}x^4+4a_{10}x^3y+4a_{01}x^3z+6a_{20}x^2y^2+12a_{11}x^2yz+6a_{02}x^2z^2+4a_{30}xy^3+12a_{21}xy^2z+12a_{12}xyz^2+$$

$$4a_{03}xz^3 + a_{40}y^4 + 4a_{31}y^3z + 6a_{22}y^2z^2 + 4a_{13}yz^3 + a_{04}z^4 = W^t \cdot C_f \cdot W$$

where $W^t = (x^2, 2xy, 2xz, y^2, 2yz, z^2)$ and

$$C_f = \begin{bmatrix} a_{00} & a_{10} & a_{01} & a_{20} & a_{11} & a_{02} \\ a_{10} & a_{20} & a_{11} & a_{30} & a_{21} & a_{12} \\ a_{01} & a_{11} & a_{02} & a_{21} & a_{12} & a_{03} \\ a_{20} & a_{30} & a_{21} & a_{40} & a_{31} & a_{22} \\ a_{11} & a_{21} & a_{12} & a_{31} & a_{22} & a_{13} \\ a_{02} & a_{12} & a_{03} & a_{22} & a_{13} & a_{04} \end{bmatrix}$$

The basic property is that if $f = l^4$ is the 4-th power of a linear form, then C_f has rank 1 (as we will see in a while with the concept of apolarity). It follows that if $f = \sum_{i=1}^5 l_i^4$ is the sum of five 4-th powers of linear forms, then

$$\mathrm{rk}\ C_f = \mathrm{rk}\ \sum_{i=1}^5 C_{l_i^4} \leq \sum_{i=1}^5 \mathrm{rk}\ C_{l_i^4} = \sum_{i=1}^5 1 = 5$$

Hence $\sigma_5(v_4(\mathbf{P}^2)) \subseteq \mathbf{P}(S^4K^3) = \mathbf{P}^{14}$ is contained in the degree 6 hypersurface given by $\det C_f = 0$, This is surprising, because the expected dimension of $\sigma_5(v_4(\mathbf{P}^2))$ is $5 * 2 + 4 = 14$. Clebsch main result was that $\sigma_5(v_4(\mathbf{P}^2))$ coincides with the "catalecticant" hypersurface with equation $\det C_f = 0$, which has degree 6 and dimension 13.

Exercise Show that there are infinitely many symmetric matrices C_f such that $f = W^t \cdot C_f \cdot W$.

Exercise Prove that $xz(xz + y^2)$ is not the sum of five 4-th powers.

Exercise What is the dual variety of catalecticant variety of plane quartics ?

A convenient way to implement the construction of the matrices of $A_\phi$on a computer is the following. Let D_i be a basis of homogeneous differential operators of degree m. Let T_j be a basis of homogeneous differential operators of degree $d - m$. Then the (i, j) entry of A_ϕ is given by $D_iT_j\phi$. With Macaulay, the above construction can be implemented with the command "diff".

If dim $V = 3$, the degrees of $\sigma_k(v_d(\mathbf{P}^2))$ has been computed by Ellingsrud and Stromme. This is useful in order to check some cases with the computer.

3.1 Interlude: Young diagrams and $SL(n)$-representations

For any filling λ of a Young diagram with d boxes, we want to define the Schur projection $c_\lambda\colon \otimes^d V \to \otimes^d V$.

Let Σ_d be the symmetric group of permutations over d elements. Due to the filling, we can consider the elements of Σ_d as permuting the boxes. Let $R_\lambda \subseteq \Sigma_d$ be the subgroup of permutations preserving each row.

Let $C_\lambda \subseteq \Sigma_d$ be the subgroup of permutations preserving each column.

For any $p \in \Sigma_d$, with abuse of notation, we call p also the induced $p \in \mathrm{End}(V^{\otimes d}$.

Let
$$c_\lambda = \sum_{\sigma\in R_\lambda}\sum_{\tau\in C_\lambda}\epsilon(\tau)\sigma\tau \in \mathrm{End}(V^{\otimes d}$$

Theorem 3.1 *The image of c_λ is a irreducible $GL(V)$-module, that we denote by $S_\lambda V$, which is nonzero iff the number of rows is smaller or equal than $\dim V$. The isomorphism class does not depend on the filling, and we can write $S_\lambda V$ just for a Young diagram λ.*

An irreducible $GL(V)$-module is isomorphic to $S_\lambda V \otimes (\det V)^a$, for some Young diagram λ and some $a \in \mathbf{Z}$.

Any $GL(V)$-modules is a sum of irreducible ones.

A irreducible $SL(V)$-modules is isomorphic to $S_\lambda V$, where the number of rows of λ is $\le \dim V - 1$. See Landsberg book or Fulton-Harris book for more details.

3.2 Inheritance

For this section, see JM lectures.

Inheritance plus the equations for $\sigma_2(v_d(\mathbf{P}^1))$, gives the equations for $\sigma_2(v_d(\mathbf{P}^n))$.

Proposition 3.2 *Let $\mathbf{P}^n = \mathbf{P}(V)$ and $k \le n$. Then for any $\phi \in S^dV$ the condition*
$$\mathit{rank}\left[A_\phi\colon V^\vee \to S^{d-1}(V)\right] \le k$$
defines the "subspace variety" consisting of hypersurfaces which are cones, with vertex a linear subspace of dimension $n-k$.

Proof We can consider a base of $V^\vee$ given by $\frac{\partial}{\partial x_i}$ for $i = 0,\dots n$. Then $A_\phi\frac{\partial}{\partial x_i} = \frac{\partial\phi}{\partial x_i}$. If the image of A_ϕ has equations $\sum_i a_{ij}\frac{\partial\phi}{\partial x_i}$ for $j = 1,\dots n-k$ then the points with coordinates $(a_{0j},\dots a_{nj})$ span the vertex of the cone $\phi = 0$. □

Corollary 3.3 *Let $\mathbf{P}^n = \mathbf{P}(V)$. Then for any $\phi \in S^dV$ the condition*
$$\mathit{rank}\left[A_\phi\colon V^\vee \to S^{d-1}(V)\right] \le 1$$
defines $v_d(\mathbf{P}^n)$.

The above description is enough to get the equations of σ_2 in higher dimension.

Theorem 3.4 *Let $\mathbf{P}^n = \mathbf{P}(V)$. The variety $\sigma_2(v_d(\mathbf{P}^n))$ is defined scheme-theoretically by the conditions*

$$rank\left[A_\phi \colon V^\vee \to S^{d-1}(V)\right] \leq 2$$

$$rank\left[A'_\phi \colon S^2V^\vee \to S^{d-2}(V)\right] \leq 2$$

Proof The first condition ensures that ϕ defines a cone with vertex a codimension two subspace, so ϕ can be defined by two homogeneous coordinates. At this point it is enough to apply the theorem for the rational normal curve $v_d(\mathbf{P}^1)$.

3.3 Apolarity

For $f \in S^dV$, we define $A(f)_{e,d-e} \colon S^eV^\vee \to S^{d-e}V$. Note that the transpose is $A(f)_{d-e,e}$.

In the classical literature $g \in S^eV^\vee$ is said to be *apolar* to f if $g \cdot f = 0$, that is if $g \in \ker A(f)_{e,d-e}$.

The application to the decomposition on sum of powers relies on a simple principle and generalizes what we have seen for binary forms to higher dimensional Veronese varieties.

Proposition 3.5 *Let $Z = \{l_1, \ldots, l_k\} \subseteq \mathbf{P}(V^\vee)$ and assume they impose independent conditions to hypersurfaces of degree d. Then $< l_1^d, \ldots, l_k^d >^\perp = H^0(I_Z(d))$*

Proof: If g is apolynomial of degree d, then g contracted with l_i^d coincides with the value of f at l_i. Then the inclusion $H^0(I_Z(d)) \subseteq < l_1^d, \ldots, l_k^d >^\perp$ is obvious and the equality follows because, by the assumption, both spaces have the same dimension. □

Proposition 3.6 *Let $f = \sum_{i=1}^k (l_i)^d$ and let $Z = \{l_1, \ldots, l_k\} \subseteq \mathbf{P}(V^\vee)$. We have $H^0(I_Z(e)) \subseteq \ker A_f$ Im $A_f \subseteq < l_1^{d-e}, \ldots, l_k^{d-e} >$.*

Moreover, if one of the previous inclusions hold, we get that there exist $c_i \in K$ such that $f = \sum_{i=1}^k c_i(l_i)^d$

Proof: The inclusion statement about the Image is obvious.

