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Group Action on Instanton Bundles over $\mathbb{P}^3$

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Abstract. Denote by $MI(k)$ the moduli space of k–instanton bundles E of rank 2 on $\mathbb{P}^3 = \mathbb{P}(V)$ and by $Z_k(E)$ the scheme of k–jumping lines. We prove that $[E] \in MI(k)$ is non–stable for the action of $SL(V)$ if $Z_k(E) \neq \emptyset$. Moreover $\dim Sym(E) \geq 1$ if length $Z_k(E) \geq 2$. We prove also that E is special if and only if $Z_k(E)$ is a smooth conic. The action of $SL(V)$ on the moduli of special instanton bundles is studied in detail.

1. Introduction

A k–instanton bundle on $\mathbb{P}^3 = \mathbb{P}(V)$ is a stable 2–bundle E such that $c_1 = 0$, $c_2 = k$ and $H^1(E(-2)) = 0$ (see [Har78] and [BH78]). The instanton bundles which are trivial on the fibers of the twistor map $\mathbb{P}^3 \to S^4$ correspond to self dual Yang Mills $Sp(1)$–connections on S^4, according to the ADHM–correspondence. The moduli space of k–instanton bundles on $\mathbb{P}^3$ will be denoted by $MI_{\mathbb{P}^3}(k)$.

It is expected to be a smooth and irreducible variety. The smoothness of $MI_{\mathbb{P}^3}(k)$ is known only for $k \leq 5$ (see [KO99] and the references there). A proof of the irreducibility of $MI_{\mathbb{P}^3}(k)$ for $k \leq 5$ has been recently found by Coanda, Tikhomirov and Trautmann ([CTT02]). $MI_{\mathbb{P}^3}(k)$ contains an irreducible component of dimension $8k - 3$, called the main component. We believe that the study of the $GL(V)$–action can be helpful in understanding the properties of $MI_{\mathbb{P}^3}(k)$.

Our aim is to study the natural action of $GL(V)$ over $MI_{\mathbb{P}^3}(k)$. In this sense this paper can be seen as the natural continuation of the study begun in [AO99] trying to answer a general problem raised by Simpson who asked about the stable points of the $GL(V)$–action on the moduli space of bundles over $\mathbb{P}(V)$. See also [Kar02] for recent contributions. In [CO00] we proved that $MI_{\mathbb{P}^3}(k)$ is an affine variety and this fact simplifies a lot the geometry of the group action. For $k \geq 3$ the generic $[E] \in MI_{\mathbb{P}^3}(k)$, at least in the main component, has trivial stabilizer $Sym(E)$ and

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hence, for $k \geq 3$, among the non–stable points $[E]$ are all the points $[E]$ such that $Sym(E)$ has positive dimension. In this work, we will focus our attention on the connected component $Sym^0(E)$ containing the identity.

A classical tool for the study of vector bundles is the scheme of jumping lines (see [OSS80]). The restriction of a k–instanton bundle E on a line L is isomorphic to $\mathcal{O}_L(-p) \oplus \mathcal{O}_L(p)$ for some p depending on L and satisfying the inequalities $0 \leq p \leq k$. It is known that $p = 0$ is the general value and if $p \geq 1$ then L is called a jumping line of order p. A useful criterion that we prove is the following (see Theorem 4.3, Example 4.5):

Theorem A. *Let E be a k–instanton bundle. If E has a jumping line of maximal order k, then $[E]$ is non–stable and the converse is not true.*

Moreover, we have found an example of a k–instanton bundle E with $Z_k(E) \neq \emptyset$ and $Sym^0(E) = 0$ (see Example 4.6).

In [ST90] was observed that the action of $SL(2) \simeq SL(U)$ over $V \simeq U \otimes U$ is useful to get the cohomology and other geometrical properties of special instanton bundles. Unfortunately special instanton bundles are not invariant for such action. Instead, they are invariant for the action of $SL(2) \simeq SL(U)$ over $V \simeq U \otimes \mathbb{C}^2$ where $SL(U)$ acts trivially over $\mathbb{C}^2$. Our starting point was the remark that the correct action to be considered is $SL(2) \times SL(2) \simeq SL(U) \times SL(U') \simeq Spin(V)$ acting over $V \simeq U \otimes U'$ for a suitable isomorphism (see Remark 4.1), hence for a suitable symmetric form on V. In fact the symmetry group of the kernel bundle considered in [ST90] is $SL(U) \times SL(U')$ and for any special instanton bundle E we get that $Sym(E)$ lies always in such a group (we believe that this fact is true for any instanton bundle). This fact conduces us to introduce the definition of fine action (see Definition 4.2). In particular a subgroup $G \subseteq GL(V)$ has a fine action if there are 2–dimensional vector spaces U and U' such that $G \subseteq GL(U) \times GL(U')$ and G acts according to fixed isomorphism $V \simeq U \otimes U'$.

In Section 5 (see Theorem 5.1 and Theorem 5.3) we will prove

Theorem B. *Let E be a k–instanton bundle. If length $Z_k(E) \geq 2$ then $Sym^0(E)$ contains G of positive dimension having a fine action.*

In addition, by means of Example 4.5, we will see that the converse of Theorem B is not true. On the other hand, we will see

Theorem C. *The following are equivalent*

(i) *$Z_k(E)$ is a conic.*

(ii) *E is a special k–instanton bundle.*

(iii) *$Sym^0(E) \supseteq SL(U')$ having a fine action.*

The equivalence between *(i)* and *(ii)* is proved in Corollary 2.8, *(ii)* implies *(iii)* is proved by (4.1) in Section 4 meanwhile *(iii)* implies *(ii)* is proved in Proposition 4.12.

We can show by explicit examples that not all actions on instantons are fine. Even more, recently KATSYLO showed us by means of an example that there are $\mathbb{C}^*$–actions which are not fine.

In Section 4 we study the group action over the subvariety $MI^{sp}_{\mathbb{P}^3}(k)$ of special k–instanton bundles (see Definition 2.2) and our main results are Corollary 4.11 and

Theorem 4.13. More precisely, it is known that every special k–instanton bundle is defined by $(k+1)$ lines through the classical Serre correspondence and that there is a pencil of lines whose branch locus can be studied with the classical machinery of binary forms and which is a $GL(V)$–invariant. In Theorem 4.13 we show that in general there are at most $\frac{1}{k+1}\binom{2k}{k}$ $GL(V)$–orbits of special k–instanton bundles with the same branch locus. We mainly apply results of TRAUTMANN ([Tra88]) that we report in a form suitable for our purposes. Although TRAUTMANN did not apply his results to instanton bundles, probably because the group action approach on moduli has only nowadays a stronger importance, some of these applications were certainly known to him.

