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Some Applications of Beilinson's Theorem to Projective Spaces and Quadrics

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Abstract. In this paper we apply the Beilinson theorem [Functional Anal. Appl. **12** (1978), 214–216] to the following problems. (1) We give sufficient cohomological conditions in order that a coherent sheaf on $\mathbb{P}^n$ or on the quadric contains as direct summand a generator of the derived category (i.e. the line bundles, the bundles of p-forms on $\mathbb{P}^n$, the spinor bundles and the bundles ψ_i introduced by Kapranov [Inv. Math. **92** (1988), 479–508]. (2) We characterize the indecomposable sheaves of order one (with respect to H^1 and H^2) on $\mathbb{P}^3$ and we show that also the diameter is one. (3) We give a new proof of the key theorem which Chang uses to characterize the arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$.

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Introduction

Beilinson's theorem was stated in 1978 [B] and has been extensively used in many works on vector bundles and other topics. The precise statement needs the language of derived categories introduced by Grothendieck [H] and it describes quite explicitly the derived category of coherent sheaves on $\mathbb{P}^n$.

The theorem is often used in a weaker form: from the cohomology of the sheaf one can construct a spectral sequence which abuts to a filtration of the sheaf itself.

The aim of this paper is to link the abstract and general context of the theorem to concrete examples and applications. Our feeling is that Beilinson's theorem plays a fundamental role in the study of sheaves on varieties. We apply Beilinson's theorem to investigate how certain vanishing of the cohomology imply conditions on the sheaves themselves and we give also new proofs of known results in this area that appear now in a common setting.

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Given a coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$, we will give in section 2 sufficient conditions involving only a finite number of suitably chosen cohomology groups in order that the sheaf $\mathscr{F}$ contains as direct summand the trivial line bundle $\mathcal{O}$ or the sheaf of p-forms Ω^p. In the case of the bundle $\mathcal{O}$ our conditions are essentially the same as in [Ei]. When $\mathscr{F}$ is a sheaf on the smooth quadric hypersurface Q_n, we will give in the sections 5 and 6 similar conditions in order that the sheaf $\mathscr{F}$ contains as direct summand the spinor bundles [O2] or the bundles ψ_i introduced by Kapranov [K].

As corollaries we find the Horrocks criterion for a coherent sheaf on $\mathbb{P}^n$ to split as a direct sum of line bundles and its generalization to quadrics given by Buchweitz, Greuel, Knörrer and Schreyer [BGS], [Kn]. On the quadrics an essential tool will be the exact resolution of the diagonal in $Q_n \times Q_n$ found by Kapranov.

In a similar way we will prove some characterizations of bundles Ω^p on $\mathbb{P}^n$ (see [Ho2]) and of the bundles ψ_i on Q_n, recovering in section 3 (Corollary 3.3) a result of Chang [C1], [C2] which is the key point in Chang's proof that arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$ are geometrically Buchsbaum (i.e. have an Ω-resolution [C2]).

Another concept we are interested in is that of order of a sheaf studied by Ellia on $\mathbb{P}^n$ [El]. We study in the section 4 the sheaves $\mathscr{F}$ of order one on $\mathbb{P}^3$, and we prove that for the indecomposable one also the diameter is one, that is there is only one $h^i(\mathscr{F}(t)) \neq 0$ for $i = 1, 2$. We generalize to arbitrary coherent sheaves the description of order one and rank two vector bundles given by Ellia [E1].

We work over the field of complex numbers. A variety is a smooth variety.

1. Preliminaries

Let us state the two forms of Beilinson's theorem that we will use. We will apply the strong form, that is a more handy version of the original theorem in [B], only when we will need it. For a detailed proof of the strong form see [AO].