The statement about the kernel is the dual one because of Prop. 3.5, applied to the transpose map. Conversely, if $H^0(I_Z(e)) \subseteq \ker A_{e,d-e}$ then there is a decomposition of f in terms of Z that it is better written as $f = \sum_{z_i \in Z} \lambda_i z_i$ for $\lambda_i \in K$.

Indeed the apolar space of polynomials of degree d which are apolar to any $g \in I_Z(e)$ it is spanned by the duals Z_i^d, since f is in this space, it is a combination of Z_i^d.

□

Let's see an example. Let $\phi = \sum_{i=1}^5 l_i^3 \in S^3\mathbb{C}^4$, as usual denote by $Z = \{l_1, \ldots, l_5\}$ the set of 5 points in the dual space and consider the catalecticant morphism

$$A(\phi)_{2,1} \colon S^2{\mathbb{C}^4}^\vee \to S^1\mathbb{C}^4$$

The kernel of $A(\phi)_{2,1}$ is a linear system of dimension 6 of quadrics, containing the 5-dimensional system of quadrics passing through the five points giving Z.

Lemma 3.7 (Lasker) *Let $f \in V$, so that $f^d \in S^dV$. The conormal space* $\left(T_{[f^d]}v_d(\mathbf{P}(V))\right)^{\perp} \subseteq \mathbf{P}(S^dV^\vee)$ *consists of all the hypersurfaces singular at $[f]$. More precisely, if we denote by $C(V^{d,n})$ the affine cone over $V^{d,n}$, then the following holds*

$$\left(T_{[f^d]}v_d(\mathbf{P}(V))\right)^{\perp} = \left(m^2_{[f]}\right)_d \subseteq S^dV^\vee$$

Proof. Let $e_0, \ldots e_n$ be a basis of V and $x_0, \ldots, x_n$ its dual basis. Due to the $GL(V)$-action it is enough to check the statement for $f = e_0$. Then $m_{[f]} = (x_1, \ldots, x_n)$, $m^2_{[f]} = (x_1^2, x_1x_2, \ldots, x_n^2)$, so that $\left(m^2_{[f]}\right)_d$ is generated by all monomials of degree d with the exception of $x_0^d, x_0^{d-1}x_1, \ldots, x_0^{d-1}x_n$.
Since $T_{e_0^d}C(v_d(\mathbf{P}(V))) =< e_0^d, e_0^{d-1}e_1, \ldots, e_0^{d-1}e_n >$ the thesis follows. □

Lasker Lemma and Terracini Lemma imply together

Theorem 3.8 *Let $f = \sum_{i=1}^k (l_i)^d$ and let $Z = \{l_1, \ldots, l_e\} \subseteq \mathbf{P}(V^\vee)$. Then the conormal space is given by the space $H^0(I_{Z^2}(d))$ of polynomials of degree d which are singular at Z.*

Proof: The orthogonal of $T_{l_1} + \ldots + T_{l_e}$ is given by $\cap_{i=1}^e T_{l_i}^{\perp}$. □

Theorem 3.9 Criterion for the catalecticant
Let $f = \sum_{i=1}^k (l_i)^d$ and let $Z = \{l_1, \ldots, l_k\} \subseteq \mathbf{P}(V^\vee)$. If the map

$$H^0(I_Z(e)) \otimes H^0(I_Z(d-e)) \to H^0(I_{Z^2}(d))$$

is surjective then the $k+1$-minors of the catalecticant cut locally $\sigma_k(v_d(\mathbf{P}^n))$ at f as scheme. If there exists a Z such that the above map is surjective, then $\sigma_k(v_d(\mathbf{P}^n))$ is one irreducible component of the variety given by the $k+1$-minors of the catalecticant $A_{e,d-e}$.

Proof: Since the catalecticant matrix has rank one on points of $(v_d(\mathbf{P}^n)$, it has rank $\leq k$ on points of $\sigma_k(v_d(\mathbf{P}^n))$. It follows that the variety defined by the $k+1$-minors of the catalecticant contains $\sigma_k(v_d(\mathbf{P}^n))$. So it is enough to see that the two conormal space coincide.

The conormal space at the variety cutted by the $k+1$-minors contains the image of

$$H^0(I_Z(e)) \otimes H^0(I_Z(d-e)) \to H^0(I_{Z^2}(d))$$

(by Cor. 2.5 and Prop. 3.6).

Since the target space is the conormal space at f of $\sigma_k(v_d(\mathbf{P}^n))$ by Thm. 3.7, the thesis follows. □

The first application is the promised proof that the minors of the catalecticant cut $v_d(\mathbf{P}^1)$ transversally.

End of proof of Thm. 1.3 Let $Z \subset \mathbf{P}^1$ be a finite set of points. Then $H^0(I_Z(e))$ consists of the polynomials of degree e whose roots contain Z. If $Z = (\alpha_1, \beta_1), \ldots (\alpha_k, \beta_k)$, then the polynomials in $H^0(I_Z(e))$ have the form $\prod_{i=1}^k(x\beta_i - y\alpha_i)g_{e-k}$ where g_{e-k} is any polynomial of degree $e-k$. The polynomials in $H^0(I_{Z^2}(d))$ have the form $\prod_{i=1}^k(x\beta_i - y\alpha_i)^2 g_{d-2k}$ where g_{d-2k} is any polynomial of degree $d-2k$. Then the surjectivity of

$$H^0(I_Z(e)) \otimes H^0(I_Z(d-e)) \to H^0(I_{Z^2}(d))$$

is obvious. $\square$

Theorem 3.10 *[Iarrobino-Kanev Theorem 4.10A] In the case $d = 2m$, in the range $k \leq \binom{m+n-1}{n}$, the $k+1$-minors of the catalecticant $A(\phi)_{m,m}$ define a variety which contains $\sigma_k(v_d(\mathbf{P}^n))$ as irreducible component.*

Proof: Let $Z = \{u_1, \ldots, u_k\} \subseteq \mathbf{P}^n$. By the Thm. 3.9, it is enough to show that

$$H^0(I_Z(m)) \otimes H^0(I_Z(m)) \xrightarrow{g} H^0(I_{Z^2}\mathcal{O}(2m))$$

is surjective. Thanks to the Theorem in [LO], we may assume that $k = \binom{m+n-1}{n}$. This case can be proved by degenerating Z to the set of vertices of $m+n-1$ general hyperplanes, every vertex being the intersection of n hyperplanes.

In order to prove this particular case, note that $h^0(I_Z(m)) = \binom{m+n}{m} - \binom{m+n-1}{m-1} = \binom{m+n-1}{m}$ and a basis of $H^0(I_Z(m))$ is given by all the polynomials obtained as product of m distinct hyperplanes among the $m+n-1$ hyperplanes. Indeed every such product forgets exactly $n-1$ hyperplanes, and then vanishes on every vertex. Moreover these monomials are independent, because for any I such that $|I| = n-1$, the monomial $\prod_{i \notin I} x_i$ is the only one which do not vanish on the line $\{x_i = 0, \text{ for } i \in I\}$, hence it cannot be a linear combination of the others. For $m = 1$ (any n) or $n = 1$ (any m) the surjectivity of g is obvious. Let us prove it, in general, by double induction on m and n. Fix one hyperplane H_1, according to it decompose $Z = Z_1 \cup Z_2$ where Z_1 consists of the $\binom{m+n-2}{n-1}$ points on H_1 and Z_2 consists of the $\binom{m+n-2}{n}$ points outside H_1. We have the following exact sequence

$$0 \longrightarrow H^0(I_{Z_2^2 \cup Z_1}(2m-1)) \xrightarrow{g} H^0(I_{Z^2}(2m)) \longrightarrow H^0(I_{Z_1^2, H_1}(2m))$$

where g is the multiplication by the equation of H_1. From this sequence it follows by induction that $h^0(I_{Z^2}(2m))$ has the expected dimension $\binom{2m+n}{n} - (n+1)\binom{m+n-1}{m-1}$, so that the sequence is exact also on the right. By the inductive assumption $H^0(I_{Z_2}(m-1)) \otimes H^0(I_{Z_2}(m-1))$ surjects over $H^0(I_{Z_2^2}(2m-2))$, hence $H^0(I_Z(m)) \otimes H^0(I_{Z_2}(m-1))$ surjects over $H^0(I_{Z_2^2 \cup Z_1}(2m-1))$. Consider the commutative diagram

$$\begin{array}{ccccccc} & & H^0(I_Z(m)) \otimes H^0(I_{Z_2}(m-1)) & \longrightarrow & H^0(I_Z(m)) \otimes H^0(I_Z(m)) & \longrightarrow & H^0(I_{Z_1,H_1}(m)) \otimes H^0(I_{Z_1,H_1}(m)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\ 0 & \longrightarrow & H^0(I_{Z_2^2 \cup Z_1}(2m-1)) & \xrightarrow{g} & H^0(I_{Z^2}(2m)) & \longrightarrow & H^0(I_{Z_1^2,H_1}(2m)) & & \end{array}$$