We want to point out that we have chosen to present the actions in matrix form, due to the fact that this is suitable in view of computer implementing ([Anc96]).

2. Generalities about instantons over $\mathbb{P}^3$

The goal of this section is to prove some general results about k–instanton bundles on $\mathbb{P}^3$ that will be needed later. To this end, we start fixing some notation and recalling some known facts (see for instance [BT87]).

Notation 2.1. $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^3}(d)$ denotes the invertible sheaf of degree d on $\mathbb{P}^3 = \mathbb{P}(V)$. For any coherent sheaf E on $\mathbb{P}^3$ we denote $E(d) = E \otimes \mathcal{O}_{\mathbb{P}^3}(d)$ and by $h^iE(d)$ the dimension of $H^i\big(\mathbb{P}^3, E(d)\big) = H^iE(d)$.

Definition 2.2. A *k–instanton bundle on* $\mathbb{P}^3$ is an algebraic vector bundle E of rank 2 with Chern classes $c_1(E) = 0$ and $c_2(E) = k$, which is stable $\big($i. e. $h^0E = 0\big)$ and satisfies the vanishing $h^1E(-2) = 0$. E is called special if, in addition, $h^0E(1) = 2$.

We will denote by $MI_{\mathbb{P}^3}(k)$ $\big($resp. $MI^{sp}_{\mathbb{P}^3}(k)\big)$ the moduli space of k–instanton (resp. special k–instanton) bundles on $\mathbb{P}^3$. By definition, $MI^{sp}_{\mathbb{P}^3}(k)$ is constructed as a closed subscheme of $MI_{\mathbb{P}^3}(k)$. Notice that any $E \in MI_{\mathbb{P}^3}(2)$ is special ([BT87]).

Let E be a k–instanton bundle. It is well–known (see [BH78]) that there is a vector space I of dimension k and a symplectic vector space (W, J) of dimension $2k+2$ such that E is the cohomology of a symplectic monad

$$I^* \otimes \mathcal{O}(-1) \xrightarrow{A^t} W \otimes \mathcal{O} \xrightarrow{A} I \otimes \mathcal{O}(1)\,. \tag{2.1}$$

The vector space $W \otimes I \otimes V = Hom(W \otimes \mathcal{O}, I \otimes \mathcal{O}(1))$ contains the subvariety $\mathcal{Q}$ given by morphisms A such that the sequence (2.1) is a complex, that is, such that $AJA^t = 0$. In that case, we shall say that A represents E. $GL(I) \times Sp(W)$ acts on $\mathcal{Q}$ by $(g, s) \cdot A = gAs$. Let $\mathcal{Q}^0$ be the open subvariety of $\mathcal{Q} \subseteq W \otimes I \otimes V$ which consists of morphisms $A \in \mathcal{Q}$ that are surjective. It is proved in [CO00] that $\mathcal{Q}^0$ is affine. In this paper the quotient of an affine scheme by a reductive group is constructed as the *Spec* of the ring of invariants, according to Mumford's GIT. This quotient is geometric if and only if all orbits are closed. In this setting, a point is called stable if its orbit is closed and has the maximal dimension. In [BH78] it is proved that $GL(I) \times Sp(W)$ acts on $\mathcal{Q}^0$ with the same stabilizer in all points given by $\mathbf{Z}_2$. Hence all points of $\mathcal{Q}^0$

are stable for the action of $GL(I) \times Sp(W)$ and $MI_{\mathbb{P}^3}(k)$ is isomorphic to the geometric quotient $\mathcal{Q}^0/GL(I) \times Sp(W)$ and it is affine. In particular, $MI^{sp}_{\mathbb{P}^3}(k)$ is also affine and it has dimension $2k + 9$ (see e. g. Proposition 2.10).

Definition 2.3. Let E be a k–instanton bundle on $\mathbb{P}^3$. A hyperplane $H \subseteq \mathbb{P}^3$ is an *unstable hyperplane* of E if $h^0 E_{|H} \neq 0$. The set of unstable hyperplanes of E has a natural structure of scheme and it will be denoted by $W(E) \subseteq \mathbb{P}^{3*}$.

For any k–instanton bundle E we have $\dim W(E) \leq 2$. By a theorem of Coanda ([Coa92]) a k–instanton bundle E is special if and only if $\dim W(E) = 2$ and in this case $W(E)$ is a smooth quadric.

Now we are going to give a similar result for the scheme of k–jumping lines.

Definition 2.4. Let E be a k–instanton bundle on $\mathbb{P}^3$ and $1 \leq p \in \mathbb{Z}$. A line l on $\mathbb{P}^3$ is called a *p–jumping line* of E if $E_{|l} = \mathcal{O}_l(-p) \oplus \mathcal{O}_l(p)$ and we denote by

$$Z_j(E) \;=\; \left\{ l \in G\big(\mathbb{P}^1, \mathbb{P}^3\big) \mid E_{|l} = \mathcal{O}_l(-i) \oplus \mathcal{O}_l(i),\ j \leq i \right\}.$$

A line in $Z_k(E)$ is called a maximal order jumping line of E.

Remark 2.5. Let $(x_0, \ldots, x_3)$ be homogeneous coordinates on $\mathbb{P}(V)$ and let $A = \sum_{i=0}^{3} A_i x_i \in W \otimes I \otimes V$ be a matrix representing E. Then it is well–known that $Z_k(E)$ is obtained by cutting the Plücker quadric $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$ with the linear space $\sum_{i<j} p_{ij} A_i J A_j^t = 0$. In particular $\deg Z_k(E) \leq 2$.

In the following result we relate the maximal order jumping lines of E with unstable hyperplanes of E.