Beilinson's theorem (weak form) [OSS]. *Let $\mathscr{F}$ be a coherent sheaf over $\mathbb{P}^n$. There is a spectral sequence $E_r^{p,q}$ with E_1-term $E_1^{p,q} = H^q(\mathbb{P}^n, \mathscr{F}(p)) \otimes \Omega^{-p}(-p)$ such that $E_\infty^{p,q} = 0$ for $p + q \neq 0$ and $\bigoplus_{p=0}^{n} E_\infty^{-p,p}$ is the associated* ***graded*** *sheaf of a filtration of $\mathscr{F}$.*

Beilinson's theorem (strong form) [B]. *Let $X = \mathbb{P}^n$, denote by $p, q : X \times X \to X$ the two projections and by Δ the diagonal in $X \times X$. For $\mathscr{F}, \mathscr{G} \in \mathrm{Coh}(X)$ let us put $\mathscr{F} \boxtimes \mathscr{G} := p^*\mathscr{F} \otimes q^*\mathscr{G}$. Let $\mathscr{F} \in \mathrm{Coh}(X)$, $t \in \mathbb{Z}$. Then*

A) *The diagonal Δ has the following resolution on $X \times X$*

$$(1.1) \qquad 0 \to \Omega^n(n) \boxtimes \mathcal{O}(-n) \xrightarrow{u_n} \dots \to \Omega^1(1) \boxtimes \mathcal{O}(-1) \xrightarrow{u_1} q^*\mathcal{O} \xrightarrow{u_0} q^*\mathcal{O}|_\Delta \to 0$$

B) *There exists a complex of vector bundles $L^{\cdot}(t)$ on X such that:*

1) $L^{\cdot}(t) \sim \mathscr{F}(t)$ *in the derived category* $\mathrm{D^b}(\mathrm{Coh}(X))$.

   In particular: $H^k(L^{\cdot}(t)) = \begin{cases} \mathscr{F}(t) & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$

2) $L^k(t) = \bigoplus_{j+k=i} X_j^i(t),\ X_j^i(t) = \Omega^j(j)^{\oplus h^i(\mathscr{F}(t-j))}$

3) *the maps* $v_j^i(t,s): X_j^i(t) \to X_{j-s}^{i-s+1}(t)$ $(s \in \mathbb{Z})$ *induced by the differentials* $L^k \to L^{k+1}$ *are zero for* $s \leq 0$

4) *the maps* $v_j^i(t,1)$ *agree with the natural maps*
   $R^i p_*(\Omega^j(j) \boxtimes \mathscr{F}(t-j)) \to R^i p_*(\Omega^{j-1}(j-1) \boxtimes \mathscr{F}(t-j+1))$
   coming from the exact sequence (which is (1.1) tensored by $q^*\mathscr{F}(t)$)

$$0 \to \Omega^n(n) \boxtimes \mathscr{F}(t-n) \xrightarrow{u_n} \dots \to \Omega^1(1) \boxtimes \mathscr{F}(t-1) \xrightarrow{u_1} q^*\mathscr{F}(t) \xrightarrow{u_0} q^*\mathscr{F}(t)|_{\Delta} \to 0$$

Definition of order of a sheaf (this definition is slightly different from other ones in the literature; for example [Ba]).

(i) on $\mathbb{P}^n$ [El]: Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$. We set $\mathrm{o}(j)(\mathscr{F}) = \inf\{k \geq 0 \mid \mathscr{M}^k \cdot \oplus_t H^j(\mathscr{F}(t)) = 0\}$ where $\mathscr{M}$ is the maximal ideal $(x_0, \dots, x_n)$ in $\mathbb{C}[x_0, \dots, x_n]$ (including the case $\mathrm{o}(j) = +\infty$). In other words $\mathrm{o}(j)(\mathscr{F}) \leq k$ means that the morphisms $H^j(\mathscr{F}(t)) \to H^j(\mathscr{F}(t+k))$ given by multiplication by any homogeneous polynomial of degree k are zero. The *order of* $\mathscr{F}$ is $\mathrm{o}(\mathscr{F}) = \max_{1 \leq j \leq n-1} \mathrm{o}(j)(\mathscr{F})$.