We have seen that h_1 is surjective, by the inductive assumption we have also that h_3 is surjective. A diagram chase shows that h_2 is surjective as we wanted. $\square$

Remark Iarrobino and Kanev prove a similar statement for odd degree. In the last sections we will see that with the Young flattening it is possible to improve this bound.

Remark There are cases covered by Thm. 3.10 where the minors of catalecticant cut several irreducible components, only one of them is the secant variety. The first case is described in example 7.11 of [Iarrobino-Kanev] and corresponds to 11-minors of $A(f)_{4,4}$ for $f \in S^8(\mathbb{C}^3)$, which define at least two components, one of them being $\sigma_{10}(v_8(\mathbf{P}^2))$.

Exercise Find an example where

$$H^0(I_Z(e)) \otimes H^0(I_Z(d-e)) \to H^0(I_{Z^2}(d))$$

is not surjective. *Hint: Take Z be given by 3 general points in $\mathbf{P}^2$ and $d = 3$, any e.*

4 The apolar ring and the Iarrobino-Kanev conditions

The subspace of $\oplus_e S^e V^\vee$ of operators g such that $g(\phi) = 0$ is an ideal I_ϕ. Note that when $e \geq \deg \phi$ then the graded part of I_ϕ coincides with $S^e V^\vee$.

The quotient ring $R_\phi = (\oplus_e S^e V^\vee) / I_\phi$ is an Artinian graded ring, which is called the *apolar ring* of ϕ.

Theorem 4.1 *Macaulay Lemma*
$R_\phi \simeq R_{\phi'} \Longleftrightarrow \phi = c\phi'$ *for some* $c \in K^*$.

Proof: Let R^i_ϕ be the graded summand of degree i of R_ϕ. Note that R^d_ϕ has dimension 1 and it can be identified with K, up to the choice of a scalar. Then the multiplication $S^d(R^1_\phi) \to R^d_\phi \simeq K$ defines a polynomial which can be identified with ϕ.
□

The Macaulay lemma says that the apolar rings contains all the necessary informations on the polynomial ϕ. For more details see [Ranestad-Schreyer], where a syzygy approach is used to compute the Waring decomposition. The dimension of the graded summands of the apolar ring, can be computed by the ranks of the several catalecticant matrices.

On $\mathbf{P}^2$, the theorem of Buchsbaum-Eisenbud describes correspondingly all the possible sequences for the ranks of catalecticant matrices.

Each sequences of ranks corresponds to several skew-symmetric resolutions. Among these resolutions, there is a minimal one.

Let see some examples

For general $\phi \in \sigma_6(v_6(\mathbf{P}^2))$ we have the resolution

$$\mathcal{O}(-9) \longrightarrow \mathcal{O}(-6)^4 \oplus \mathcal{O}(-4)^3 \longrightarrow \mathcal{O}(-5)^3 \oplus \mathcal{O}(-3)^4 \longrightarrow I_\phi \longrightarrow 0$$

For general $\phi \in \sigma_7(v_6(\mathbf{P}^2))$ we have the resolution

$$\mathcal{O}(-9) \longrightarrow \mathcal{O}(-6)^3 \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-4) \longrightarrow \mathcal{O}(-5) \oplus \mathcal{O}(-4) \oplus \mathcal{O}(-3)^3 \longrightarrow I_\phi \longrightarrow 0$$

For general $\phi \in \sigma_8(v_6(\mathbf{P}^2))$ we have the resolution

$$\mathcal{O}(-9) \longrightarrow \mathcal{O}(-6)^2 \oplus \mathcal{O}(-5)^3 \longrightarrow \mathcal{O}(-4)^3 \oplus \mathcal{O}(-3)^2 \longrightarrow I_\phi \longrightarrow 0$$

For general $\phi \in \sigma_9(v_6(\mathbf{P}^2))$ we have the resolution

$$\mathcal{O}(-9) \longrightarrow \mathcal{O}(-6) \oplus \mathcal{O}(-5)^6 \longrightarrow \mathcal{O}(-4)^6 \oplus \mathcal{O}(-3) \longrightarrow I_\phi \longrightarrow 0$$

For general $\phi \in \sigma_{10}(v_6(\mathbf{P}^2)) = \mathbf{P}^{27}$ we have the resolution

$$\mathcal{O}(-9) \longrightarrow \mathcal{O}(-5)^9 \longrightarrow \mathcal{O}(-4)^9 \longrightarrow I_\phi \longrightarrow 0$$

Often important informations can be achieved by the sequence of ranks of the several catalecticants, of different size.

Indeed, the polynomials such that their sequence of ranks of the several catalecticants is fixed, make a irreducible variety. This makes a description with *open and closed conditions.*

This applies in particular to $\sigma_k(v_d(\mathbf{P}^n))$, which have an open part which can be described exactly by asking, for $2e \leq d$ that rk $A(f)_{e,d-e} = \min\left(k, \binom{n+e}{e}\right)$

Example This example is due to S. Diesel, see also Iarrobino-Kanev , example 7.11.

The sequence of ranks of rk $A(f)_{e,8-e}$ for general $f \in \sigma_{10}(v_8(\mathbf{P}^2))$ is (for $e = 0, 1, \ldots$) $1, 3, 6, 10, 10, 10, 6, 3, 1$.

There is another irreducible component, containing $x^3y^3z^2$ such that the sequence of ranks is $1, 3, 6, 9, 10, 9, 6, 3, 1$.

The general element of this second component has border rank bigger than 10.

The **Conca-Valla formula** describes the dimension of these components

Let $H = (h_0, h_1, \ldots\ldots)$ be the finite sequence of the ranks of the catalecticants. This is the Hilbert function of the apolar ring. Let $p_i = h_i - 3h_{i-1} + 3h_{i-2} - h_{i-3}$ the third difference sequence.

Let $Gor(H)$ be the variety of polynomials $\phi \in \mathbf{P}(S^d(\mathbb{C}^3))$ such that the apolar ring of ϕ has Hilbert function H.

Theorem 4.2 *Iarrobino-Diesel*

$Gor(H)$ is an irreducible quasi-projective variety

Theorem 4.3 *Conca-Valla*

if $d = 2t$ is even, $\dim Gor(H) = \left(h_t + 3h_{t-1} - \sum_{i=0}^{j} h_i p_i\right)/2$

while if $d = 2t+1$ is odd, $\dim Gor(H) = \left(3h_t + h_{t-1} - \sum_{i=0}^{j} h_i p_i\right)/2$

Remark By analizing possible Hilbert functions occurring, we have found in [Landsberg-Ottaviani] the equations for several $\sigma_k(v_d(\mathbf{P}^2))$ for $k \leq 6$.

Let f be a hypersurface of degree $d$and let z be a point. Let's recall that the polar $P_z(f)$ is the degree $d-1$ hypersurface $\sum z_i \frac{\partial f}{\partial x_i}$ (this does not depend on the choice of coordinates).

Consider, for a cubic f, the locus of x such that the polar $P_x(f)$ is singular. This coincides with the Hessian of f.