Lemma 2.6. *Let E be a k–instanton bundle on $\mathbb{P}^3$ and $l \in Z_k(E)$. Then any $H \supseteq l$ is an unstable hyperplane of E, i. e., $H \in W(E)$.*

Proof. Since $H \supseteq l$ we have the following exact sequence

$$0 \longrightarrow E(-1)_{|H} \longrightarrow E_{|H} \longrightarrow E_{|l} \longrightarrow 0\,.$$

Assume that H is not an unstable plane. Taking cohomology to the above exact sequence we get the injection $H^0 E_{|l} \hookrightarrow H^1 E(-1)_{|H}$. Since by Serre's duality $h^2 E(-1)_{|H} = h^0 E(-2)_{|H} = 0$, this gives us

$$k + 1 \;=\; h^0 E_{|l} \;\leq\; h^1 E(-1)_{|H} \;=\; -\chi(E(-1)_{|H}) \;=\; k$$

which is a contradiction. Hence H is an unstable hyperplane of E. □

Remark 2.7. In particular, since for any k–instanton bundle E on $\mathbb{P}^3$ we have $h^0 E_{|H} \leq 1$ ([KO99]; Theorem 1.2), we get immediately from the above proof that $Z_j(E) = \emptyset$ for any $j \geq k + 1$.

Corollary 2.8. *Let E be a k–instanton bundle on $\mathbb{P}^3$. We have $\dim Z_k(E) \leq 1$. E is special if and only if $\dim Z_k(E) = 1$ and in this case $Z_k(E)$ is a smooth conic in $G\big(\mathbb{P}^1, \mathbb{P}^3\big)$.*

Proof. It follows from Lemma 2.6 that if $\dim Z_k(E) \geq 1$ then $\dim W(E) \geq 2$. Hence, by [Coa92], E is a special instanton bundle. □

We know ([Har78] for $k = 2$ and [BT87] in general) that if $E \in MI^{sp}_{\mathbb{P}^3}(k)$ then $Z_k(E)$ is a smooth conic on $G(\mathbb{P}^1, \mathbb{P}^3)$ and that $W(E)$ is a smooth quadric. In addition, $Z_k(E)$ corresponds to one of the two rulings of $W(E)$ and the generic $s \in H^0E(1)$ vanishes on k disjoint lines of the other ruling. Moreover, it is well–known that each $E \in MI^{sp}_{\mathbb{P}^3}(k)$ can be identified with a g^1_{k+1} without base points on $Z_k(E) \simeq \mathbb{P}^1$ and that we can choose this line as the line of all hyperplanes containing the line $\{x_0 = x_1 = 0\}$. We will define the branch locus of this g^1_{k+1} as the branch locus of the bundle. It is defined up to $SL(2)$–action.

Notation 2.9. We will denote by G^k the open subset in $G(\mathbb{P}^1, \mathbb{P}^{k+1})$ which consists of g^1_{k+1} without fixed points and we will denote by $\mathcal{W} \subseteq Gr(\mathbb{P}^2, \mathbb{P}^5)$ the open subset of planes which cut the Plücker quadric in a smooth conic.

The above discussion shows the following

Proposition 2.10. ([HN82], [BT87]; Coroll. 4.7.) *The morphism $\pi_1 \colon MI^{sp}_{\mathbb{P}^3}(k) \to \mathcal{W}$ which takes E to the plane spanned by the conic $Z_k(E)$ is a fibration such that all the fibers are isomorphic to G^k.*

In [AO98]; Proposition 1.1, special k–instanton bundles were characterized in terms of their monads. However that description, as the one given in [BT87], is not useful for our purposes because the monads considered there were not symplectic and the techniques that we develop in Section 4 do not apply. So we have to introduce an alternative description. With this aim we introduce some notations

Definition 2.11. A $(k+1) \times (k+1)$–*Hankel matrix* $H = (\alpha_{ij}) = (\alpha_{i+j-2})$ is one having equal elements along each diagonal line parallel to the secondary diagonal.

Let $X = \left[x^k, -kx^{k-1}y, \binom{k}{2}x^{k-2}y^2, \ldots, (-1)^k y^k\right]$ and H be a $(k+1) \times (k+1)$–Hankel matrix. Following [Tra88]; 2.4, we denote by f_H the form of degree $2k$ defined by $X \cdot H \cdot X^t$.

Definition 2.12. Denote by Δ the invariant for f_H defined by the determinant

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_k \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{k+1} \\ \vdots & \vdots & & \vdots \\ \alpha_k & \alpha_{k+1} & \cdots & \alpha_{2k} \end{vmatrix}.$$

Let $(x_0, \ldots, x_3)$ be homogeneous coordinates on $\mathbb{P}(V)$. We set

$$I_k(x_0, x_1) := \begin{pmatrix} x_0 & x_1 & & \\ & \ddots & \ddots & \\ & & x_0 & x_1 \end{pmatrix} \quad \text{and} \quad \widetilde{I_k}(x_0, x_1) := \begin{pmatrix} x_1 & & & \\ x_0 & x_1 & & \\ & \ddots & \ddots & \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix}.$$

Notation 2.13. Given H a nondegenerate $(k+1) \times (k+1)$–Hankel matrix, we denote

$$A \; := \; [I_k(x_0, x_1) | I_k(x_2, x_3) \cdot H] \; . \tag{2.2}$$

Lemma 2.14. *Let J be the standard skew–symmetric nondegenerate matrix and let A be as in* (2.2)*. The following holds*

(i) $AJA^t = 0$.

(ii) *Let E be a k–instanton bundle on $\mathbb{P}^3$ defined by A. Then $h^0E(1) = 2$ and E is special.*

(iii) *Conversely, every special k–instanton bundle is the cohomology bundle of a monad* (2.2) *for a suitable system of coordinates.*

Proof. (i) is a straightforward verification. Let

$$T_k \; = \; \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}$$

and let

$$\tilde{A} \; = \; \left[\widetilde{I_k}(x_0, x_1) \cdot T_{k+1} \cdot H^{-1} | \widetilde{I_k}(x_2, x_3) \cdot T_{k+1} \right] .$$

Then $AJ\tilde{A}^t = 0$. Since $\tilde{A}$ has $(k+2)$ rows it follows $h^0E(1) = 2$ which proves (ii). When H moves among the nondegenerate Hankel matrices, the moduli space of bundles given by (2.2) is isomorphic to $\mathbb{P}(S^{2k}U) \setminus \{\Delta = 0\}$ (see Corollary 3.2). By [Tra88]; Proposition 2.2, $\mathbb{P}(S^{2k}U) \setminus \{\Delta = 0\}$ is isomorphic to G^k. Since this is exactly a fiber of the map π_1 of Proposition 2.10, we obtain (iii). □

We will end with the following result which is a straightforward verification and we left it to the reader.

