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On 3-Folds in $\mathbb{P}^5$ which are Scrolls

GIORGIO OTTAVIANI [1]

Introduction

A smooth 3-fold $X \subset \mathbb{P}^5$ is called a *scroll over a surface B* if there exists a morphism $p : X \to B$ such that $p^{-1}(b) \subset \mathbb{P}^5$ is a projective line for every point $b \in B$. In equivalent way $X = \mathbb{P}(E) \xrightarrow{p} S$ where E is a rank 2 vector bundle over B. X is embedded as a $\mathbb{P}^1$-scroll by the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) = H$. We set $d = \deg X$. There are four examples, all classical, of such scrolls in $\mathbb{P}^5$:

(a) $d = 3$, **Segre scroll** $\mathbb{P}^1 \times \mathbb{P}^2$ over $B = \mathbb{P}^2$, with $E = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. A resolution of the ideal sheaf is

$$0 \to \mathcal{O}(-3)^2 \xrightarrow{\phi} \mathcal{O}(-2)^3 \to \mathcal{I}_X \to 0$$

with ϕ generic. *This is the only scroll in* $\mathbb{P}^5$ *over a curve with a linear* $\mathbb{P}^2$ *as fiber* [16].

(b) $d = 6$, **Bordiga scroll** over $B = \mathbb{P}^2$ with E stable bundle on $\mathbb{P}^2$ with $c_1(E) = 4$, $c_2(E) = 10$. A resolution is

$$0 \to \mathcal{O}(-4)^3 \xrightarrow{\phi} \mathcal{O}(-3)^4 \to \mathcal{I}_X \to 0$$

with ϕ generic.

(c) $d = 7$ **Palatini scroll** ([20], pag. 381) (studied also by Okonek [18]) over $B =$ cubic surface in $\mathbb{P}^3$ with E bundle on the cubic surface with $c_1(E) = \mathcal{O}(2)$, $c_2(E) = 5$. A resolution is

$$0 \to \mathcal{O}(-4)^4 \xrightarrow{\phi} \Omega^1(-2) \to \mathcal{I}_X \to 0$$

with ϕ generic.

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(d) $d = 9$, **K3 scroll** over $B = \{\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^5) \cap \mathbb{P}^8\}$, which is K3 by the adjunction formula. X is formed by the lines corresponding to the points of B, then $E = U^*$ where U is the universal bundle on $\mathrm{Gr}(1,5)$, $c_1(E) = \mathcal{O}(1)$, $c_2(E) = 5$. A resolution is ([9]):

$$0 \to \mathcal{O}(-4)^9 \xrightarrow{\phi} \Omega^3 \to \mathcal{I}_X \to 0$$

with ϕ generic.

(a) is linked to (b) in the complete intersection of two cubics, (c) is linked to (d) in the complete intersection of two quartics.

The aim of this paper is to prove the following theorem, which solves problem 8 of Schneider's list [22]:

THEOREM. *Let X be a smooth 3-fold in $\mathbb{P}^5$ which is a scroll over a surface. Then X is one of the examples* (a), (b), (c), (d).

Partial results about scrolls in $\mathbb{P}^5$ were obtained by Beltrametti, Schneider and Sommese [6], [5]. In particular in [5] it is proved that $\deg X \leq 24$ and a list of possible invariants is given. Our approach is independent.

One motivation for such a classification is the main result of [2]: there are only finitely many families of 3-folds in $\mathbb{P}^5$ which are not of general type. So, theoretically, one could ask for the complete list of non-general type 3-folds in $\mathbb{P}^5$.

Moreover, scrolls occurr as special cases in adjunction theory [23]. More precisely, apart from a small list of well known examples, the only 3-folds X such that the adjunction map ϕ_{K_X+2H} (H = hyperplane divisor) drops dimension are scrolls over surfaces and quadric bundles over curves. There is only one quadric bundle in $\mathbb{P}^5$ over a curve: its degree is 5 [3]. In [3] it is considered also the case when ϕ_{K_X+H} drops dimension.

Surfaces embedded in $\mathbb{P}^4$ which are scrolls are classified by Lanteri and Aure ([15], [1]) (this classification implies in particular that the only $\mathbb{P}^2$-scroll in $\mathbb{P}^5$ is $\mathbb{P}^1 \times \mathbb{P}^2$). Codimension two submanifolds in $\mathbb{P}^n$ for $n \geq 6$ have $H^2(X, \mathbb{C}) = \mathbb{C}$, hence they cannot be scrolls. So we obtain

COROLLARY. *The following is the complete list of all codimension two projective submanifolds which are $\mathbb{P}^k$-scrolls:*

(i) *the rational cubic ruled surface $S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{P}^2(x_0) \xrightarrow{|2H - x_0|} \mathbb{P}^4$ with resolution*

$$0 \to \mathcal{O}(-3)^2 \to \mathcal{O}(-2)^3 \to \mathcal{I}_S \to 0;$$

(ii) *the elliptic quintic scroll in $\mathbb{P}^4$ with resolution*

$$0 \to \mathcal{O}(-3)^5 \to \Omega^2_{\mathbb{P}^4} \to \mathcal{I}_S \to 0$$

or

$$0 \to \mathcal{O}(-3)^2 \to F(-2) \to \mathcal{I}_S \to 0$$

where F is the (normalized) Tango bundle on $\mathbb{P}^4$ [19].

(iii) *one of the examples* (a), (b), (c), (d).

REMARK. Beltrametti and Sommese [4] have shown that, when $X \subset \mathbb{P}^5$ and $\deg X \geq 4$, the definition of scroll over a surface B is equivalent to the following one (more natural in *adjunction theory*): there exists a morphism $p : X \to B$ over a normal surface and an ample line bundle L on B such that $K_X + 2H = p^*L$. Hence our theorem applies as well in the adjunction theoretic setting.

The proof of the theorem is given in Section 2. More informations about the four examples are contained in Section 3. In Section 4 we give some results concerning the number of the equations defining these examples. In particular we exhibit some examples of surfaces in $\mathbb{P}^4$ and 3-folds in $\mathbb{P}^5$ that are defined by the maximum number of equations and not less. In the last section we study the case of the general embedding of a $\mathbb{P}^1$-bundle in $\mathbb{P}^5$.

I wish to thank C. Ciliberto for stimulating discussions and for pointing to me the classical papers [7] and [20].

1. - Preliminaries

We work over the field of complex numbers.