(ii) on a projective variety X: Let $\mathscr{F}$ be a coherent sheaf on X. Let L be an ample line bundle. We set $\mathrm{o}(j)(\mathscr{F}) = \inf\{m \geq 0 \mid \forall k \geq m\ \forall s \in H^0(L^k)\ \forall q \in \mathbb{Z}$ the natural map $H^j(\mathscr{F} \otimes L^q) \xrightarrow{\otimes s} H^j(\mathscr{F} \otimes L^{q+k})$ is zero$\}$ (including the case $\mathrm{o}(j) = +\infty$). The *order of* $\mathscr{F}$ is $\mathrm{o}(\mathscr{F}) = \max_{1 \leq j \leq \dim X - 1} \mathrm{o}(j)(\mathscr{F})$. We note that the order depends on the bundle L chosen.

When $X = \mathbb{P}^n$ and $L = \mathscr{O}(1)$ the two definitions agree because $H^0(\mathscr{O}(m)) = S^m H^0(\mathscr{O}(1))$.

In general if L is very ample and X embedded by L is projectively normal, then $\mathrm{o}(j)(\mathscr{F}) = \inf\{m \mid \forall s \in H^0(L^m)\ \ \forall q \in \mathbb{Z}$ the natural map $H^j(\mathscr{F} \otimes L^q) \xrightarrow{\otimes s} H^j(\mathscr{F} \otimes L^{q+m})$ is zero$\}$. This simpler definition works on quadrics.

Definition of diameter. Let $\mathscr{F}$ be a coherent sheaf on the projective variety X. Let L be an ample line bundle. If $h^j(\mathscr{F} \otimes L^t) = 0\ \forall t$ we set $\mathrm{d}(j)(\mathscr{F}) = 0$. Otherwise we set $\mathrm{d}(j)(\mathscr{F}) = \sup\{t \in \mathbb{Z} \mid h^j(\mathscr{F} \otimes L^t) \neq 0\} - \inf\{t \in \mathbb{Z} \mid h^j(\mathscr{F} \otimes L^t) \neq 0\} + 1$ (including the case $d(j) = +\infty$). The *diameter of* $\mathscr{F}$ is $d(\mathscr{F}) = \max_{1 \leq j \leq \dim X - 1} \mathrm{d}(j)(\mathscr{F})$. Obviously $\mathrm{d}(j)(\mathscr{F}) \geq \mathrm{o}(j)(\mathscr{F})$ and $\mathrm{d}(\mathscr{F}) \geq \mathrm{o}(\mathscr{F})$.

Definition of g.skyscraper sheaf. We call a sheaf $\mathscr{F}$ with dim supp $\mathscr{F} = 0$ a *generalized (g.) skyscraper sheaf.*

Lemma 1.1. *Let $\mathscr{F}$ be a torsion-free sheaf on the variety X. Let $\mathcal{O}(1)$ be an ample line bundle on X. Then $H^0(\mathscr{F}(t)) = 0$ for $t \ll 0$.*

Proof. We may suppose $h^0(\mathscr{F}(t))$ to be constant for $t \ll 0$. Let H be a general hyperplane, from the sequence ([S] pag. 277) $0 \to H^0(\mathscr{F}(t-1)) \to H^0(\mathscr{F}(t)) \to H^0(\mathscr{F}(t)_H)$ we obtain that any section of $\mathscr{F}(t)$ is generically zero. As $\mathscr{F}$ is torsion-free we have locally the embedding $\mathscr{F}_U \to \mathcal{O}_U^{\oplus n}$, so that by the identity principle any section of $\mathscr{F}(t)$ is zero everywhere.

Lemma 1.2. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$ of order 1. Let p,q be the natural projections of $\mathbb{P}^n \times \mathbb{P}^n$ onto $\mathbb{P}^n$. Then each arrow of the exact Koszul complex (1.1):*

$$(1.1) \qquad 0 \to \Omega^n(n) \boxtimes \mathcal{O}(-n) \to \Omega^{n-1}(n-1) \boxtimes \mathcal{O}(-n+1) \to \cdots \\ \cdots \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

induces the zero morphism

$$\begin{array}{ccc} R_j p_*(\Omega^i(i) \boxtimes \mathscr{F}(-i)) & \to & R_j p_*(\Omega^{i-1}(i-1) \boxtimes \mathscr{F}(-i+1)) \\ \| & & \| \\ \Omega^i(i)^{h^i(\mathscr{F}(-i))} & & \Omega^{i-1}(i-1)^{h^j(\mathscr{F}(-i+1))} \end{array} .$$