Lemma 2.15. *Let E be a special k–instanton bundle defined by the monad* (2.2)*. Then $W(E)$ is given by all the hyperplanes which are tangent to the quadric $x_0x_3 - x_1x_2$ and $Z_k(E)$ is the conic obtained by cutting the Plücker quadric with the plane $\{p_{02} = p_{13} = p_{03} + p_{12} = 0\}$.*

3. Useful facts about $SL(2)$–actions

For any $g \in sl(U)$, we denote by $s^k g \in sl(S^kU)$ the image of g through the Lie algebra representation of $sl(U)$ given by the k–symmetric power.

Lemma 3.1. *Let H be a $(k+1) \times (k+1)$–Hankel matrix determined by $\alpha = (\alpha_0, \ldots, \alpha_{2k})$ and consider $g \in sl(U)$. Then the following holds*

(i) *$H' = (s^k g)^t \cdot H + H \cdot s^k g$ is a Hankel matrix.*

(ii) *If $\alpha' = (\alpha'_0, \ldots, \alpha'_{2k})$ are the coefficients of H' then $(s^{2k} g)^t \cdot (\alpha)^t = (\alpha')^t$.*

(iii) *The natural action of $sl(U)$ on forms of degree $2k$ takes f_H to $f_{H'}$.*

Proof. It follows from direct computation. □

Corollary 3.2. *The $(k+1)\times(k+1)$–Hankel matrices forms an invariant subspace of $\big(S^2\big(S^kU\big)\big)$ isomorphic to $S^{2k}U$.*

Proof. It is an immediate consequence of Lemma 3.1. □

There is an analog description for the Lie group $SL(U)$. If $g\in SL(U)$, we denote by $S^kg\in SL\big(S^kU\big)$ the image of g through the group representation of $SL(U)$ given by the k–symmetric power. As a consequence of the fact that $S^k(\exp g)=\exp\big(s^kg\big)$, from Lemma 3.1 we obtain

Lemma 3.3. *Let H be a $(k+1)\times(k+1)$–Hankel matrix determined by $\alpha=(\alpha_0,\dots,\alpha_{2k})$ and consider $g\in SL(U)$. Then the following holds*
(i) *$H''=(S^kg)^t\cdot H\cdot S^kg$ is a Hankel matrix.*
(ii) *If $\alpha''=\big(\alpha''_0,\dots,\alpha''_{2k}\big)$ are the coefficients of H'' then $\big(S^{2k}g\big)^t\cdot(\alpha)^t=(\alpha'')^t$.*
(iii) *The natural action of $SL(U)$ on forms of degree $2k$ takes f_H to $f_{H''}$.*

We have a filtration of $SL(U)$–invariant closed subvarieties $H_{2k}\subseteq\dots\subseteq H_2\subseteq H_1=\mathbb{P}\big(S^{2k}(U)\big)$ where $H_j=\big\{f\in S^{2k}(U)\mid f \text{ has a root of multiplicity}\geq j\big\}$. H_{k+1} consists of the locus where all the $SL(U)$–invariant vanish (i. e. not semistable points) and the $SL(U)$–quotient map is defined on $\mathcal{P}^0=\mathbb{P}\big(S^{2k}(U)\big)\setminus H_{k+1}$. It is well–known that the orbit of $f\in\mathcal{P}^0$ is not closed in $\mathcal{P}^0$ if and only if $f\in H_k$ (i. e. non–stable points). In fact, the orbits of points in H_k all contain in their closure the orbit of $x^ky^k\in H_k$, which is the only orbit which is two–dimensional (all the other orbits are three–dimensional). This particular $SL(U)$–orbit corresponds to a particular $SL(V)$–orbit of special instanton bundles, that we will define in a while.

Now consider the analogous filtration $W_{k+1}\subseteq\dots\subseteq W_2\subseteq W_1=\mathbb{P}\big(S^{k+1}(U)\big)$ where $W_j=\big\{f\in S^{k+1}(U)\mid f \text{ has a root of multiplicity}\geq j\big\}$.

It is well–known (and easy to check) that a pencil $L\in Gr\big(\mathbb{P}^1,\mathbb{P}\big(S^{k+1}(U)\big)\big)$ has a fixed point (that is it does not belong to G^k) if and only if L is contained in some osculating space to the rational normal curve W_{k+1}. Moreover, the branch locus of L can be identified with $B_L\in\mathbb{P}\big(S^{2k}(U)\big)$ being B_L isomorphic to $L\cap W_2$. It follows the basic fact that $L\cap W_{s+1}\neq\emptyset$ if and only if $B_L\in H_s$, for $1\leq s\leq k$ and we also remark that $L_\alpha\cap W_{k+1}\neq\emptyset$ if and only if $\alpha\in H_k$ (compare it with [Tra88]; Prop. 2.6). The map $G^k\to\mathbb{P}\big(S^{2k}(U)\big)$ which takes L to B_L extends to a map $R:Gr\big(\mathbb{P}^1,\mathbb{P}\big(S^{k+1}(U)\big)\big)\to\mathbb{P}\big(S^{2k}(U)\big)$ which is finite and equivariant (see [Tra88][1]; Remark 2.5, and also our proof of Theorem 4.13). In particular, a pencil $L\in G^k$ is non–stable if and only if $L\cap W_{k+1}\neq\emptyset$. Notice once more that all the orbits of L such that $L\cap W_{k+1}\neq\emptyset$ contain in the closure the unique orbit given by the chordal variety to W_{k+1}.

We resume the above discussion in the following

[1] The only correction to be done in the diagram at page 41 of [Tra88] is that the arrow H must be dotted.