1.1 Riemann-Roch. *Let X be a 3-fold in $\mathbb{P}^5$, L be a line bundle on X*

$$\chi(\mathcal{O}_X) = \frac{c_1c_2}{24}$$

$$\chi(\mathcal{O}_X(L)) = \chi(\mathcal{O}_X) + \frac{1}{12}(15-d)H^2L + \frac{1}{6}K^2L + \frac{1}{2}HKL - \frac{1}{4}KL^2 + \frac{L^3}{6}.$$

The following smoothing criterion is well known and relies on generic smoothness. It was essentially noticed in [14]. A proof of a more general statement can be found in [8].

1.2 (Kleiman). *Let E, F be vector bundles on $\mathbb{P}^n$, $n \leq 5$. If* $\operatorname{rank} F = \operatorname{rank} E + 1$ *and $E^* \otimes F$ is globally generated, then the generic morphism $\phi : E \to F$ degenerates on a smooth codimension two subvariety X, and we have the exact sequence*

$$0 \to E \to F \to \mathcal{I}_X(c_1(F) - c_1(E)) \to 0.$$

1.2 is a basic tool that allows to construct many projective submanifolds. In particular it applies to the examples (a), ...,(d).

1.3 Resolutions of Linked Subvarieries [21]. *Let V, W be two smooth subvarieties of codimension two in $\mathbb{P}^n$, $n \leq 5$, which are linked in the complete* *intersection of two hypersurfaces of degree* a, b. *If* $\mathcal{I}_V$ *has the locally free resolution*

$$0 \to E \to F \to \mathcal{I}_V \to 0,$$

then we obtain

$$0 \to F^*(-a-b) \to E^*(-a-b) \oplus \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \mathcal{I}_W \to 0.$$

1.4 (Severi). *Let* S *be a smooth surface in* $\mathbb{P}^4$. *We have* $H^1(\mathcal{I}_S(1)) = 0$ *unless* S *is the Veronese surface of degree 4, image of* $\phi_{\mathcal{O}_{\mathbb{P}^2}(2)}$.

The following lemma is a special case of a theorem of Roth (see also 5.4).

LEMMA 1.5. *Let* S *be a surface in* $\mathbb{P}^4$ *of degree* ≥ 5. *If the generic hyperplane section* C *is contained in a quadric of* $\mathbb{P}^3$, *then* S *is contained in a quadric of* $\mathbb{P}^4$.

PROOF. Consider the cohomology sequence associated to the sequence

$$0 \to \mathcal{I}_S(1) \to \mathcal{I}_S(2) \to \mathcal{I}_{C,\mathbb{P}^3}(2) \to 0.$$

By 1.4, $H^1(\mathcal{I}_S(1)) = 0$. Hence $H^0(\mathcal{I}_{C,\mathbb{P}^3}(2)) \neq 0$ implies $H^0(\mathcal{I}_S(2)) \neq 0$.

1.6 (Halphen) (see e.g. [10]). *If a curve in* $\mathbb{P}^3$ *of degree* d *and genus* g *is not contained in a quadric, then we have*

$$g \leq \frac{d^2}{6} - \frac{d}{2} + 1.$$

LEMMA 1.7 (Ellingsrud-Peskine) ([10], Lemma 1). *If a surface in* $\mathbb{P}^4$ *of degree* d *is contained in a quadric, and* g *is the genus of the hyperplane section, we have*

$$\frac{d(d-5)}{2} \leq 2g - 2.$$

2. - Proof of the theorem

PROPOSITION 2.1. *Let* $X = \mathbb{P}(E) \xrightarrow{p} B$. *The Chern classes of* X *are the following*

$$c_1 = 2H - p^*c_1(E) + p^*c_1(B)$$

$$c_2 = 2H \cdot p^*c_1(B) - p^*c_1(E) \cdot p^*c_1(B) + p^*c_2(B)$$

$$c_3 = 2H \cdot p^*c_2(B).$$

PROOF. We have

$$c_1((p^*E^*)\otimes \mathcal{O}_{\mathbb{P}(E)}(1)) = 2H - p^*c_1(E)$$
$$c_2((p^*E^*)\otimes \mathcal{O}_{\mathbb{P}(E)}(1)) = 0$$

(this is sometimes called the Leray-Hirsch equation).

Now we compute the Chern classes from the sequence

$$0 \to \mathcal{O} \to (p^*E^*)\otimes \mathcal{O}_{\mathbb{P}(E)}(1) \to T\mathbb{P}(E) \to p^*TB \to 0.$$

PROPOSITION 2.2. *Let $X \subset \mathbb{P}^5$ be a $\mathbb{P}^1$-scroll, and C be the generic curve section of genus g. We have*

$$g = \frac{d(2d-11)(d-5)}{11d-63} + 1.$$

PROOF. Consider the exact sequence

$$0 \to TX \to T\mathbb{P}^5 \to N_{X,\mathbb{P}^5} \to 0.$$

The Proposition 2.1 and the self-intersection formula $c_2(N_{X,\mathbb{P}^5}) = dH^2$ give

$$c_1(N_{X,\mathbb{P}^5}) = 4H + p^*c_1(E) - p^*c_1(B)$$

$$15H^2 = c_2 + c_1\cdot(4H + p^*c_1(E) - p^*c_1(B)) + dH^2 \tag{2.1}$$

$$20H^3 = c_3 + c_2\cdot(4H + p^*c_1(E) - p^*c_1(B)) + c_1\cdot dH^2. \tag{2.2}$$

After substituting the formulas of Proposition 2.1, the equation (2.1) becomes

$$\begin{aligned}(7-d)H^2 = {}& p^*c_2(B) - 2H\cdot p^*c_1(E) - (p^*c_1(E))^2 + 4H\cdot p^*c_1(B)\\ &+ p^*c_1(E)\cdot p^*c_1(B) - (p^*c_1(B))^2.\end{aligned}$$

Intersecting respectively with the cycles H, $p^*c_1(E)$, $p^*c_1(B)$, we get the three equations

$$\begin{aligned}d(7-d) = {}& H\cdot p^*c_2(B) - 2H^2\cdot p^*c_1(E) - H\cdot(p^*c_1(E))^2\\ &+ 4H^2\cdot p^*c_1(B) + H\cdot p^*c_1(E)\cdot p^*c_1(B) - H\cdot(p^*c_1(B))^2\end{aligned} \tag{L1}$$

$$(7-d)H^2\cdot p^*c_1(E) = -2H\cdot(p^*c_1(E))^2 + 4H\cdot p^*c_1(E)\cdot p^*c_1(B) \tag{L2}$$

$$(7-d)H^2\cdot p^*c_1(B) = -2H\cdot p^*c_1(E)\cdot p^*c_1(B) + 4H\cdot(p^*c_1(B))^2. \tag{L3}$$