4.1 De Paolis algorithm

Let C be a general cubic curve defined by $F \in S^3(V)$. Given a general line $l_0 \subset \mathbf{P}^2$ there are three uniquely defined lines l_i for $i = 1, \ldots 3$ (that we identify with their equations) such that (with suitably chosen constants defining the equations)

$$F = \sum_{i=0}^{3} l_i^3$$

The three lines can be found in the following way. If $l_0 \cap H(C) = \{P_1, P_2, P_3\}$, denote by Q_i the singular point of the conic $P_{P_i}C$. Then $l_i = < P_i, Q_j >$ with any $j \neq i$.

Proof. Assume $F = \sum_{i=0}^{3} l_i^3$ and let $P_1 = l_0 \cap l_1$. Then $P_{P_1}C$ has equation $l_2^3 + l_3^3 = 0$ and it is singular on $l_2 \cap l_3$ which we denoted Q_1. In particular P_1 belongs to $H(C)$. The same argument works for P_2 and P_3, so that the points Q_i are uniquely determined by the points P_i, which in turn are the intersection points of l_0 with $H(C)$.

4.2 Sylvester Pentahedral Theorem

Theorem 4.4 *Let $f \in S^3\mathbb{C}^4$ be general. There is a unique decomposition $f = \sum_{i=1}^{5} l_i^3$ where the 10 vertices points of the 5 planes l_i coincide with the 10 points such that rk $P_x(f) \leq 2\}$.*

Proof: Let's prove the uniqueness, which is an extension of De paolis algorithm. Assume we have one decomposition $f = \sum_{i=1}^{5} l_i^3$. Consider $\{x \in \mathbf{P}^3 | \text{rk } P_x(f) \leq 2\}$.

It is given by the intersection of a linear $\mathbf{P}^3$ with the variety $DS_2 \subseteq \mathbf{P}^9$ which has codimension 3 and degree 10. Hence the intersection it is given by 10 points. But the 10 points are already obtained by $\{l_i = l_j = l_k = 0\}$. Indeed in these 10 points, 3 among the 5 summands are killed and we are left with a quadric of rank 2. Hence there are no other points. Conversely these 10 points determine l_i.

In the following subsections we see some examples on how we can use of the equations of secant varieties in order to find the tensor decomposition.

4.3 The example of $3 \times 3 \times 5$ tensors

We report on a work of ten Berge.

Jos M. F. ten Berge Partial uniqueness in CANDECOMP/PARAFAC Journal of Chemometrics (2004) Volume 18 Issue 1, Pages 12 - 16

This example is very interesting. Let $\phi \in A \otimes B \otimes C$ where $\dim A = \dim B = 3$, $\dim C = 5$. Let $X = \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^4$, note that $\sigma_5(X)$ fills the ambient space (over the complex numbers) and that a general $\phi \in A \otimes B \otimes C$ has rank 5 and has finitely decompositions as sum of five decomposable tensors. We wonder how to find these decompositions. Again the equations of the secant varieties, coming from the minors of $\mathrm{Hom}(A^\vee \otimes B^\vee, C)$, are useful.

Consider the contraction morphism $A^\vee \otimes B^\vee \to C$

For general ϕ, the kernel has codimension 5, dually the cokernel has codimension 4 and meets the Segre variety in 6 points, because $6 = \deg P^2 \times P^2$. It turns out that, on the complex numbers, six different decompositions hold, any two of them share four summands! Indeed the cokernel , which has codimension 4 and projective dimension 4, i t is spanned by any 5 of the six intersection points. ten Berge says that in this case we have "partial uniqueness".

ten Berge proves that the only typical ranks over the real numbers are 5 and 6.

We can state the result in a dual formulation with the image that is we consider $\mathbb{C}^{5\vee} \to \mathbb{C}^3 \otimes \mathbb{C}^3$

and the image meets the Segre varietyin six points.

Exercise Generalize to the case $3 \times a \times (2a-1)$, $a \geq 2$. It is the first balanced case.

Prove that a general $\phi \in \mathbb{C}^{2a-1} \otimes \mathbb{C}^3 \otimes \mathbb{C}^a$ has rank $2a-1$ and there are exactly

$$\binom{a(a+1)/2}{2a-1}$$

possible decompositions. For $a=4$ they are $\binom{10}{3}=120$, each of them choosen among 10.

Moreover s

Exercise Generalize the previous exercise to the case $a \times b \times [(a-1)(b-1)+1]$. (it is the border case of balanced, see next sections)

In particular prove that uniqueness holds in cases

$2x2x2$ (generic rank is 2),

$3x3x2$ (generic rank is 3)

$n \times n \times 2$ (generic rank is n, related to Kronecker normal form in Weyman's lectures)

4.4 Symultaneous decompositions

Theorem 4.5 *Richmond, Roberts (i) Given general $f \in S^3\mathbb{C}^2$, $g \in S^4\mathbb{C}^2$ there exist a unique symultaneous decomposition with three summands $f = \sum_{r=1}^{3} l_i(x,y)^3, g = \sum_{r=1}^{3} l_i(x,y)^4$.*

(ii) Given general $f \in S^2\mathbb{C}^3$, $g \in S^3\mathbb{C}^3$ there exist a unique symultaneous decomposition with four summands $f = \sum_{r=1}^{4} l_i(x,y,z)^2, g = \sum_{r=1}^{4} l_i(x,y,z)^3$.

Proof: In case (i) there is a unique apolar cubic, with three zeroes.

In case (ii) there is a pencil of apolar conics, which has four base points. □

Theorem 4.6 *Given general $f, g \in S^3\mathbb{C}^3$, it does not exist a symultaneous decoposition with five summands $f = \sum_{r=1}^{5} l_i(x,y,z)^3, g = \sum_{r=1}^{5} l_i(x,y,z)^3$.*

5 The unbalanced case for the Segre varieties, the Alexander-Hirschowitz Theorem

In this section, $X = \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k}$ with $k \geq 3$ and $n_1 \leq \ldots \leq n_k$. We classify Segre varieties, X, for which $\sigma_r(X)$ is defective with $r \leq 6$.

Following [BCS], the typical tensor rank of a format $(n_1, \ldots, n_k)$ is the smallest integer s such that $\sigma_s(\mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k})$ fills the ambient space, and it is denoted by $\underline{R}(n_1, \ldots, n_k)$. Equivalently, the generic tensor in $V_1 \otimes \ldots \otimes V_k$ where $\dim V_i = n_i + 1$ is the sum of $\underline{R}(n_1, \ldots, n_k)$ (and not less) tensors of rank one. Obviously we have

$$\left\lceil \frac{\prod(n_i+1)}{1+\sum n_i} \right\rceil \leq \underline{R}(n_1, \ldots, n_k)$$

and in particular

$$\left\lceil \frac{(n+1)^k}{nk+1} \right\rceil \leq \underline{R}(n^k).$$

The following lemma is well-known (see [CGG1, Proposition 3.3]).

Lemma 5.1 *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $1 \leq n_1 \leq \cdots \leq n_k$. Suppose that*

$$\prod_{i=1}^{k-1}(n_i+1) - \sum_{i=1}^{k-1} n_i < d < \min\left\{\prod_{i=1}^{k-1}(n_i+1), n_k+1\right\}.$$

Then X has a defective d-secant variety.

Proof: Pick d general points on X where d satisfies the conditions of the Lemma. Since $d < n_k+1$, there exists a subvariety $V = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{k-1}} \times \mathbb{P}^{d-1} \subseteq X$, which contains these d points. Let $N(d) = d\prod_{i=1}^{k-1}(n_i+1)-1$ and $N = \prod_{i=1}^{k}(n_i+1)-1$. The span of V is $\mathbb{P}^{N(d)} \subseteq \mathbb{P}^N$. Thus, the linear subspace spanned by the tangent spaces of X at the d points has dimension at most $F(d)-1$, where $F(d) = d\left[\prod_{i=1}^{k-1}(n_i+1) + (n_k+1-d)\right]$. Then, by the assumption as given above, we have

$$\begin{aligned} d\left(\sum_{i=1}^{k} n_i + 1\right) - F(d) &= d\left(\sum_{i=1}^{k} n_i + 1\right) - d\left[\prod_{i=1}^{k-1}(n_i+1) + (n_k+1-d)\right] \\ &= d\left[\sum_{i=1}^{k-1} n_i - \prod_{i=1}^{k-1}(n_i+1) + d\right] > 0 \end{aligned}$$

and

$$\begin{aligned} \prod_{i=1}^{k}(n_i+1) - F(d) &= d^2 - d\left[\prod_{i=1}^{k-1}(n_i+1) + (n_k+1)\right] + \prod_{i=1}^{k}(n_i+1) \\ &= \left[d - \prod_{i=1}^{k-1}(n_i+1)\right][d-(n_k+1)] > 0. \end{aligned}$$

So $F(d) < \min\left\{d\left(\sum_{i=1}^{k} n_i + 1\right), \prod_{i=1}^{k}(n_i+1)\right\}$. An application of Terracini's lemma shows that X has a defective d-secant variety.