Proof. Observe that the morphism

$$\begin{array}{ccc} \Omega^i(i) \boxtimes \mathscr{F}(-i) & \xrightarrow{u_i} & \Omega^{i-1}(i-1) \boxtimes \mathscr{F}(-i+1) \\ \| & & \| \\ \wedge^i[\Omega^1(1) \boxtimes \mathcal{O}(-1)] \otimes q^*\mathscr{F} & & \wedge^{i-1}[\Omega^1(1) \boxtimes \mathcal{O}(-1)] \otimes q^*\mathscr{F} \end{array}$$

is the tensor product of the identity on $q^*\mathscr{F}$ and of the contraction map for the section $\sum x_i^* \boxtimes x_i \in H^0(T\mathbb{P}^n(-1) \boxtimes \mathcal{O}(1))$ with x_i basis of $H^0(\mathbb{P}^n, \mathcal{O}(1)) \simeq H^0(\mathbb{P}^n, T\mathbb{P}^n(-1))^*$.

Then $R_j p_* u_i : \Omega^i(i) \otimes_{\mathbb{C}} H^j(\mathscr{F}(-i)) \to \Omega^{i-1}(i-1) \otimes_{\mathbb{C}} H^j(\mathscr{F}(-i+1))$ is the contraction map by the element $\sum x_i^* \otimes x_i$ and then it is zero because $\mathscr{F}$ has order 1.

For further reference we state the following easy lemma.

Lemma 1.3. *Let A_j $(0 \le j \le n)$, B_k $(0 \le k \le m)$ be bundles on the variety X such that $\mathrm{Ext}^i(A_j, B_k) = 0$ for $i > 0, \forall j,k$. If there are morphisms such that the following sequences are exact*

$$0 \to K \to A_n \to A_{n-1} \to \ldots \to A_2 \to A_1 \to A_0 \to 0$$
$$0 \to B_n \to B_{n-1} \to \ldots \to B_2 \to B_1 \to B_0 \to C \to 0$$

then

(i) *K is locally free*

(ii) $\mathrm{Ext}^i(K, C) = H^i(K^* \otimes C) = 0$ *for* $i \ge 0$.

Proof. (i) follows cutting into short exact sequences. Dualizing the first sequence and tensoring by B_k we get $H^i(K^* \otimes B_k) = 0$ for $i > 0, \forall k$. Tensoring the second sequence by K^* we get (ii).

Lemma 1.4. *Let $\phi : \mathcal{O} \to \Omega^p_{\mathbb{P}^n}(p+1)$ be a section, for any hyperplane H consider the restriction $(\phi^1_H, \phi^2_H) : \mathcal{O}_H \to \Omega^p_H(p+1) \oplus \Omega^{p-1}_H(p)$. If $\phi^1_H \equiv 0$ or $\phi^2_H \equiv 0$ for the generic H then $\phi \equiv 0$.*

Proof. Consider that $H^0(\mathbb{P}^n, \Omega^p(p+1)) = \wedge^{p+1} V$ where $V = H^0(\mathbb{P}^n, \mathcal{O}(1))$. There exists a basis $v_0, \ldots, v_n$ of V such that $\phi^1_H \equiv 0$ for $H = H_i$ zero loci of the section v_i $(i = 0, \ldots, n)$. We can write the section ϕ as a linear combination $\phi = \sum a_{i_1,\ldots,i_{p+1}} v_{i_1} \wedge \ldots \wedge v_{i_{p+1}}$. As $\phi^1_{H_i} \equiv 0$ we have that $a_{i_1,\ldots,i_{p+1}} = 0$ when $i \notin \{i_1, \ldots, i_{p+1}\}$. Then all $a_{i_1,\ldots,i_{p+1}}$ are zero and $\phi \equiv 0$. If $\phi^2_H \equiv 0$ the argument is similar ($\phi^2_{H_i} \equiv 0$ means that $a_{i_1,\ldots,i_{p+1}} = 0$ when $i \in \{i_1, \ldots, i_{p+1}\}$).