In the same way, (2.2) gives

$$\begin{aligned}2d^2 + 20d = {}& 6H\cdot p^*c_2(B) + (8+d)H^2\cdot p^*c_1(B) - 2H\cdot c_1(B)\cdot c_1(E)\\ &- 2H\cdot(p^*c_1(B))^2 - dH^2\cdot p^*c_1(E).\end{aligned} \tag{L4}$$

Moreover, the Leray-Hirsch equation is

$$H^2 - H \cdot p^*c_1(E) + p^*c_2(E) = 0 \tag{2.3}$$

$$\Big(\text{that is } c_2((p^*E^*) \otimes \mathcal{O}_{\mathbb{P}(E)}(1)) = 0\Big).$$

Intersecting respectively with the cycles $p^*c_1(E)$, $p^*c_1(B)$, we get the two equations

$$\text{(L5)} \qquad H \cdot (p^*c_1(E))^2 = H^2 \cdot p^*c_1(E)$$

$$\text{(L6)} \qquad H \cdot p^*c_1(E) \cdot p^*c_1(B) = H^2 \cdot p^*c_1(B).$$

Now we solve the system (L1), ...,(L6). Substituting (L5) and (L6) in (L1), (L2), (L3) and (L4), we get

$$\begin{aligned} d(7-d) = H \cdot p^*c_2(B) - 3H^2 \cdot p^*c_1(E) + 5H^2 \cdot p^*c_1(B) \\ - H \cdot (p^*c_1(B))^2 \end{aligned} \tag{2.4}$$

$$(9-d)H^2 \cdot p^*c_1(E) = 4H^2 \cdot p^*c_1(B) \tag{2.5}$$

$$(9-d)H^2 \cdot p^*c_1(B) = 4H \cdot (p^*c_1(B))^2 \tag{2.6}$$

$$\begin{aligned} 2d^2 + 20d = 6H \cdot p^*c_2(B) + (6+d)H^2 \cdot p^*c_1(B)2H \cdot (p^*c_1(B))^2 \\ - dH^2 \cdot p^*c_1(E). \end{aligned} \tag{2.7}$$

Hence from (2.5) and (2.6):

$$H^2 \cdot p^*c_1(B) = \frac{(9-d)}{4} H^2 \cdot p^*c_1(E) \tag{2.8}$$

$$H \cdot (p^*c_1(B))^2 = \frac{(9-d)}{4} H^2 \cdot p^*c_1(B) = \frac{(9-d)^2}{16} H^2 \cdot p^*c_1(E) \tag{2.9}$$

and substituting these values into (2.4) and (2.7), we get the two equations

$$\begin{aligned} d(7-d) = H \cdot p^*c_2(B) - 3H^2 \cdot p^*c_1(E) + \frac{5(9-d)}{4} H^2 \cdot p^*c_1(E) \\ - \frac{(9-d)^2}{16} H^2 \cdot p^*c_1(E) \end{aligned} \tag{2.10}$$

$$\begin{aligned} 2d^2 + 20d = 6H \cdot p^*c_2(B) + (6+d)\frac{(9-d)}{4} H^2 \cdot p^*c_1(E) \\ - \frac{(9-d)^2}{8} H^2 \cdot p^*c_1(E) - dH^2 \cdot p^*c_1(E). \end{aligned}$$

We eliminate $H \cdot p^* c_2(B)$ from these two last equations and we get

$$6d(7-d) + 2d^2 - 20d = \left[-18 + \frac{15(9-d)}{2} - \frac{3(9-d)^2}{8} - \frac{(6+d)(9-d)}{4} + \frac{(9-d)^2}{8} + d \right] H^2 \cdot p^* c_1(E)$$

that is, after simplifications,

$$H^2 \cdot p^* c_1(E) = \frac{8d(2d-11)}{11d-63}. \tag{2.11}$$

Now from the adjunction formula we have

$$g - 1 = \frac{H^2 K}{2} + d \qquad \text{(by Proposition 2.1)}$$

$$= \frac{1}{2} H^2(-2H + p^* c_1(E) - p^* c_1(B)) + d$$

$$= \frac{1}{2}(H^2 \cdot p^* c_1(E) - H^2 \cdot p^* c_1(B)) \qquad \text{(by (2.8))}$$

$$= \frac{1}{2} H^2 \cdot p^* c_1(E) \left(1 + \frac{d-9}{4}\right) = H^2 \cdot p^* c_1(E) \frac{(d-5)}{8} \qquad \text{(by (2.11))}$$

$$= \frac{d(d-5)(2d-11)}{11d-63} \qquad \text{q.e.d..}$$

PROOF OF THE THEOREM. We may suppose $d \geq 6$. Suppose first that C is not contained in a quadric. Then from 1.6 and Proposition 2.2 we have

$$\frac{d(2d-11)(d-5)}{11d-63} \leq \frac{d^2}{6} - \frac{d}{2}$$

that is

$$d^2 - 30d + 141 \leq 0$$

which gives $d \leq 24$. But it is easy to check that $g - 1 = \dfrac{d(d-5)(2d-11)}{11d-63}$ is an integer in this bound only for $d = 6, 7, 9, 21$. But for, $d = 21$, $H^2 \cdot p^* c_1(E) = 31$ from (2.11) and in this case the solution of $H \cdot p^* c_2(B)$ that we obtain from (2.10) is not an integer.

If otherwise C is contained in a quadric, from 1.5 we get that also the surface section of X is contained in a quadric. Hence the Lemma 1.7 gives

$$\frac{d(d-5)}{2} \leq \frac{2d(2d-11)(d-5)}{11d-63}$$

that is $3d - 19 \leq 0$. Now the classification in [6] concludes the proof.

REMARK 2.3. The complete solution of the system (L1), ...,(L6) and Riemann-Roch formula 1.1 give, after some computations, the equality $\chi(\mathcal{O}_X(1))= 6+\dfrac{(d-3)(d-6)(d-7)(d-9)}{3(11d-63)}$. This shows immediately the role of the four values $d=3,\ 6,\ 7,\ 9$.

3. - More on the four examples

The geometrical description of the scroll structure of the example (a) is clear, and is given by Chang [9] for the example (d). We give here an explicit description of the scroll structure of the examples (b) and (c), which is different from the adjunction method employed by Okonek [18]. We follow strictly the ideas of G. Castelnuovo [7].