Definition 5.2 *Suppose* $\mathbf{n} = (n_1, \dots, n_k)$ *with* $n_1 \leq \cdots \leq n_k$.

- $\mathbf{n}$ *is called balanced if* $n_k \leq \prod_{i=1}^{k-1}(n_i + 1) - \sum_{i=1}^{k-1} n_i$.
- $\mathbf{n}$ *is called unbalanced if* $n_k - 1 \geq \prod_{i=1}^{k-1}(n_i + 1) - \sum_{i=1}^{k-1} n_i$.

Thus Lemma 5.1 states that if $\mathbf{n} = (n_1, \dots, n_k)$ is unbalanced then $\mathbb{P}^{\mathbf{n}}$ is defective. The following proposition is often useful.

Proposition 5.3 *Let* $\mathbf{n} = (n_1, \dots, n_k)$ *be balanced. If* $s \leq n_k$ *then* $\sigma_s(\mathbb{P}^{\mathbf{n}})$ *has the expected dimension (and does not fill the ambient space).*

The following theorem sets completely the defective behaviour of higher secant varieties in the unbalanced cases. This has also been observed as part of Theorem 2.4 in [CGG4].

Theorem 5.4 *Let* $\mathbf{n} = (n_1, \dots, n_k)$ *be unbalanced.*

(i) $\sigma_s(\mathbb{P}^{\mathbf{n}})$ *has the expected dimension (and does not fill the ambient space) if and only if* $s \leq \prod_{i=1}^{k-1}(n_i + 1) - \sum_{i=1}^{k-1} n_i$.

(ii) $\underline{R}(\mathbf{n}) = \min\{n_k + 1, \prod_{i=1}^{k-1}(n_i + 1)\}$

In particular unbalanced implies defective. The equations of $\sigma_s(\mathbb{P}^{\mathbf{n}})$ *have always degree* $s+1$ *and come from the flattening* $\left(\prod_{i=1}^{k-1}(n_i + 1)\right) \times (n_k+1)$. *See the Theorem 2.4 of [CGG] arXiv 0609054.*

5.1 The classification of defective σ_r for $r \leq 6$

Theorem 5.5 *Let* $k \geq 3$.
$\sigma_2(\mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k})$ *is never defective.*
$\sigma_3(\mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k})$ *is non-defective with the following exceptions:*
$(n_1, n_2, n_3) = (1, 1, a)$ *with* $a \geq 3$ *(unbalanced)*
and $(n_1, n_2, n_3, n_4) = (1, 1, 1, 1)$.

Theorem 5.6 $\sigma_4(\mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k})$ *is non-defective with the following exceptions:*
$(n_1, n_2, n_3) = (1, 2, a)$ *with* $a \geq 4$ *(unbalanced)*
and $(n_1, n_2, n_3) = (2, 2, 2)$.

Proposition 5.7 *If* $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^n$ *then*

(i) X *has a defective* $2n+1$*-secant variety.*

(ii) The codimension of $\sigma_{2n+1}(X)$ *is 2.*

(iii) The codimension of $\sigma_k(X)$ *is the expected one for* $k \neq 2n+1$, *in particular* σ_{2n+2} *fills the ambient space.*

Proposition 5.8 $\dim\sigma_5(\mathbf{P}^2\times\mathbf{P}^3\times\mathbf{P}^3)=43$, *while the expected dimension is* 44. *This is proved by showing that through* 5 *general points there is a rational normal curve of degree* 8.

$\dim\sigma_k(\mathbf{P}^2\times\mathbf{P}^3\times\mathbf{P}^3)$ *has the expected dimension for all* $k\neq 5$, *in particular* σ_6 *fills the ambient space.*

Theorem 5.9 $\sigma_5(\mathbf{P}^{n_1}\times\ldots\times\mathbf{P}^{n_k})$ *is non-defective with the following exceptions:*
$(n_1,n_2,n_3)=(2,3,3)$
$(n_1,n_2,n_3)=(1,2,a)$ *with* $a\geq 5$ *(unbalanced)*
$(n_1,n_2,n_3)=(1,3,a)$ *with* $a\geq 5$ *(unbalanced)*
$(n_1,n_2,n_3,n_4)=(1,1,2,2)$

Theorem 5.10 $\sigma_6(\mathbf{P}^{n_1}\times\ldots\times\mathbf{P}^{n_k})$ *is non-defective with the following exceptions (all unbalanced):*
$(n_1,n_2,n_3)=(1,3,a)$ *with* $a\geq 6$
$(n_1,n_2,n_3)=(1,4,a)$ *with* $a\geq 6$
$(n_1,n_2,n_3)=(2,2,a)$ *with* $a\geq 6$
$(n_1,n_2,n_3,n_4)=(1,1,1,a)$ *with* $a\geq 6$

A nice example is the format $2\times 2\times(n+1)$. Take up to size 4 minors of the $4\times(n+1)$ matrix.

5.2 The results and the conjectures on the dimension

Theorem 5.11 (Alexander-Hirschowitz) *Let* X *be a general collection of* k *double points in* $\mathbf{P}^n=\mathbf{P}(V)$ *and let* $S^dV^\vee$ *be the space of homogeneous polynomials of degree* d. *Let* $I_X(d)\subseteq S^dV^\vee$ *be the subspace of polynomials through* X, *that is with all first partial derivatives vanishing at the points of* X. *Then the subspace* $I_X(d)$ *has the expected codimension* $\min\left((n+1)k,\binom{n+d}{n}\right)$ *except in the following cases*

	exp.codim	*codim*	*equation*
$\sigma_k(\mathbf{P}^n,\mathcal{O}(2))\quad 2\leq k\leq n$	$\max(\frac{(n+1)(n+2-2k)}{2},0)$	$\binom{n-k+2}{2}$	$(k+1)-$ *minors*
$\sigma_{\frac{1}{2}n(n+3)}(\mathbf{P}^n,\mathcal{O}(4))\quad n=2,3,4$	0	1	*catalecticant inv.*
$\sigma_7(\mathbf{P}^4,\mathcal{O}(3))$	0	1	*invariant of degree* 15

We remark that the case $n=1$ is the only one where the assumption that X is general is not necessary. The degree 15 invariant comes from the cube of the determinant of a 45×45 matrix.

Conjecture Let $k\geq 3$. Other than $(\mathbf{P}^2)^3$ and $(\mathbf{P}^1)^4$, every Segre variety of the form $(\mathbf{P}^n)^k$ is nondefective.

Conjecture Let the number of factors $k\geq 3$. All defective Segre varieties fall into the following 4 classes:

1. X is unbalanced.

2. $X=\mathbf{P}^2\times\mathbf{P}^n\times\mathbf{P}^n$ with n even.

3. $X = \mathbf{P}^2 \times \mathbf{P}^3 \times \mathbf{P}^3$.

4. $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^n$.

$\sigma_{2n+1}(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^n)$ has codimension 2 (the expected value is one) and there are two equations of degree $2(n+1)$ defining it, which are the two determinants of the two flattenings $\mathbb{C}^2 \otimes \mathbb{C}^{n+1} \to \mathbb{C}^2 \otimes \mathbb{C}^{n+1}$.

When $X = \mathbf{P}^2 \times \mathbf{P}^n \times \mathbf{P}^n$ with n even, then $\sigma_{\frac{3n}{2}+1}(X)$ is a hypersurface of degree $3(n+1)^2$.