Corollary 3.4. *All points of G^k are semistable for the action of $SL(U)$ and the only non–stable ones are the pencils with a point of multiplicity $(k+1)$. All the non–stable orbits contain in the closure the unique orbit of pencils with two points of multiplicity $(k+1)$.*

4. The action on $MI_{\mathbb{P}^3}(k)$ and $MI^{sp}_{\mathbb{P}^3}(k)$

Along this section, we will keep the notations introduced before. Since a morphism between k–instanton bundles lifts to a morphism between the corresponding monads, we get that the action of $GL(V)$ over $MI_{\mathbb{P}^3}(k)$ lifts to the action of $GL(I) \times Sp(W) \times GL(V)$ over $\mathcal{Q}^0$. This means that if E and E' are k–instanton bundles in the same $GL(V)$–orbit then any two representatives $A, A' \in \mathcal{Q}^0$ are in the same $GL(I)\times Sp(W)\times GL(V)$–orbit. The (connected components containing the identity of the) stabilizers and the stable points of the two actions correspond to each other (this technique was used in [AO99]). We underline that since $\mathcal{Q}^0$ is affine (it is the complement in $\mathcal{Q}$ of an invariant hypersurface, see [CO00]), the stable points in $\mathcal{Q}^0$ for the action of $GL(I) \times Sp(W) \times GL(V)$ are well defined. The same for the stable points in $MI_{\mathbb{P}^3}(k)$ for the action of $GL(V)$. In addition, the two GIT–quotient $MI_{\mathbb{P}^3}(k)//GL(V)$ and $\mathcal{Q}^0//GL(I) \times Sp(W) \times GL(V)$ are isomorphic.

Remark 4.1. Let U and U' be vector spaces of dimension 2. Fix three isomorphisms $V \simeq U \otimes U'$, $W \simeq S^kU \otimes U'$ and $I \simeq S^{k-1}U$. Then, there is an induced natural multiplication map $A \in Hom(W \otimes \mathcal{O}, I \otimes \mathcal{O}(1))$ which is $GL(U) \times GL(U')$–invariant. Indeed it can be shown that $\ker A$ is a rank $k+2$ bundle on $\mathbb{P}(V)$ with symmetry group in $GL(V)$ isomorphic to $GL(U) \times GL(U')/\mathbb{C}^*$. In the case $U = U'$, this construction was considered in [ST90], where $\ker A$ was called the kernel bundle. We could work in $SL(V)$, which is the universal cover of $Aut(\mathbb{P}(V))$, and in this case the symmetry group of $\ker A$ is $Spin(V) \simeq SL(U)\times SL(U')$, but we prefer to consider the GL groups, which is a not essential variation.

After this remark, we are led to pose the following definition, which will play an important role in the sequel.

Definition 4.2. A subgroup $G \subseteq GL(V)$ is said to have a *fine action* over $MI_{\mathbb{P}^3}(k)$ if there are vector spaces U and U' of dimension 2 such that $G \subseteq GL(U) \times GL(U')$ and there exists a lifted action of G over $W \otimes I \otimes V$ which is determined by some fixed isomorphisms $V \simeq U \otimes U'$, $W \simeq S^kU \otimes U'$, $I \simeq S^{k-1}U$.

Theorem 4.3. *Let E be a k–instanton bundle such that $Z_k(E) \neq \emptyset$. Then $[E]$ is a non–stable point in $MI_{\mathbb{P}^3}(k)$ for the $GL(V)$–action.*

Proof. By [Ski97], E is represented by

$$A \;=\; [I(x_0, x_1)|H(x_0, x_1, x_2, x_3)]$$

where H is a $k \times (k+1)$ matrix of linear forms. Let $\lambda \simeq \mathbb{C}^* \subseteq SL(U')$ acting in such a way that the splitting $W = {U'}^{k+1}$ defines two eigenspaces in W with positive and negative weights that correspond to the two blocks in which are divided the columns of A. Moreover, the positive eigenspace on $V = U' \oplus U'$ is generated by x_0, x_1 and the negative one by x_2, x_3. The action on I is trivial. $\mathbb{C}^*$ has a fine action according to Definition 4.2 and we get

$$\lim_{t \to 0} \lambda(t) \cdot A \ = \ [I(x_0, x_1)|H(0, 0, x_2, x_3)]$$

which still represents a k–instanton. $\square$

Remark 4.4. By [Rao97] we know that if $Z_k(E) \neq \emptyset$ then $[E]$ is a smooth point in $MI_{\mathbb{P}^3}(k)$. It should be interesting to know if the unstable points for the $GL(V)$–action over $MI_{\mathbb{P}^3}(k)$ are smooth points.

Let us see, by means of an example, that the converse of Theorem 4.3 is no longer true.

Example 4.5. Let

$$A \ = \ \begin{pmatrix} 3x_0 & x_1 & & 4x_3 & & & 5x_3 & x_2 \\ & 5x_0 & x_1 & & & 5x_3 & x_2 & \\ & & 5x_0 & x_1 & 3x_3 & x_2 & & 4x_0 \end{pmatrix}.$$

It is easy to check that $AJA^t = 0$ and that the associated morphism is surjective $\big($the rank is 3 on every $(x_0, \ldots, x_3) \in \mathbb{P}^3\big)$ so that A represents a 3–instanton bundle E. Moreover $Z_3(E) = \emptyset$ and $\mathbb{C}^* \subseteq Sym(E)$ having a fine action. The weights are as follows:

- on $I : -2, 0, 2\,,$
- on $W : 5, 3, 1, -1, -5, -3, -1, 1\,,$
- on $V : x_0(-3), x_1(-1), x_2(1), x_3(3)\,,$

which correspond to weights -1 and 1 on U and -2 and 2 on U' with the isomorphisms $I \simeq S^2U$, $W \simeq S^3U \otimes U'$, $V \simeq U \otimes U'$. In particular $[E]$ is a non–stable point for the $GL(V)$–action over $MI_{\mathbb{P}^3}(k)$ and this shows that the converse of Theorem 4.3 is not true.

Example 4.6. The generic 3–instanton bundle E with Z_3 given by one point has $Sym^0(E) = 0$. This can be checked with the help of Macaulay system ([BS]) by using Skiti monad ([Ski97]) with a generic Hankel matrix. Moreover, it seems likely that the generic k–instanton bundle E with $Z_k \neq \emptyset$ has $Sym^0(E) = 0$. On the other hand, there are examples of k–instanton bundles such that $\dim Sym(E) = 2$.

Keeping the notations introduced in Section 2, we recall a result from [CO00].