3.1. In example (b) (Bordiga scroll) the morphism $F:\mathcal{O}(-4)^3\to\mathcal{O}(-3)^4$ is given by a generic 4×3 matrix $F=[f_{ij}]$, where $f_{ij}\in H^0(\mathbb{P}^5,\mathcal{O}(1))$. The four 3×3 minors of F are the equations of X. If $x'\in X$ then $\operatorname{rk} F\le 2$ and there exists $(\lambda_1,\lambda_2,\lambda_3)$ such that

$$\sum_{j=1}^{3}\lambda_j f_{ij}(x')=0 \qquad \text{for } i=1,\ 2,\ 3,\ 4. \tag{3.1}$$

The morphism $p:X\to\mathbb{P}^2$ defined by $p(x')=(\lambda_1,\lambda_2,\lambda_3)$ gives the structure of scroll and (3.1) are the equations of the fiber of p over $(\lambda_1,\lambda_2,\lambda_3)$.

3.2. In example (c) (Palatini scroll), let $\mathbb{P}^5=\mathbb{P}(V)$. With the help of the twisted dual Euler sequence on $\mathbb{P}^5$

$$0\to\Omega^1(2)\to V^*\otimes\mathcal{O}(1)\to\mathcal{O}(2)\to 0,$$

we have the natural identification $\mathrm{Hom}(\mathcal{O},\Omega^1(2))\simeq\overset{2}{\wedge}V^*$, where a morphism $\mathcal{O}(-4)\to\Omega^1(-2)$ is given (after choosing a basis in V and its dual basis in V^*) by a skew-symmetric 6×6 matrix $A=[a_{ij}]$ $(a_{ij}\in\mathbb{C})$ and corresponds to the morphism $V\to V^*$ given by

$$(x_1,\dots,x_6)\to\left(\sum a_{1i}x_1,\dots,\sum a_{6i}\right).$$

This morphism defines a linear line complex in $\mathbb{P}^5$.

The map $\phi:\mathcal{O}(-4)^4\to\Omega^1(-2)$ of the example (c) is given by four generic skew-symmetric matrixes $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}]$, $D=[d_{ij}]$. The equations of the degeneracy locus X of ϕ are the fifteen 4×4 minors of the

4×6 matrix

$$F = \begin{bmatrix} \sum a_{1i}x_i \cdots \sum a_{6i}x_i \\ \sum b_{1i}x_i \cdots \sum b_{6i}x_i \\ \sum c_{1i}x_i \cdots \sum c_{6i}x_i \\ \sum d_{1i}x_i \cdots \sum d_{6i}x_i \end{bmatrix}.$$

If $x' \in X$ then there exist $(\lambda, \mu, \nu, \psi)$ such that

$$\sum_{i=1}^{6} (\lambda a_{ji} + \mu b_{ji} + \nu c_{ji} + \psi d_{ji})x'_i = 0 \qquad \text{for } j = 1, \dots, 6. \tag{3.2}$$

Hence $\lambda A + \mu B + \nu C + \psi D$ has to be a degenerate skew-symmetric matrix, and its pfaffian has to vanish. Let $(\lambda, \mu, \nu, \psi)$ be homogeneous coordinates in $\mathbb{P}^5$ and set

$$S = \{(\lambda, \mu, \nu, \psi) |\ \mathrm{Pfaff}(\lambda A + \mu B + \nu C + \psi D) = 0\} \subset \mathbb{P}^3.$$

S is a cubic surface. The morphism $p : X \to S$ defined by $p(x') = (\lambda, \mu, \nu, \psi)$ gives the structure of scroll because (3.2) are the equations of the fiber of p over $(\lambda, \mu, \nu, \psi)$ and a skew-symmetric matrix has always even rank, hence every fiber of p is a line in $\mathbb{P}^5$. X is in fact (as in [20]) the locus of the lines which are center of the degenerate linear complexes in the system spanned by the four complexes corresponding to A, B, C, D.

It is interesting to remark that the Palatini scroll is the only known smooth 3-fold in $\mathbb{P}^5$ such that the hyperquadrics cut on it a non-complete linear system (see [22], Problem 5).

3.3. In the examples (b), (c) and (d) the morphism $\phi : X \to B$ is associated to the line bundle $K_X + 2H$. In the example (a) is associated to $-K_X - 2H$, while $\phi_{K_X+3H} : X \to \mathbb{P}^1$.

In the natural embedding $f : B \to \mathrm{Gr}(1,5)$, $f(B)$ has bidegree (α, β) where $\alpha = \#\{\mathbb{P}^1 | \mathbb{P}^1 \cap \overline{\mathbb{P}^2} \neq \emptyset,\ \overline{\mathbb{P}^2} \text{ fixed}\} = d$, $\beta = \#\{\mathbb{P}^1 | \mathbb{P}^1 \subset \overline{\mathbb{P}^4},\ \overline{\mathbb{P}^4} \text{ fixed hyperplane}\} =$ number of points blown-up in the morphism $p|_{X \cap H} : X \cap H \to B$. For example the hyperplane sections of the Bordiga scroll are Bordiga surfaces which are isomorphic to the plane blown-up in ten points. From the Leray-Hirsch equation (2.3) we have $\beta = c_2(E)$.

We summarize the numerical invariants of the four examples in the following table (see also [18], [9]). The Hilbert polynomials can be computed from the resolutions of the ideal sheaves.

Table 1

	$d=3$ Segre	$d=6$ Bordiga	$d=7$ Palatini	$d=9$ K3
H^2K	-8	-8	-8	-4
HK^2	21	9	7	-6
K^3	-54	-6	-2	12
B	$\mathbb{P}^2$	$\mathbb{P}^2$	cubic in $\mathbb{P}^3$	surf. section of $\mathrm{Gr}(1,5)$
E	$\mathcal{O}(1)^2$	stable $c_1=4$, $c_2=10$	stable $c_1=\mathcal{O}(2)$, $c_2=5$	f^*U^*, U = univ. bundle
bidegree in $\mathrm{Gr}(1,5)$	$(3,1)$	$(6,10)$	$(7,5)$	$(9,5)$
$\chi(\mathcal{O}_{X\cap H})$	1	1	1	2
g	0	3	4	8
$\chi(\mathcal{O}_X(t))$	$\frac{(t+1)^2(t+2)}{2}$	$(t+1)(t^2+t+1)$	$\frac{(t+1)(7t^2+5t+6)}{6}$	$\frac{(t+1)(3t^2-t+4)}{2}$

4. - On the number of equations defining codimension two submanifolds

DEFINITION 4.1. A subvariety $X \subset \mathbb{P}^n$ is said to be the scheme-theoretic intersection of the p hypersurfaces with equations $f_1, \dots, f_p$ if

$$X = \mathrm{Proj}\, \frac{\mathbb{C}[x_0, \dots, x_n]}{(f_1, \dots, f_p)}.$$

In equivalent way, we have that $f_1, \dots, f_p$ generate the ideal sheaf $\mathcal{I}_X$, that is

$$\bigoplus_{i=1}^{p} \mathcal{O}(-d_i) \xrightarrow{(f_1,\dots,f_p)} \mathcal{I}_X \to 0 \quad \text{with } d_i = \deg f_i. \tag{4.1}$$

4.2. *Any subvariety $X \subset \mathbb{P}^n$ is the scheme-theoretic intersection of at most $n+1$ hypersurfaces* ([11], Example 9.1.3). The number of generators of $\mathcal{I}_X$ should not be confused with the number of generators of the homogeneous ideal $I_X \subset \mathbb{C}[x_0, \dots, x_n]$, which is unbounded.