Regarding Grassmannians, there is the following conjecture proposed in [BDG] (conj. 4.1)

Conjecture (Baur-Draisma-de Graaf) Let $k \geq 2$. $\sigma_s(Gr(k,n))$ has the expected dimension with the only exceptions:

		codim	exp. codim
(1)	$\sigma_3(Gr(2,6))$	1	0
(2)	$\sigma_3(Gr(3,7))$	20	19
(2′)	$\sigma_4(Gr(3,7))$	6	2
(3)	$\sigma_4(Gr(2,8))$	10	8

The equation of $\sigma_3(Gr(2,6))$ is an invariant of degree 7, which is the cube of the determinant of a 21×21 matrix.

Theorem 5.12 *If $\omega \in \wedge^3\mathbf{C}^7$ consider the contraction operator*

$$\phi_\omega \colon \wedge^2 \mathbf{C}^7 \to \wedge^5\mathbf{C}^7$$

The equation of $\sigma_3(Gr(\mathbf{P}^2, \mathbf{P}^6))$ is given by a $SL(7)$-invariant polynomial P_7 of degree seven such that

$$\det(\phi_\omega) = 2\,[P_7(\omega)]^3$$

There are geometric explanations for each of the exceptions listed in the Conjecture.

Another case of interest are the Segre-Veronese case, which is a mixture of the previous two. There is a particular interest in the case $U \otimes S^2V$, which can be considered as partially symmetric tensors. The "rank 1" variety is $\mathbf{P}(U) \times \mathbf{P}(V)$ embedded with $\mathcal{O}(1,2)$.

After the work of several people, among them Catalisano, Geramita, Gimigliano, Chipalkatti, Carlini Abo,Brambilla, the last two authors proposed a conjecture that the list of defective $\sigma_k(X)$ where $X = \mathbf{P}^n \times \mathbf{P}^m$ is embedded by $\mathcal{O}(1,2)$ is given by

the unbalanced cases

$(n,m,k) = (2, 2s+1, 3s+2)$ for $s \geq 1$

$(n,m,k) = (4,3,6)$

There are several positive results supporting this conjecture.

5.3 Asymptotic result for Segre varieties

For three factor "hypercubic" Segre varieties, there is a precise result, due to Lickteig and Strassen.

Theorem 5.13 *Strassen-Lickteig $\sigma_k((\mathbf{P}^n)^3)$ has always the expected dimension for any $n \neq 2$.*

Theorem 5.14 *Let $X = (\mathbf{P}^n)^k$, $k \geq 3$. Let s_k and δ_k be defined by*

$$s_k = \left\lfloor \frac{(n+1)^k}{nk+1} \right\rfloor \quad \text{and} \quad \delta_k \equiv s_k \ mod\ (n+1) \quad \text{with} \quad \delta_k \in \{0, \ldots, n\}.$$

(i) If $s \leq s_k - \delta_k$ then $\sigma_s(X)$ has the expected dimension.

(ii) If $s \geq s_k - \delta_k + n + 1$ then $\sigma_s(X)$ fills the ambient space.

The case of format $2 \times 2 \times \ldots \times 2$ (k times) has been completely solved by Catalisano, Geramita, Gimigliano. It turns out that the only defective case is when $k = 4$.

There is also a general positive result in [AOP] that we mention here

Theorem 5.15 *If n is odd then the Segre variety $\mathbf{P}^k \times (\mathbf{P}^n)^{k+1}$ is perfect.*

6 The real case

We begin with some results about zeroes of real polynomials in one variable. Equivalently of homogeneous polynomials in two variables.

The discriminant of a polynomial $F(x,y) = \sum_{i=0}^{d} a_i x^{d-i} y^i = a_0 \prod_{i=1}^{r}(x - \alpha_i y)$ is the homogeneous polynomial of degree $2(d-1)$.

This is the resultant of f and of its first derivative. More geometrically, it is the resultant of F_x and F_y. So the discriminant does not vanish exactly when F_x and F_y have no nonconstant common factor.

Geometrically, the discriminant is the equation of the variety of $n-2$ osculating spaces to the rational normal curve. We can express $\Delta = a_0^{2(d-1)} \prod_{i<j}(\alpha_i - \alpha_j)^2$.

Outside $\Delta = 0$, the polynomials have distinct roots. The hypersurface $\Delta = 0$ leaves a complement which has exactly $\lfloor \frac{n}{2} \rfloor + 1$ connected components (with respect to the euclidean topology). Each component is labelled by the number of pairs of conjugate roots, which goes from 0 to $\lfloor \frac{n}{2} \rfloor$. The case with zero pairs is particularly important, in this case all the roots are real.

This components have a intricate structure.

The picture for $n = 2$ already reveals an interesting structure.

S_1 is convex, S_0 is not convex.

Namely:

there are lines such that all polynomials in the line have no real roots, so these lines are contained in S_0.

Every line meets S_0, and even the interior of S_0. There are no lines contained in S_1.

Question Does every hyperplane meet (the interior of) S_0

Define $p_k = \sum_{i=0}^{d} \alpha_i^k$, power sums

Define the matrix P of size $\{0..x\}$ such that $P_{ij} = p_{i+j}$. These coefficients can be expressed as polynomial in a_i by the Newton identities.

This is a symmetric (Hankel) matrix.

Theorem 6.1 Sylvester *If P has signature (p,q) then $F(x,y)$ has $d-q$ real roots.*

In particular P is semidefinite positive if and only if F has only real roots.

The condition of the theorem can be rephrased.

Theorem 6.2 Newton *Let $F(x,y) = \sum_{i=0}^{d} a_i x^{d-i} y^i$ be a polynomial with all real roots. Then $\begin{vmatrix} a_i & a_{i+1} \\ a_{i+1} & a_{i+2} \end{vmatrix} \leq 0$ for every $i = 0..d-2$.*

If the roots of F are distinct, then the inequalities are strict.

Proof: If F has all real roots, the same is true for both F_x and F_y (Rolle theorem). Then $F_{x^{d-i-2}y^i}(x,y) = \frac{(d-i)!i!}{2} a_i x^2 + 2a_{i+1}xy + a_{i+2}y^2$ has only real roots. □

Note how the projective approach has made possible the above simple proof.

The Newton conditions are necessary, but are sufficient only for $n = 2$.

Exercise

1. Exhibit a real polynomial F of degree 3, where the Newton conditions are satisfied, but such that F has only one real root.

2. Show that the determinant of the Sylvester matrix P is equal to the discriminant. item Draw a picture in the (a, b) plane of the regions where $x^4 + ax^2 + b$ has 0, 2 or 4 real roots.

The real Sylvester algorithm proceeds exactly as in the complex case, but we have to check if the polynomials in the kernel have all real roots.

Definition 6.3 *A rank r is called typical it the locus $\{x|rk(x) = r\}$ has non empty interior.*

Theorem 6.4 *There are two typical ranks for real polynomials of degree 3, namely 2 and 3. Precisely, a real polynomial of degree 3*
has rank 3 if and only if it has 3 real roots
has rank 2 if and only if it has 2 real roots

Theorem 6.5 *There are two typical ranks for real polynomials of degree 4, namely 3 and 4. Precisely, a real polynomial of degree 4*
has rank 4 if and only if it has 4 real roots
has rank 3 if and only if it has 0 or 2 real roots

Theorem 6.6 *There are three typical ranks for real polynomials of degree 5, namely 3, 4 and 5.*
Moreover a real polynomial of degree 5 has rank 5 if and only if it has 5 real roots
the typical rank 4 can appear in both cases with 0 or 2 real roots.
the typical rank 3 can appear in both cases with 0 or 2 real roots.

Theorem 6.7 Causa-Re *Let $d \geq 3$. The following are equivalent.*
(i) All the roots of F are real.
(ii) For every $(\lambda, \mu) \neq (0, 0)$ all the roots of $\lambda F_x + \mu F_y$ are real.