Theorem 4.7. *There is a $Sp(W) \times SL(I) \times SL(V)$–invariant homogeneous polynomial D over $W \otimes I \otimes V$ of degree $2k(k+1)$ such that $A \in \mathcal{Q}^0$ if and only if $A \in \mathcal{Q}$ and $D(A) \neq 0$.*

Corollary 4.8. *Let $A \in \mathcal{Q}^0$ representing a k–instanton bundle E and $\lambda : \mathbb{C}^* \to Sp(W) \times SL(I) \times SL(V)$ be a morphism such that its image is contained in $Sym(E)$. Then, if $\lim_{t \to 0} \lambda(t) \cdot A$ exists, it belongs to $\mathcal{Q}^0$.*

Proof. It follows from the fact that $D(\lambda(t) \cdot A)$ is constant with respect to t. □

Remark 4.9. After Corollary 4.8, a classification of all $\mathbb{C}^*$–invariant k–instanton bundles should give the classification of unstable points in $MI_{\mathbb{P}^3}(k)$, but we postpone this study.

For special k–instanton bundles the description of the group action is quite precise. By the above geometric description, if $g^*E = E$, then g leaves the smooth quadric $W(E)$ fixed and does not exchange the two rulings. Hence, $g \in SL(U) \times SL(U')$, for some complex vector spaces U and U' of dimension two, acting over $\mathbb{P}^3 = \mathbb{P}(U \otimes U')$ and one has to check how $SL(U) \times SL(U')$ acts on the space G^k. Notice that the first $SL(U')$ does not change anything so, for any $E \in MI^{sp}_{\mathbb{P}^3}(k)$

$$SL(U') \subseteq Sym(E) \tag{4.1}$$

$SL(V)$ acts transitively on the open subset $\left[\mathbb{P}^9\right]^0$ given by smooth quadrics and even on $\mathcal{W}$ which is a $2:1$ covering of $\left[\mathbb{P}^9\right]^0$. Looking at the Proposition 2.10 it follows that

$$MI^{sp}_{\mathbb{P}^3}(k)//SL(V) \;\cong\; G^k//SL(U) \tag{4.2}$$

(see also next Lemma 4.10). This description was performed for $k = 2$ by Hartshorne ([Har78]) and in the general case by Spindler and Trautmann ([ST90]), although they considered $U = U'$.

Let us now see that under this isomorphism, the isomorphism class of the bundle is uniquely determined by the $SL(2)$–class of its associated Hankel matrix. Indeed we have

Lemma 4.10. *The isomorphism $MI^{sp}_{\mathbb{P}^3}(k)//SL(V) \cong G^k//SL(U)$ takes the $SL(V)$ class of the bundle to the $SL(2)$–class of its associated Hankel matrix given in* (2.2).

Proof. Given E a special k–instanton bundle choose coordinates such that $\{x_0x_3 - x_1x_2 = 0\} \in W(E)$. Take two k–jumping lines L_1 and L_2 such that $H^0\left(E_{|L_1}\right)$ and $H^0\left(E_{|L_2}\right)$ are orthogonal spaces into W with respect to J (here $H^0\left(E_{|L_i}\right)$ are considered into W by the monad (2.1)). Moreover we can assume that $L_1 = \{x_0 = x_1 = 0\}$ and $L_2 = \{x_2 = x_3 = 0\}$. With this choice of coordinates there exists a matrix A representing E as in (2.2) containing a Hankel matrix H. $SL(2)$ acts on these choices, the $SL(2)$–class of H is uniquely determined and characterizes the $SL(V)$–orbit of E. □

As a consequence we obtain the following nice description (compare it with Corollary 3.4)

Corollary 4.11. *Let $MI^{sp}_{\mathbb{P}^3}(k)$ be the moduli space of special k–instanton bundles on $\mathbb{P}^3$. Then the only non–stable points correspond to bundles having a section vanishing on a line with multiplicity $(k+1)$. All the non–stable orbits contain in the closure the unique orbit of bundles having two distinct sections each one vanishing on a (different) line with multiplicity $(k+1)$.*

Proof. It follows from the isomorphism (4.2), Corollary 3.4 and Lemma 4.10. □

Proposition 4.12. *Let E be a k–instanton bundle such that $SL(U') \subseteq Sym(E)$ having a fine action. Then E is special.*

Proof. By the assumption there is a $\mathbb{C}^* \subseteq SL(U')$ with a two–dimensional eigenspace in V of positive weight. Then there is a matrix representing E such that in the first $k \times (k+1)$ submatrix only the coordinates of this eigenspace appear. It follows that the line spanned by this coordinates is a k–jumping line. Since $Z_k(E)$ is $SL(U')$–invariant and not empty, it is one–dimensional and the result follows from Corollary 2.8. □

Let us now briefly describe different group actions of $G \subset Sym(E)$, being E a special k–instanton bundle represented by a matrix A (see Notation 2.13). In general we have $SL(2) \subseteq Sym(E)$ acting in the following way. If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2)$ then the action on W is determined by

$$\begin{bmatrix} \alpha \cdot Id & \beta \cdot H \\ \gamma \cdot H^{-1} & \delta \cdot Id \end{bmatrix} \in Sp(W)$$

while the action on I is trivial and the action on V is given by $x_0 \mapsto \alpha x_0 + \gamma x_2$, $x_1 \mapsto \alpha x_1 + \gamma x_3$, $x_2 \mapsto \beta x_0 + \delta x_2$, $x_3 \mapsto \beta x_1 + \delta x_3$.

Let us now describe some particular case. Consider the Hankel matrix H given by $\alpha_i = \delta_{i,k}$. It corresponds to the form $x^k y^k$ which, for $k \geq 2$, is the only form such that $\Delta \neq 0$ and such that the stabilizer has dimension 4. In this case, $\mathbb{C}^* \cdot SL(2) \subseteq Sym(E)$.

A matrix description of the instanton bundle E with $Sym(E) = \mathbb{C}^* \cdot SL(2)$ is

$$(4.3) \qquad \begin{pmatrix} x_0 & x_1 & & & & & x_3 & x_2 \\ & \cdot & \cdot & & & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & & \\ & & & x_0 & x_1 \; x_3 & x_2 & & \end{pmatrix}.$$

Finally, let us mention that bundles E having two distinct sections each one vanishing on a (different) line with multiplicity $(k+1)$ are quite interesting, because they are characterized by the property $Sym(E) \simeq SL(U) \cdot \mathbb{C}^*$ with a fine action.