4.3. In the case X is locally Cohen-Macaulay of codimension two, the kernel of the morphism in (4.1) is locally free. Hence varieties defined by a small number of equations give vector bundles of small rank. We will see in the following that all the bundles obtained by the examples considered in this paper are well known.

EXAMPLE 4.4. The Segre scroll $\mathbb{P}^1 \times \mathbb{P}^2$ is the scheme-theoretic intersection of three quadrics, which are the three minors of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}.$$

This matrix defines a morphism ϕ like in the introduction (example (a)).

In the same way the Bordiga scroll (example (c)), is the scheme-theoretic intersection of 4 cubics.

There are many examples of curves in $\mathbb{P}^3$ which cannot be the scheme-theoretic intersection of only 3 surfaces (e.g. [11], Example 9.1.2). These curves are the "most general" considering 4.2.

In this section we present a surface in $\mathbb{P}^4$, Example 4.10, (respectively a threefold in $\mathbb{P}^5$, Example 4.12) which cannot be the scheme-theoretic intersection of 4 (respectively 5) hypersurfaces. We prove these facts as a straightforward application of the Segre-Fulton formula for the equivalence of a component in an intersection product ([11] Proposition 9.1.1, see also [12]). The following theorems 4.5, 4.6, 4.7 are special cases of the Segre-Fulton formula.

THEOREM 4.5 ([11], Example 9.1.1; [12] 1.10). *Let $C \subset \mathbb{P}^3$ a smooth* curve *of degree d and genus g which is the scheme-theoretic intersection of* three *surfaces of degree n_1, n_2, n_3. We have*

$$d(n_1 + n_2 + n_3) + 2 - 2g - 4d = n_1 n_2 n_3.$$

THEOREM 4.6 (see [11], Example 9.1.5). *Let $S \subset \mathbb{P}^4$ be a smooth* surface *of degree d and let g be the genus of $S \cap H$.*

(i) *If S is the scheme-theoretic intersection of* four *hypersurfaces of degree n_1, n_2, n_3, n_4, we have*

$$\left(\sum_{i<j} n_i n_j\right) d + \left(\sum n_i\right)(2 - 2g - 4d) + (10d + 10g + c_2(S) - 10) = n_1 n_2 n_3 n_4. \tag{4.2}$$

(ii) *If S is the scheme-theoretic intersection of* three *hypersurfaces of degree n_1, n_2, n_3, we have*

$$\begin{cases} \left(\sum_{i<j} n_i n_j\right) d + \left(\sum n_i\right)(2-2g-4d) + (10d+10g+c_2(S)-10) = 0 \\ \left(\sum n_i\right) d + 2 - 2g - 4d = n_1 n_2 n_3. \end{cases}$$

THEOREM 4.7. *Let $X \subset \mathbb{P}^5$ be a smooth* 3-fold *of degree d. Let K be the canonical bundle and g the genus of $X \cap H^2$.*

(i) *If X is the scheme-theoretic intersection of* five *hypersurfaces of degree n_1, n_2, n_3, n_4, n_5, we have*

$$\begin{aligned} &\left(\sum_{i<j<k} n_i n_j n_k\right) d + \left(\sum_{i<j} n_i n_j\right)(2-2g-4d) \\ &\quad + \left(\sum n_i\right)(12d - d^2 + 12(2g-2) + K^2 H) \\ &\quad + c_3(X) + 6d^2 - 32d - 57(2g-2) - 6K^2 H = n_1 n_2 n_3 n_4 n_5. \end{aligned} \tag{4.3}$$

(ii) *If X is the scheme-theoretic intersection of* four *hypersurfaces of degree n_1, n_2, n_3, n_4, we have*

$$\begin{aligned} &\left(\sum_{i<j<k} n_i n_j n_k\right) d + \left(\sum_{i<j} n_i n_j\right)(2-2g-4d) \\ &\quad + \left(\sum n_i\right)(12d - d^2 + 12(2g-2) + K^2 H) \\ &\quad + c_3(X) + 6d^2 - 32d - 57(2g-2) - 6K^2 H = 0 \end{aligned}$$

$$\begin{aligned} &\left(\sum_{i<j} n_i n_j\right) d + \left(\sum n_i\right)(2-2g-4d) \\ &\quad + (12d - d^2 + 12(2g-2) + K^2 H) = n_1 n_2 n_3 n_4. \end{aligned}$$

(iii) *If X is the scheme-theoretic intersection of* three *hypersurfaces of degree n_1, n_2, n_3, we have*

$$(n_1n_2n_3)d + \left(\sum_{i<j} n_in_j\right)(2-2g-4d)$$
$$+\left(\sum n_i\right)(12d - d^2 + 12(2g-2) + K^2H)$$
$$+c_3(X) + 6d^2 - 32d - 57(2g-2) - 6K^2H = 0$$

$$\left(\sum_{i<j} n_in_j\right)d + \left(\sum n_i\right)(2-2g-4d) + (12d - d^2 + 12(2g-2) + K^2H) = 0$$

$$\left(\sum n_i\right)d + 2 - 2g - 4d = n_1n_2n_3.$$

REMARK 4.8. In order to apply the previous theorem in concrete cases, it is useful the formula (set $S = X \cap H$)

$$K^2H = \frac{1}{2}d(d+1) - 9(g-1) + 6\chi(\mathcal{O}_S).$$

EXAMPLE 4.9. *The Veronese surface S in $\mathbb{P}^4$ is the scheme-theoretic intersection of four cubics.* We have

$$0 \to F^*(-4) \to \mathcal{O}(-3)^4 \to \mathcal{I}_S \to 0 \tag{4.4}$$

where F is the Tango bundle on $\mathbb{P}^4$, in fact S is linked to the elliptic quintic scroll in the complete intersection of two cubics (apply 1.3).