As a consequence we have

Theorem 6.8 *A general real polynomial of degree d has rank d if and only if it has all real roots.*

Exercises

1. Find a decomposition of $x^3 - 3xy^2$

2. Find a decomposition of x^2y. Generalize to $x^n y$.

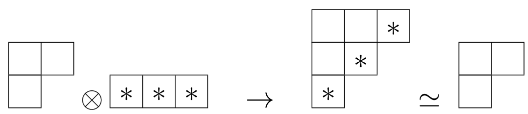
3. Let $F(x, y) = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i$ $(d \geq 3)$ and let $G(b_0, b_1, b_2, b_3) := \begin{bmatrix} b_0 & b_2 \\ b_1 & b_3 \end{bmatrix}^2 - 4 \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \cdot \begin{bmatrix} b_0 & b_1 \\ b_1 & b_2 \end{bmatrix}$ Prove the following generalization of Newton criterion. If F has real roots then $G(a_i, a_{i+1}, a_{i+2}, a_{i+3}) \leq 0$ for $i = 0 \ldots (d-3)$. If F has distinct roots, the strict inequalities hold.

7 Strassen equation and generalizations

Le teorie vanno e vengono, ma le formule restano. G.C. Rota 1991

7.1 The Aronhold invariant

Map $S^3V \subseteq Hom(EndV, EndV)$ by the following construction. If $\phi = v^3$ is decomposable then $M \in EndV$ goes to the map which takes $w \in W$ to $(M(v) \wedge v \wedge w)v$ and it is extended by linearity. The map factorizes to $S^3V \subseteq Hom(ad\ V, ad\ V)$ corresponding to the picture



where $ad\ V \simeq S_{2,1}V$ is the space of traceless endomorphisms.

Thus, when $n = 2$, $\phi \in S^3V$ gives rise to an element of $\mathbb{C}^9 \otimes \mathbb{C}^9$. In bases, if we write

$$\begin{aligned}\phi =& \phi_{000}x_0^3 + \phi_{111}x_1^3 + \phi_{222}x_2^3 + 3\phi_{001}x_0^2x_1 + 3\phi_{011}x_0x_1^2 + 3\phi_{002}x_0^2x_2 \\ &+ 3\phi_{022}x_0x_2^2 + 3\phi_{112}x_1^2x_2 + 3\phi_{122}x_1x_2^2 + 6\phi_{012}x_0x_1x_2,\end{aligned}$$

the corresponding matrix is:

$$\begin{bmatrix} & & & \phi_{002} & \phi_{012} & \phi_{022} & -\phi_{010} & -\phi_{011} & -\phi_{012} \\ & & & \phi_{012} & \phi_{112} & \phi_{122} & -\phi_{011} & -\phi_{111} & -\phi_{112} \\ & & & \phi_{012} & \phi_{112} & \phi_{222} & -\phi_{012} & -\phi_{112} & -\phi_{122} \\ -\phi_{002} & -\phi_{012} & -\phi_{022} & & & & \phi_{000} & \phi_{001} & \phi_{002} \\ -\phi_{012} & -\phi_{112} & -\phi_{122} & & & & \phi_{001} & \phi_{011} & \phi_{012} \\ -\phi_{012} & -\phi_{112} & -\phi_{222} & & & & \phi_{001} & \phi_{011} & \phi_{022} \\ \phi_{010} & \phi_{011} & \phi_{012} & -\phi_{000} & -\phi_{001} & -\phi_{002} & & & \\ \phi_{011} & \phi_{111} & \phi_{112} & -\phi_{001} & -\phi_{011} & -\phi_{012} & & & \\ \phi_{012} & \phi_{112} & \phi_{122} & -\phi_{001} & -\phi_{011} & -\phi_{022} & & & \end{bmatrix}.$$

Al the principal Pfaffians of size 8 of the this matrix coincide, up to scale, with the classical *Aronhold invariant.*

The construction shows how the Aronhold invariant is analogous to the Strassen invariant in $S^9(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ that was discovered by Strassen.

An interpretation is the following. For ϕ such that the Aronhold invariant vanishes, we have $A_\phi \colon S_{21}V \to S_{21}V$, which is skew-symmetric.

The following holds (lemma 2.2 in [Ot])

Lemma 7.1 *Let $\phi = w^3$ with $w \in W$. Then $rkA_\phi = 2$. More precisely*

$$KerA_{w^3} = \{M \in ad\ W | w \textit{ is an eigenvector of } M\}$$

Note the following

Corollary 7.2 *Let $\phi = \sum_{i=1} w_i^3$ with $w_i \in W$ independent vectors. Then the Aronhold invariant vanishes at ϕ and $rkA_\phi = 6$. More precisely*

$$KerA_\phi = \{M \in ad\, W | w_i \text{ are eigenvectors of } M\}$$

Consider now the Euler sequence on $\mathbf{P}(V)$.

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \otimes V^\vee \longrightarrow T\mathbf{P}(V) \longrightarrow 0$$

We get $End(V) \longrightarrow H^0(T\mathbf{P}(V))$ and more precisely ad $(V) \simeq H^0(T\mathbf{P}(V))$, where ad (V) is the space of traceless maps.

The map $A \in$ ad (V) goes to a section s_A, whose zero locus consists of the eigenvectors of A.

Any $\phi \in S^3(\mathbb{C}^3)$ induces a skew-symmetric morphism

$$A_\phi \colon H^0(T\mathbf{P}(V)) \to H^0(T\mathbf{P}(V))$$

which can be seen from the natural map $\wedge^2 H^0(T\mathbf{P}(V)) \to H^0(\wedge^2 T\mathbf{P}(V)) = H^0(\mathcal{O}(3))$

Note that Pf (A_ϕ) is exactly the Aronhold invariant.

It follows the following algorithm to decompose a plane cubic which satisfies the Aronhold invariant.

Theorem 7.3 *Let $\phi \in S^3(\mathbb{C}^3)$ such that Pf $(A_\phi) = 0$ and assume that rk $(A_\phi) = 6$. Then every section in $\ker A_\phi$ has the same zero locus $Z = \{l_1, l_2, l_3\}$. It follows that l_i can be suitably represented in such a way that $\phi = \sum_{i=1}^3 l_i^3$*

The previous theorem is another display of how the equations of the secant varieties allow to decompose the tensors.

7.2 Young flattening and other examples

Example

Let $E = Q(m)$ on $X = \mathbf{P}^2$, it is presented by

$$L_1 = \mathcal{O}(m) \otimes U \xrightarrow{p} \mathcal{O}(m+1) \otimes U^\vee = L_0$$

with p represented by the matrix

$$\begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}$$

Let now $L = \mathcal{O}(2m+1)$, so that (E, L) is a skew-symmetric pair. For any $\phi \in S^{2m+1}U^\vee = V$, A_ϕ is the skew-symmetric morphism from $H^0(Q(m))$ to its dual $H^0(Q^\vee(m+1))^\vee$ and $H^0(p)_\phi$ is represented by the matrix

$$\begin{pmatrix} 0 & C_{m,m}(\phi_2) & -C_{m,m}(\phi_1) \\ -C_{m,m}(\phi_2) & 0 & C_{m,m}(\phi_0) \\ C_{m,m}(\phi_1) & -C_{m,m}(\phi_0) & 0 \end{pmatrix}$$

Note that when $\phi = x_0^{2m+1}$ is the power of a linear form then the rank of the above matrix is 2, which indeed is the rank of $E = Q(m)$.

With the notations above, $H^0(L_1) = S^m U^\vee \otimes U = S_{m+1,m}U \oplus S^{m-1}U^\vee$ and $H^0(E) = S_{m+1,m}U$ corresponds to the first summand.

The elements of $S^m U^\vee \otimes U = Hom(S^m U, U)$ can be represented by matrices

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ f_0 & f_1 & f_2 \end{bmatrix}$$

where x_i is a basis of $U^\vee$ and $f_i \in S^m U^\vee$. According to this identification, the inclusion $i\colon S^{m-1}U^\vee \to S^m U^\vee \otimes U$ is given by $h \mapsto (f_0, f_1, f_2) = (hx_0, hx_1, hx_2)$.

Then $H^0(E) = S_{m+1,m}U$ corresponds to equivalence classes of triples (f_0, f_1, f_2) with $\deg f_i = m$ where $(f_0, f_1, f_2) \sim (g_0, g_1, g_2)$ if there is h such that $f_i - g_i = hx_i$ for $i = 0, \ldots, 2$.