Lemma 1.5. *Let $\phi : \Omega^r(r)^{\oplus a} \to \Omega^s(s)^{\oplus b}$ with $s < r$ and $s > 0$ be a morphism, for any hyperplane H consider the restriction*

$$(\phi^1_H, \phi^2_H) : \Omega^r_H(r)^{\oplus a} \oplus \Omega^{r-1}_H(r-1)^{\oplus a} \to \Omega^s_H(s)^{\oplus b} \oplus \Omega^{s-1}_H(s-1)^{\oplus b}.$$

(i) *If $\phi^2_H \equiv 0$ or $\phi^1_H \equiv 0$ for the generic H then $\phi \equiv 0$;*
(ii) *If $\phi_H|_{\Omega^r_H(r)^{\oplus a}} \equiv 0$ or $\phi_H|_{\Omega^{r-1}_H(r-1)^{\oplus a}} \equiv 0$ for the generic H then $\phi \equiv 0$;*

Proof. After twisting we consider $\phi : \Omega^r(r+1)^{\oplus a} \to \Omega^s(s+1)^{\oplus b}$. As $\Omega^r(r+1)^{\oplus a}$ is globally generated it suffices to prove that for every $\psi : \mathcal{O} \to \Omega^r(r+1)^{\oplus a}$ we have that $\phi \circ \psi : \mathcal{O} \to \Omega^s(s+1)^{\oplus b}$ is zero. Then (i) follows from Lemma 1.3. (ii) is dual to (i).

The following proposition is probably well known. In the case $\mathcal{F}$ torsion-free see Lemma 1.1 in [HH].

Proposition 1.6. *Let $\mathcal{F}$ be a coherent sheaf on a projective variety X. The following conditions are equivalent:*

(i) *$\mathcal{F}$ is a direct sum of a vector bundle and a g.skyscraper sheaf*
(ii) *$H^i(\mathcal{F} \otimes L^t) = 0$ for $t \ll 0$ $0 < i < \dim X$, $\forall L$ ample line bundle (i.e. $\mathrm{d}(\mathcal{F}) < +\infty$ w.r.t. to all L)*
(iii) *$H^i(\mathcal{F} \otimes L^t) = 0$ for $t \ll 0$ $0 < i < \dim X$, for a fixed L ample line bundle (i.e. $\mathrm{d}(\mathcal{F}) < +\infty$ w.r.t. to a fixed L)*
(iv) *$\mathcal{F}$ has finite order with respect to each ample line bundle L*
(v) *$\mathcal{F}$ has finite order with respect to a fixed ample line bundle L.*

Proof. (i) $\Rightarrow$ (ii) by Serre duality and Theorem B. (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is trivial. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) is trivial. It remains to show (v) $\Rightarrow$ (i). Let $\mathrm{d} = \dim X$, $\mathrm{o}(\mathcal{F}) = k$, ω be the canonical bundle of X and set $\mathcal{O}(1) = L$. By Serre duality we obtain that the natural arrows $\mathrm{Ext}^j(\mathcal{F}(-n), \omega) \to \mathrm{Ext}^j(\mathcal{F}(-n-k), \omega)$ are equal to zero for $0 < j < \dim X$, $\forall n$ and that $\dim(\mathrm{Ext}^d(\mathcal{F}(-n), \omega))$ is a constant for $n \gg 0$. For $n \gg 0$

we have $\mathrm{Ext}^j(\mathcal{F}(-n),\omega) = H^0(\mathcal{E}xt^j(\mathcal{F},\omega)(n))$. This implies that $\mathcal{E}xt^j(\mathcal{F},\omega)=0$ for $0<j<\dim X$ and that $\chi[\mathcal{E}xt^d(\mathcal{F},\omega)(n)]$ is a constant so that $\dim \mathrm{Supp}\,\mathcal{E}xt^d(\mathcal{F},\omega)=0$ ([S] p. 276). We call $V=\mathrm{Supp}\,\mathcal{E}xt^d(\mathcal{F},\omega)$. Then for $x\notin V$ we have $(\mathcal{E}xt^j(\mathcal{F},\omega))_x=0$ for $j>0$ so that $\mathcal{F}$ is locally free at x. This implies that $\mathrm{Supp}\,\mathrm{T}(\mathcal{F})\subset V$ has dimension zero. Then from the sequence