From Theorem 4.3 it follows that *S is not the scheme-theoretic intersection of three hypersurfaces, and if four hypersurfaces define S, then they must be all cubics.* In fact, let us suppose that there exist four hypersurfaces defining S of degrees n_1, n_2, n_3, n_4. By the resolution (4.4) $H^0(\mathcal{I}_S(2)) = 0$, hence we have $n_i \geq 3$. The equation obtained by (4.2) is

$$4\left(\sum n_in_j\right) - 14\sum n_i + 33 = n_1n_2n_3n_4$$

that, with the substitution $n_i = \lambda_i + 3$, becomes

$$\lambda_1\lambda_2\lambda_3\lambda_4 + \left(\sum \lambda_i\lambda_j\lambda_k\right) + 5\left(\sum \lambda_i\lambda_j\right) + 5\left(\sum \lambda_i\right) = 0$$

that has the only nonnegative solution $\lambda_i = 0$.

EXAMPLE 4.10. *The elliptic quintic scroll is not the scheme-theoretic intersection of four hypersurfaces.* In fact the equation obtained by (4.2) is

$$5\left(\sum n_in_j\right) - 20\sum n_i + 50 = n_1n_2n_3n_4$$

that, with the substitution $n_i = \lambda_i + 3$, becomes

$$\lambda_1\lambda_2\lambda_3\lambda_4 + 3\left(\sum \lambda_i\lambda_j\lambda_k\right) + 4\left(\sum \lambda_i\lambda_j\right) + 2\left(\sum \lambda_i\right) + 1 = 0$$

that has no nonnegative solutions.

EXAMPLE 4.11. *The Palatini scroll is not the scheme-theoretic intersection of four hypersurfaces, and if five hypersurfaces define it, they must be all quartics.*

In fact the equation obtained by (4.3) is

$$7\left(\sum n_in_jn_k\right) - 34\left(\sum n_in_j\right) + 114\sum n_i - 296 = n_1n_2n_3n_4n_5$$

that, with the substitution $n_i = \lambda_i + 4$, becomes

$$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 + 4\left(\sum \lambda_i\lambda_j\lambda_k\lambda_h\right) + 9\left(\sum \lambda_i\lambda_j\lambda_k\right) + 14\left(\sum \lambda_i\lambda_j\right) + 128\left(\sum \lambda_i\right) = 0$$

which has the only nonnegative solution $\lambda_i = 0$.

The Palatini scroll X is really the scheme-theoretic intersection of five quartics: we have the sequence

$$0 \to F^*(-5) \to \mathcal{O}(-4)^5 \to \mathcal{I}_X \to 0$$

where F is the Tango bundle on $\mathbb{P}^5$ (the argument is analog to that of Example 4.9).

The same statement as above applies as well at the K3 scrool (Example (d)) which is defined by five quartics by the sequence (N = nullcorrelation bundle):

$$0 \to N(-4) \to \mathcal{O}(-4)^5 \to \mathcal{I}_X \to 0.$$

EXAMPLE 4.12. *The 3-fold in $\mathbb{P}^5$ defined by the resolution*

$$0 \to \mathcal{O}^5 \to \mathcal{O}(1)^6 \to \mathcal{I}_X(6) \to 0$$

is not the scheme-theoretic intersection of five hypersurfaces. In fact from the resolution one computes the Hilbert polynomial which is $\frac{5}{2}t^3 - 5t + \frac{25}{2}t - 4$. The numerical invariants are: $d = 15$, $H^2K = 20$, $HK^2 = 15$, $K^3 = 6$, $g = 26$, $c_3 = -306$. The equation obtained by (4.3) is

$$15\left(\sum n_in_jn_k\right) - 110\left(\sum n_in_j\right) + 570\sum n_i - 2376 = n_1n_2n_3n_4n_5$$

that, with the substitution $n_i = \lambda_i + 5$, becomes

$$(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1)(\lambda_4 + 1)(\lambda_5 + 1) = 0$$

that has no nonnegative solutions.

The examples of this section should be compared with the following higher-dimensional result of Netsvetaev [17], which in turn is related to the well known Hartshorne conjecture on complete intersections.

THEOREM 4.13 (Netsvetaev). *Let $X \subset \mathbb{P}^n$ be a codimension two submanifold, $n \geq 6$. If X can be defined by p equations with $p \leq n - 1$, then X is a complete intersection.*

Hence Netsvetaev theorem cannot hold in $\mathbb{P}^4$ or $\mathbb{P}^5$ even with the weaker assumption $p \leq n$, while Hartshorne conjecture would be equivalent to prove Netsvetaev theorem with the assumption $p \leq n + 1$.

5. - $\mathbb{P}^1$-bundles in $\mathbb{P}^5$

We consider now any possible embedding of a $\mathbb{P}^1$-bundle $\mathbb{P}(E) \xrightarrow{p} B$ over a surface B in $\mathbb{P}^5$. We set $J = \mathcal{O}_{\mathbb{P}(E)}(1)$. Any line bundle on $\mathbb{P}(E)$ is of the form $nJ + p^*L$, with $n \geq 1$ and L some line bundle on B. In this case the fibers of $\mathbb{P}(E)$ are embedded as rational curves of degree n. Our result is the following

THEOREM 5.1. *Let $X = \mathbb{P}(E)$ be embedded in $\mathbb{P}^5$ by some line bundle $nJ + p^*L$. Then we have one of the following cases:*

(i) *$n = 1$; X is a scroll, that is one of the four examples* (a), (b), (c), (d).

(ii) *$n = 2$, $d = 12$; X is a conic bundle over a quartic surface of $\mathbb{P}^3$.*

REMARK 5.2. In [3] it is constructed a 3-fold in $\mathbb{P}^5$ with $d = 12$ which is a conic bundle over a quartic surface of $\mathbb{P}^3$ with all the fibers smooth. We do not know if such an example is a $\mathbb{P}^1$-bundle.

REMARK 5.3. It seems that the analog problem to find the possible embeddings of surfaces in $\mathbb{P}^4$ which are $\mathbb{P}^1$-bundle over a curve is open, see [13].

The proof of Theorem 5.1 is analog to the proof of the classification of scrolls given in Section 2. The main difference is that, in order to exclude some numerical possibilities, we need a computer. Moreover the computations are much heavier and we will only sketch the main steps.

We have first to collect the following facts, which are analogous to 1.5, 1.6, 1.7, that were stated separately only to ease the proof of the theorem in the introduction.