We will end this section giving a more precise description of the correspondence, introduced in Section 2, between special k–instanton bundles and linear systems g^1_{k+1} without base points. More precisely we will prove $\big(\mathbb{P}^{2k,ss}$ denotes the subscheme of semistable points for the $SL(U)$–action$\big)$

Theorem 4.13. *Let $MI^{sp}_{\mathbb{P}^3}(k)$ be the moduli space of special k–instanton bundles on $\mathbb{P}^3$. There is a natural morphism*

$$\phi \ : \ MI^{sp}_{\mathbb{P}^3}(k) \ \longrightarrow \ \mathbb{P}^{2k,ss}//SL(U)$$

which takes E to the branch locus defined by its pencil of sections and which factors through

$$\phi' \ : \ MI^{sp}_{\mathbb{P}^3}(k)//SL(V) \ \longrightarrow \ \mathbb{P}^{2k,ss}//SL(U)\,,$$

being ϕ' a finite morphism on its image.

The above theorem appears in [New81] for $k = 2$ and in this case $MI_{\mathbb{P}^3}(2)//SL(V)$ is isomorphic to $\mathbb{A}^1$.

This result answers the question posed by HARTSHORNE in [Har78b] concerning the relation between the cross–ratio of the four branch points of a g^1_3 and the orbits for the action of $SL(U)$. Indeed, HARTSHORNE asked if the cross–ratio determined uniquely the orbit and NEWSTEAD showed that for each value of the cross–ratio there are exactly two orbits in G^2 with one exception (see [New81] for more details).

Proof. By the isomorphism (4.2), $MI^{sp}_{\mathbb{P}^3}(k)//SL(V) \cong G^k//SL(U)$. Moreover, by Lemma 4.10, this isomorphism takes the $SL(V)$–class of the bundle to the $SL(2)$–class of its associated Hankel matrix given in (2.2). Hence, we will prove that there exists a finite morphism

$$\psi \ : \ G^k/SL(U) \quad \longrightarrow \quad \mathbb{P}^{2k,ss}//SL(U)\,.$$

To this end, let $G = Gr\big(\mathbb{P}^1, \mathbb{P}\big(S^{k+1}U\big)\big)$ and denote by $p_{i,j}$ the Plücker coordinates on G. Define

$$\begin{array}{rccc} R: & G & \longrightarrow & \mathbb{P}^{2k} = \mathbb{P}\big(S^{2k}U\big)\,, \\ & (p_{0,1}, \dots, p_{k,k+1}) & \longrightarrow & (q_1 : \dots : q_{2k+1}) \end{array}$$

where, for any $1 \le m \le 2k+1$, $q_m = \sum_{\mu+\nu=m,\ \mu<\nu} (\nu-\mu) p_{\mu,\nu}$ (compare it with [Tra88]; Pag. 41 – 42).

Claim: R is induced by the projection from the linear subspace defined by

$$q_0 \ = \ \dots \ = \ q_{2k+1} \ = \ 0$$

which is disjoint from G.

Proof of the Claim. First of all recall that, for any $\{i_0, i_1, i_2, i_3\} \subset \{1, \dots, k\}$, the Plücker coordinates $p_{i,j}$ verify the following Plücker relations

$$p_{i_0,i_1} p_{i_2,i_3} - p_{i_0,i_2} p_{i_1,i_3} + p_{i_0,i_3} p_{i_1,i_2} \ = \ 0\,.$$

We will prove by induction on $m = \mu + \nu$, that if $q_1 = \dots = q_{2k+1} = 0$ then $p_{i,j} = 0$ for any pair (i, j). If $m = 1$, the assumption $q_1 = 0$ implies $p_{0,1} = 0$. Let us assume $p_{\mu,\nu} = 0$ for $\mu < \nu$ such that $\mu + \nu = m$ and we will see that $p_{\mu,\nu} = 0$ for any $\mu < \nu$ such that $\mu + \nu = m + 1$. Considering the Plücker relations for a suitable sets of indices and by induction hypothesis we get $p_{i,j} p_{k,l} = 0$, for any $i < j$, $k < l$ with $i + j = k + l = m + 1$. Hence, from the assumption

$$0 \ = \ q_{m+1} \ = \ \sum_{\mu+\nu=m+1,\ \mu<\nu} (\nu-\mu) p_{\mu,\nu}$$

we deduce that $p_{\mu,\nu} = 0$ for any $\mu < \nu$ with $\mu + \nu = m + 1$, which proves what we want.

It follows from the claim that R is a finite morphism of degree $\frac{1}{k+1}\binom{2k}{k}$ (it is the degree of G) and it is $SL(U)$–equivariant. Moreover, since $G^k \subset G$, by [Tra88]; Pag. 41, R induces a finite morphism on its image

$$\psi \ : \ G^k//SL(U) \quad \longrightarrow \quad \mathbb{P}^{2k,ss}//SL(U)$$

which, by [Tra88]; Proposition 2.2, sends the $SL(U)$–class of the linear system g^1_{k+1} associated to the $SL(V)$–class of a bundle E, to the branch locus defined by its pencil of sections. □

Remark 4.14. The degree of ϕ' is at most the degree of R which is $\frac{1}{k+1}\binom{2k}{k}$. In the case $k = 2$ the degree of ϕ' is exactly 2.

Remark 4.15. A general special instanton bundle with $c_2 = 2$ has symmetry group $SL(2)\cdot G_8$ where G_8 is binary dihedral of order 8. If the quartic form corresponding to $(\alpha_0, \dots, \alpha_4)$ satisfies $I = 0$ then $Sym(E) \cong SL(2)\cdot G_{24}$ where G_{24} is binary tetrahedral of order 24. If the bundle has only one section vanishing on a line with multiplicity 4 then the quartic form has a double root and $Sym(E) \cong SL(2)\cdot G_{16}$ where G_{16} is binary dihedral of order 16. The bundles with two sections each one vanishing on a different line with multiplicity 4 have $Sym(E) \cong SL(2)\cdot \mathbb{C}^*$ and they are characterized by the property $\dim Sym(E) \geq 4$.

5. Link between Sym and Z_k

The goal of this section is to study the symmetry group $Sym(E)$ of a k–instanton bundle E such that $Z_k(E) \neq \emptyset$.