The map $H^0(\mathcal{O}(m)) \otimes U \longrightarrow H^0(Q(m))$ corresponds to the map sending (f_0, f_1, f_2) to its equivalence class. The section vanishes when

$$(f_0, f_1, f_2) \cdot \begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix} = 0$$

The morphism A_ϕ induces a map $B\colon \wedge^2 H^0(Q(m)) = \wedge^2 S_{m+1,m}U \to S^{2m+1}U^\vee = H^0(\mathcal{O}(2m+1))$.

Lemma 7.4 *B can be described in the following way, according to the above description. On the decomposable elements we have*

$$(f_0, f_1, f_2) \wedge (g_0, g_1, g_2) \mapsto \det \begin{bmatrix} x_0 & x_1 & x_2 \\ f_0 & f_1 & f_2 \\ g_0 & g_1 & g_2 \end{bmatrix}$$

Then B is extended by linearity.

Proof:

$$\det \begin{bmatrix} x_0 & x_1 & x_2 \\ f_0 & f_1 & f_2 \\ g_0 & g_1 & g_2 \end{bmatrix} = (f_0, f_1, f_2) \cdot \begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix} \cdot \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$$

□

Note that $(f_0, f_1, f_2) \in \ker H^0(p)_\phi$ if the three minors of

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ f_0 & f_1 & f_2 \end{bmatrix}$$

applied to ϕ (as differential operators) all vanish.

In [Landsberg-Ottaviani] it is proved

Theorem 7.5 *Let $t \le \binom{m+2}{2}$ and let $\phi = \sum_{i=1}^t l_i^{2m+1}$. Then the subpfaffians of size $(2t+2)$ of A_ϕ define local equations for $\sigma_k(v_{2m+1}(\mathbf{P}^2))$ at ϕ.*

The previous result can be generalized to $\mathbf{P}^n$.

In the even case $\mathbf{P}^{2a} = \mathbf{P}(U)$ we can consider $E = \wedge^a Q(m)$.

Theorem 7.6 *Let $n = 2a$ and $t \leq \binom{m+n}{n}$ and let $\phi = \sum_{i=1}^t l_i^{2m+1}$. Then the subpfaffians of size $(\binom{n}{a}t + 2)$ of A_ϕ for odd a (resp. the minors of size $(\binom{n}{a}t + 1)$ of A_ϕ for even a) define local equations for $\sigma_t(v_{2m+1}(\mathbf{P}^n))$ at ϕ, in the sense that the zero locus of such pfaffians contains $\sigma_t(v_{2m+1}(\mathbf{P}^n))$ as irreducible component . Note that for $a = 1$ they give equations of degree $t + 1$.*

Theorem 7.7 *Let $n = 2a+1$, let $t \leq \binom{m+n}{n}$ and let $\phi = \sum_{i=1}^t l_i^{2m+1}$. Then the minors of size $(\binom{n}{a}t+1)$ of A_ϕ define local equations for $\sigma_t(v_{2m+1}(\mathbf{P}^n))$ at ϕ, in the sense that the zero locus of such minors contains $\sigma_t(v_{2m+1}(\mathbf{P}^n))$ as irreducible component .*

7.3 Eigenvectors and sections of the tangent bundle

Consider the Euler sequence on $\mathbf{P}(V)$.

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \otimes V^\vee \longrightarrow T\mathbf{P}(V) \longrightarrow 0$$

We get $End(V) \longrightarrow H^0(T\mathbf{P}(V))$ and more precisely ad $(V) \simeq H^0(T\mathbf{P}(V))$, where ad (V) is the space of traceless maps.

The map $A \in$ ad (V) goes to a section s_A, whose zero locus consists of the eigenvectors of A.

This description can be generalized to generalized eigenvectors. Twisting the Euler sequence we get

$$0 \longrightarrow \mathcal{O}(m) \longrightarrow \mathcal{O}(m+1) \otimes V^\vee \longrightarrow T\mathbf{P}(V)(m) \longrightarrow 0$$

We get $S^{m+1}(V) \otimes V^\vee \longrightarrow H^0(T\mathbf{P}(V))$. Since $S^{m+1}(V) \otimes V^\vee = S^m(V) \oplus S_{m+2,1^{n-1}}$ and more precisely we get $S_{m+2,1^{n-1}} \simeq H^0(T\mathbf{P}(V)(m))$.

$A \in S_{m+2,1^{n-1}}$ corresponds to a map $f_A \colon S^{m+1}(V^\vee) \to V^\vee$. A nonzero vector $v \in V^\vee$ is called a *generalized eigenvector* if there exists $\lambda \in K$ such that $f_a(v^{m+1}) = \lambda v$. λ is a generalized eigenvalue. For $m = 0$ these are exactly the usual eigenvectors and eigenvalues.

It turns out that f_A goes to a section s_A of $T\mathbf{P}(V)(m)$, whose zero locus consists of the generalized eigenvectors of A.

In particular, since in the case $\dim V = 3$ we have $c_2(T\mathbf{P}(V)(1)) = 7$, we get that the general $f_A \colon S^{m+1}(V^\vee) \to V^\vee$ has seven generalized eigenvectors.

Find Sturmfels-Cartwright formula for the number of generalized eigenvectors.

Example Every plane quintic ϕ defines a skew-symmetric morphism $A_\phi \colon H^0(Q(2)) \to H^0(Q(2))^\vee$. For a general ϕ the kernel has dimension one and it is spanned by $s \in H^0(Q(2))$. The seven points where s vanishes correspond to the seven summands of fifth powers that uniquely give ϕ. They are the seven generalized eigenvectors of the kernel.

Cayley-Bacharach property says that giventhere exists a section of $T\mathbf{P}^2(m)$ vanishing exactly on Z if and only if for any $Z' \subset Z$ such that $\text{length}(Z)' = \text{length}(Z) - 1$ we have that every curve of degree $2m$ passing through Z' contains Z.

In particular, given Z consisting of 7 general points, there is always a section of $T\mathbf{P}^2(1)$ vanishing on Z.

Given Z consisting of 13 general points, there is a section of $T\mathbf{P}^2(2)$ vanishing on Z if and only if every quartic containing 12 of the points, contains also the last one. This gives a condition on the 13-ples of points which can be obtained in this way.

Example [Cubic 3folds revisited]

Let $E = \Omega^2(4)$ on $X = \mathbf{P}^4$, the pair $(E, \mathcal{O}(3))$ is a symmetric pair presented by

$$L_1 = \mathcal{O}(1) \otimes \wedge^3 U \xrightarrow{p} \mathcal{O}(2) \otimes \wedge^2 U = L_0$$

with p represented by the 10×10 symmetric matrix

$$\begin{bmatrix}
 & & & & & & & x_4 & -x_3 & x_2 \\
 & & & & & -x_4 & x_3 & & & -x_1 \\
 & & & & x_4 & & -x_2 & & x_1 & \\
 & & & & -x_3 & x_2 & & -x_1 & & \\
 & & x_4 & -x_3 & & & & & & x_0 \\
 & -x_4 & & x_2 & & & & & -x_0 & \\
 & x_3 & -x_2 & & & & & x_0 & & \\
x_4 & & & -x_1 & & & x_0 & & & \\
-x_3 & & x_1 & & & -x_0 & & & & \\
x_2 & -x_1 & & & x_0 & & & & &
\end{bmatrix}$$

Let $L = \mathcal{O}(3)$, for any $\phi \in S^3U^\vee$ the map A_ϕ is the skew-symmetric morphism from $H^0(\Omega^2(4))$ to its dual $H^0(\Omega^3(5))^\vee$ which have both dimension 45 and $H^0(p)_\phi$ is represented by the 50×50 block matrix where $\pm x_i$ is replaced with $\pm C_{1,1}(\phi_i)$, which are 5×5 symmetric catalecticant matrices of the quadric ϕ_i.

In [] it is shown how a matrix representing A_ϕ can be obtained by the matrix representing $H^0(p)_\phi$ by deleting five suitably chosen rows and columns.

Note that when $\phi = x_0^3$ is the power of a linear form then the rank of the above matrix is 6, which is the rank of $\Omega^2(3)$.

Here $\det(A_\phi)$ is the cube of the degree 15 equation of $\sigma_7(v_3(\mathbf{P}^4))$.

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