5.4 (Roth) (see e.g. [10]). *Let S be a surface in $\mathbb{P}^4$ of degree ≥ 10. If the generic hyperplane section C is contained in a cubic of $\mathbb{P}^3$, then S is contained in a cubic of $\mathbb{P}^4$.*

5.5 (Halphen) (see e.g. [10]). *If a curve in $\mathbb{P}^3$ of degree d and genus g is not contained in a cubic, then we have*

$$g - 1 \leq \frac{d^2}{8}.$$

LEMMA 5.6 (Ellingsrud-Peskine) ([10], Lemma 1). *If a surface in $\mathbb{P}^4$ of degree d is contained in a cubic, and g is the genus of the hyperplane section, we have*

$$\frac{d^2 - 7d}{6} \leq g - 1.$$

PROOF OF THEOREM. 5.1. We have as in Proposition 2.1:

$$c_1(X) = 2J - p^*c_1(E) + p^*c_1(B)$$

$$c_2(X) = 2J \cdot p^*c_1(B) - p^*c_1(B) \cdot p^*c_1(E)$$

$$c_3(X) = 2J \cdot p^*c_2(B)$$

$$d = (nJ + p^*L)^3 = n^3J^3 + 3n^2J^2 \cdot p^*L + 3nJ \cdot (p^*L)^2. \tag{5.1}$$

The Leray-Hirsch equation is

$$J^2 - J \cdot p^*c_1(E) + p^*c_2(E) = 0$$

and intersecting respectively with the cycles $p^*c_1(B)$, $p^*c_1(E)$, p^*L, it gives the three equations

$$\begin{cases} J^2 \cdot p^*c_1(B) - J \cdot p^*c_1(B) \cdot p^*c_1(E) = 0 \\ J^2 \cdot p^*c_1(E) - J \cdot (p^*c_1(E))^2 = 0 \\ J^2 \cdot p^*L - J \cdot p^*L \cdot p^*c_1(E) = 0. \end{cases} \tag{5.2}$$

Exactly as in Section 2 we get, from the sequence

$$0 \to TX \to T\mathbb{P}^5 \to N_{X,\mathbb{P}^5} \to 0,$$

the two equations:

$$\begin{aligned} &(-15n^2 + n^2d + 12n - 4)J^2 + (-30n + 2dn + 12)J \cdot p^*L \\ &\quad + (4 - 6n)J \cdot p^*c_1(E) + (6n - 2)J \cdot p^*c_1(B) - (p^*c_1(E))^2 \\ &\quad + p^*c_1(B) \cdot p^*c_1(E) - 6p^*L \cdot p^*c_1(E) + 6p^*c_1(B) \cdot p^*L \\ &\quad - (p^*c_1(B))^2 + p^*c_2(B) + (d - 15)(p^*L)^2 = 0 \end{aligned} \tag{5.3}$$

(terms of second degree)

$$
(5.4)\qquad \begin{aligned}
20d = {} & 6nJ \cdot p^*c_2(B) + (dn^2 + 12n - 4)J^2 \cdot p^*c_1(B) \\
& + (12 + 2nd)J \cdot p^*c_1(B) \cdot p^*L + (4 - 6n)p^*c_1(B) \cdot p^*c_1(E) \\
& - 2J \cdot (p^*c_1(B))^2 + 2dn^2J^3 + 4ndJ^2 \cdot p^*L + 2dJ \cdot (p^*L)^2 \\
& - dn^2J^2 \cdot p^*c_1(E) - 2dnJ \cdot p^*L \cdot p^*c_1(E)
\end{aligned}
$$

(term of third degree).

Now cut (5.3) respectively with J, $p^*c_1(B)$, $p^*c_1(E)$, p^*L, and obtain the four new equations:

$$
(5.5\text{i})\qquad \begin{aligned}
& (-15n^2 + n^2d + 12n - 4)J^3 + (-30n + 2dn + 12)J^2 \cdot p^*L \\
& \quad + (4 - 6n)J^2 \cdot p^*c_1(E) + (6n - 2)J^2 \cdot p^*c_1(B) - J \cdot (p^*c_1(E))^2 \\
& \quad + J \cdot p^*c_1(B) \cdot p^*c_1(E) - 6J \cdot p^*L \cdot p^*c_1(E) + 6J \cdot p^*c_1(B) \cdot p^*L \\
& \quad - J(p^*c_1(B))^2 + J \cdot p^*c_2(B) + (d - 15)J \cdot (p^*L)^2 = 0
\end{aligned}
$$

$$
(5.5\text{ii})\qquad \begin{aligned}
& (-15n^2 + n^2d + 12n - 4)J^2 \cdot p^*c_1(B) \\
& \quad + (-30n + 2dn + 12)J \cdot p^*c_1(B) \cdot p^*L + (4 - 6n)J \cdot p^*c_1(B) \cdot p^*c_1(E) \\
& \quad + (6n - 2)J \cdot (p^*c_1(B))^2 = 0
\end{aligned}
$$

$$
(5.5\text{iii})\qquad \begin{aligned}
& (-15n^2 + n^2d + 12n - 4)J^2 \cdot p^*c_1(E) \\
& \quad + (-30n + 2dn + 12)J \cdot p^*c_1(E) \cdot p^*L + (4 - 6n)J \cdot (p^*c_1(E))^2 \\
& \quad + (6n - 2)J \cdot p^*c_1(B) \cdot p^*c_1(E) = 0
\end{aligned}
$$

$$
(5.5\text{iv})\qquad \begin{aligned}
& (-15n^2 + n^2d + 12n - 4)J^2 \cdot p^*L + (-30n + 2dn + 12)J \cdot (p^*L)^2 \\
& \quad + (4 - 6n)J \cdot p^*c_1(E) \cdot p^*L + (6n - 2)J \cdot p^*c_1(B) \cdot p^*L = 0.
\end{aligned}
$$

Consider these four equations, (5.1) and (5.4) and eliminate from these six equations $J \cdot p^*c_1(B) \cdot p^*c_1(E)$, $J \cdot (p^*c_1(E))^2$, $J \cdot p^*c_1(E) \cdot p^*L$ using (5.2). Now eliminate $J \cdot p^*c_2(B)$ from (5.4) and (5.5). Set $J^3 = x$, $J^2 \cdot p^*L = y$, $J \cdot (p^*L)^2 = z$, $J^2 \cdot p^*c_1(E) = w$, $J^2 \cdot p^*c_1(B) = u$, $Jp^*c_1(B) \cdot p^*L = a$, $J \cdot (p^*c_1(B))^2 = b$. We get the system of five equations

$$
(\text{S1})\qquad d = n^3x + 3n^2y + 3nz
$$

$$\text{(S2)}\quad \begin{aligned} 20d = {} & (12n + dn^2 - 36n^2)u + (2dn - 36n + 12)a + (6n - 2)b \\ & + (90n^3 - 6n^3 d - 72n^2 + 24n + 2dn^2)x \\ & + (180n^2 - 12dn^2 - 36n + 2dn)y \\ & + (90n - 6dn + 2d)z + (36n^2 - 18n + dn^2)w \end{aligned}$$