Theorem 5.1. *Let E be a k–instanton bundle on $\mathbb{P}^3$ such that $Z_k(E)$ contains a double point. Then there exists $\mathbb{C} \subseteq Sym^0(E)$, where $\mathbb{C}$ has a fine action.*

Proof. By assumption and [Ski97] there is monad representing E with

$$A = \left(\begin{array}{cccc|c} x_0 & x_1 & & & \\ & \ddots & \ddots & & H(x_0,x_1,x_2,x_3) \\ & & x_0 & x_1 & \end{array}\right)$$

and the line $\{x_0 = x_1 = 0\} \in Z_k(E)$ with coordinates $(p_{01}, \dots p_{13}, p_{23}) = (0, \dots, 0, 1)$ corresponds to a double point. By Remark 2.5, $Z_k(E)$ is given by $\sum_{i<j} p_{ij} A_i J A_j^t = 0$. The crucial point is that

$$A' = \left(\begin{array}{cccc|c} x_0 & x_1 & & & \\ & \ddots & \ddots & & H(0,0,x_2,x_3) \\ & & x_0 & x_1 & \end{array}\right)$$

represents another instanton E'. Moreover

$$\begin{aligned} A_0 J A_2^t &= A'_0 J A'^t_2\,, & A_0 J A_3^t &= A'_0 J A'^t_3\,, \\ A_1 J A_2^t &= A'_1 J A'^t_2\,, & A_1 J A_3^t &= A'_1 J A'^t_3\,, \\ A_2 J A_3^t &= A'_2 J A'^t_3 = 0\,, \end{aligned}$$

and ${A'}_0 J {A'}_1^t = 0$ while, in general, $A_0 J A_1^t \neq 0$. By assumption the variety given by $\sum_{i<j} p_{ij} A_i J A_j^t = 0$ contains a line tangent to the Plücker quadric $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$ in the point $(0, \ldots, 0, 1)$. Hence the system

$$p_{02}A_0JA_2^t + p_{03}A_0JA_3^t + p_{12}A_1JA_2^t + p_{13}A_1JA_3^t \;=\; 0$$

has a nonzero solution $(\tilde{p}_{02}, \tilde{p}_{03}, \tilde{p}_{12}, \tilde{p}_{13})$ which gives the line $(0, s\tilde{p}_{02}, s\tilde{p}_{03}, s\tilde{p}_{12}, s\tilde{p}_{13}, t)$ where $(s,t) \in \mathbb{P}^1$. Notice that this system is the same which gives solutions in the unknowns $(p_{01}, \ldots, p_{13}, p_{23})$ for $Z_k(E')$ which has now the solutions

$$\big(u, s\tilde{p}_{02}, s\tilde{p}_{03}, s\tilde{p}_{12}, s\tilde{p}_{13}, t\big)$$

where $(u,s,t) \in \mathbb{P}^2$. Therefore, $Z_k(E')$ is obtained by cutting the Plücker quadric with a plane which, in particular, means that it is a conic. Hence, by Corollary 2.8, E' is special and by the Lemma 2.14 there exists a nondegenerate $(k+1) \times (k+1)$–Hankel matrix K such that $I_k(x_2, x_3) \cdot K = H(0, 0, x_2, x_3)$. Define the morphism

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & Sp(W) \\ t & \longmapsto & S_t \end{array}$$

where S_t is represented by the matrix $S_t = \begin{bmatrix} Id & tK \\ 0 & Id \end{bmatrix}$. We have

$$A' \cdot S_t \;=\; \left(\begin{array}{cccc|c} x_0 & x_1 & & & \\ & \ddots & \ddots & & H(0,0,tx_0+x_2,tx_1+x_3) \\ & & x_0 & x_1 & \end{array} \right).$$

Hence

$$A \cdot S_t \;=\; \left(\begin{array}{cccc|c} x_0 & x_1 & & & \\ & \ddots & \ddots & & H(x_0,x_1,tx_0+x_2,tx_1+x_3) \\ & & x_0 & x_1 & \end{array} \right)$$

and $\mathbb{C} \subseteq Sym^0(E)$ as we wanted. □

Remark 5.2. With the notations of the above proof, there is a subgroup λ isomorphic to $\mathbb{C}^*$ given by $x_0 \mapsto tx_0$, $x_1 \mapsto tx_1$, $x_2 \mapsto t^{-1}x_2$, $x_3 \mapsto t^{-1}x_3$ such that, by Corollary 4.8, $\lim_{t\to 0} \lambda(t) \cdot E$ is still a k–instanton bundle with four–dimensional symmetry group as in (4.3).

Theorem 5.3. *Let E be a k–instanton bundle on $\mathbb{P}^3$ such that $Z_k(E)$ contains two distinct points. Then there exists $\mathbb{C}^* \subseteq Sym^0(E)$, where $\mathbb{C}^*$ has a fine action.*

Proof. By assumption and [Ski97], there is monad representing E with

$$A \;=\; \left(\begin{array}{cccc|c} x_0 & x_1 & & & \\ & \ddots & \ddots & & H(x_0,x_1,x_2,x_3) \\ & & x_0 & x_1 & \end{array} \right)$$

and the line $L_1 = \{x_0 = x_1 = 0\} \in Z_k(E)$. Since $Z_k(E)$ contains two distinct points, we can assume that $L_2 = \{x_2 = x_3 = 0\} \in Z_k(E)$. Moreover, from the fact that

$h^0(E|L_2) = k+1$ we deduce that there exists a nondegenerate $(k+1)\times(k+1)$–Hankel matrix M such that $H(x_0, x_1, 0, 0) = I_k(x_0, x_1) \cdot M$. Defining $S = \begin{bmatrix} Id & -M \\ 0 & Id \end{bmatrix}$ we get

$$A \cdot S \;=\; [I_k(x_0, x_1)| - M \cdot I_k(x_0, x_1) + H(x_0, x_1, x_2, x_3)] \;=\; \left[I_k(x_0, x_1)|\widetilde{H}(x_2, x_3)\right]$$

where $\widetilde{H}(x_2, x_3)$ is a matrix of linear forms only in x_2, x_3. Therefore, there exists $\mathbb{C}^*$ acting on U with weights $-1, 0$ and on U' with weights $0, 0$ such that $\mathbb{C}^* \subset Sym(E)$ having a fine action, which proves what we want. □

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