$$\text{(S3)}\qquad (-15n + dn + 6)(nu + 2a) + (6n - 2)b = 0$$

$$\text{(S4)}\qquad (-15n + dn + 6)(nw + 2y) + (6n - 2)u = 0$$

$$\text{(S5)}\qquad (-15n + dn + 6)(ny + 2z) + (6n - 2)a = 0.$$

Now consider that from the adjunction formula

$$\begin{aligned} g - 1 &= d + \frac{1}{2} H^2 K \\ &= d + \frac{1}{2}\big(-2J + p^* c_1(E) - p^* c_1(B)\big) \cdot (n^2 J^2 + 2nJ \cdot p^* L + (p^* L)^2) \\ &= d - n^2 x - ny - z - na + \frac{1}{2} n^2 w - \frac{1}{2} n^2 u. \end{aligned}$$

The system $(S*)$ is linear in the seven unknowns x, y, z, u, w, a, b. Luckily the informations are enough to express g only in terms of d, n. We will see this fact in elementary way.

We consider first the case

$$-15n + dn + 6 = 0.$$

The only pairs of (d, n) satisfying these equality are (9,1), (12,2), (13,3) and (14,6). We want to exclude the case (14,6). We have $a = b = u = 0$ and the remaining equations of $(S*)$ are

$$\text{(A)}\qquad 70 = -36x + 96y + 16z + 171w$$

$$\text{(B)}\qquad 7 = 108x + 54y + 9z.$$

Moreover from (5.6)

$$g - 1 = 14 - 36x - 6y - z - 18w.$$

Consider that (A) $\cdot \dfrac{2}{19}$ + (B) $\cdot \left(\dfrac{17}{57}\right)$ gives $\dfrac{7 \cdot 103}{3 \cdot 19} = -36x - 6y - z + 18w$, that is $g - 1 = 14 + \dfrac{7 \cdot 103}{3 \cdot 19}$ which is a contradiction.

In the same way we exclude the case (13,3).

For $d = 12$, $n = 2$, we have in the same way the equations

$$\text{(C)} \qquad 4x + 6y + 3z - 6 = 0$$

$$\text{(D)} \qquad 2y + z + w - 4 = 0.$$

Moreover, from (5.6),

$$g - 1 = 12 - 4x - 2y - z + 2w.$$

Hence considering $-(\mathrm{C}) + 2(\mathrm{D})$, we get

$$-4x - 2y - z + 2w = 2$$

that gives $g = 15$ and $H^2K = 4$. In the same way we can compute $HK^2 = -12$, $K^3 = 12$. Hence $(K_X + H)^3 = 0$, $(K_X + H)^2 H = 4$ (compare with [3]). This means that the morphism associated to the line bundle $K_X + H$ maps X onto a quartic surface of $\mathbb{P}^3$.

If $-15n + dn + 6 \neq 0$, we sketch how to solve the system $(S*)$.

First solve (S1) and (S5) in y, z. Now solve (S3), (S4) in u, w and substitute y in the expression of w. Substitute the expressions obtained of y, z and w in terms of d, n, x, a, b in (5.6). We find that x and a disappear (this should not be suprising because the unknowns of the system $(S*)$ are not determined by X, for example we can tensor E with a line bundle). Substitute again the values obtained of u, y, z and w in (S2). x and a disappear once again. In order to solve now (S2) in b we need

$$dn(21n^2 - 12n + 2) - 9(35n^3 - 48n^2 + 24n - 4) \neq 0$$

but this is easily checked to be true. At last substitute in (5.6) the value of b obtained. We obtain the expression:

(5.7)

$$g - 1 = \frac{d[d^2n^3(3n-1) - dn^2(59n^2 54n+16) + 210n^4 - 213n^3 + 34n^2 + 36n - 12]}{n[dn(21n^2 - 12n + 2) - 9(35n^3 - 48n^2 + 24n - 4)]}.$$

We remark that, when $d \to \infty$,

$$g \sim d^2 \cdot \frac{n(3n-1)}{21n^2 - 12n + 2}.$$

Now we distinguish two cases.

Suppose first that C is not contained in a cubic. Then from fact 5.5 and (5.7) we have

$$\frac{d^2n^3(3n-1) - dn^2(59n^2 - 54n + 16) + 210n^4 - 213n^3 + 34n^2 + 36n - 12}{n[dn(21n^2 - 12n + 2) - 9(35n^3 - 48n^2 + 24n - 4)]} \leq \frac{d}{8}$$

that is, after some computations,

(5.8)

$$d \leq \frac{157n^3 - 88n + 36 + \sqrt{67^2 n^6 - 6432n^5 + 9808n^4 - 10136n^3 + 6464n^2 - 2496n + 528}}{2n(3n^2 + 4n - 2)}.$$

If otherwise C is contained in a cubic, from (5.4) we get that also Y is contained in a cubic. Hence Lemma 5.6 and (5.7) give

$$\frac{d-7}{6} \leq \frac{d^2 n^3(3n-1) - dn^2(59n^2 - 54n + 16) + 210n^4 - 213n^3 + 34n^2 + 36n - 12}{n[dn(21n^2 - 12n + 2) - 9(35n^3 - 48^2 + 24n - 4)]}$$

that is, after some computations,

(5.9)

$$d \leq \frac{54n^3 - 96n^2 + 67n - 18 + \sqrt{81n^6 + 540n^5 + 162n^4 - 2064n^4 + 2305n^2 - 1044n + 180}}{n(3n^2 - 6n + 2)}.$$

By Section 2 we may suppose $n \geq 2$. It is easy to check that for $n = 2$ the bounds (5.8) and (5.9) give $d \leq 72$, while for $n \geq 3$ they give $d \leq 37$. In the case $n = 2$, the only values of $d \leq 72$ such that g is an integer (see (5.7)) are $d = 12$, 22. But we can check that, for $d = 22$, the expression of $\chi(\mathcal{O}_B) = \dfrac{c_1^2(B) + c_2(B)}{12} = \dfrac{J \cdot (p^* c_1(B))^2 + J \cdot p^* c_2(B)}{12}$ which can be derived from $(S*)$ is not an integer (see also [3]). Let now $n \geq 3$. For every $d = 11, \ldots, 37$, we can check with the help of a computer that the expression of g cannot be integer for any n (for d fixed, g approaches finite values when n goes to infinity!). This concludes the proof.

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