$$0 \to \mathrm{T}(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}/\mathrm{T}(\mathcal{F}) \to 0$$

we obtain that also $\mathcal{F}/\mathrm{T}(\mathcal{F})$ has finite order and then by Lemma 1.1 and the previous argument $\mathcal{F}/\mathrm{T}(\mathcal{F})$ is locally free. At last

$$\mathrm{Ext}^1(\mathcal{F}/\mathrm{T}(\mathcal{F}),\mathrm{T}(\mathcal{F})) = H^1(\mathrm{T}(\mathcal{F})\otimes[\mathcal{F}/\mathrm{T}(\mathcal{F})]^*)=0$$

so that $\mathcal{F}=\mathrm{T}(\mathcal{F})\oplus[\mathcal{F}/\mathrm{T}(\mathcal{F})]$ as we wanted.

2. Sheaves on $\mathbb{P}^n$

Theorem 2.1 (see [Ei]). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Suppose that for some $t\in\mathbb{Z}$ $h^i(\mathcal{F}(t-i-1))=0$ for $0\le i<n$. Then $\mathcal{F}$ contains $\mathcal{O}(-t)^{h^0(\mathcal{F}(t))}$ as direct summand.*

Proof. We may suppose $t=0$. We write the Beilinson theorem in its strong form for the sheaf $\mathcal{F}$. $\mathcal{O}^{h^0(\mathcal{F})}$ is a direct summand of $X_0^0(0)$. No nonzero map $v_0^0(0,s)$ starts at $\mathcal{O}^{h^0(\mathcal{F})}$ and all the maps $v_i^{i-1}(0,i)$ for $0\le i<n$ which end at $\mathcal{O}^{h^0(\mathcal{F})}$ are zero by the hypothesis. Then $\mathcal{O}^{h^0(\mathcal{F})}$ is a direct summand also of $H^0(L^\cdot(0))=\mathcal{F}$, as we wanted.

Theorem 2.2. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ such that $h^i(\mathcal{F}(*))=0$ for $0<i<n$. Then $\mathcal{F}$ is a direct sum of line bundles and of a g.skyscraper sheaf.*

Proof. We may suppose that $\mathcal{F}$ is indecomposable, so that it will suffice to prove that $\mathcal{F}$ is a line bundle or is a g.skyscraper sheaf. If $\mathcal{F}$ is torsion-free by Lemma 1.1 exists an integer m such that $h^0(\mathcal{F}(m-1))=0$, $h^0(\mathcal{F}(m))\neq 0$. Then in this case the result follows by Theorem 2.1. If otherwise $\mathcal{F}$ has torsion we write $\forall t$ the E_1 step of Beilinson sequence which abuts to $\mathcal{F}(t)$:

$$\begin{array}{ccccccccc}
\Omega^n(n)^{h^n(\mathcal{F}(t-n))} & \to & \Omega^{n-1}(n)^{h^n(\mathcal{F}(t-n+1))} & \to & \dots & \to & \Omega^1(1)^{h^n(\mathcal{F}(t-1))} & \to & \mathcal{O}^{h^n(\mathcal{F}(t))} \\
0 & & 0 & & & & 0 & & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
0 & & 0 & & & & 0 & & 0 \\
\Omega^n(n)^{h^0(\mathcal{F}(t-n))} & \to & \Omega^{n-1}(n)^{h^0(\mathcal{F}(t-n+1))} & \to & \dots & \to & \Omega^1(1)^{h^0(\mathcal{F}(t-1))} & \to & \mathcal{O}^{h^0(\mathcal{F}(t))}
\end{array}$$

The first row is always exact with at most one exception at $\Omega^n(n)^{h^n(\mathcal{F}(t-n))}$, and we denote the kernel of the first row by K_t. In the same way we denote the cokernel of the last row at $\mathcal{O}^{h^0(\mathcal{F}(t))}$ by C_t. By Lemma 1.3 K_t is a direct summand of $\mathcal{F}(t)$. As $\mathcal{F}(t)$ has torsion and it is indecomposable it follows $K_t=0$. Then all the first row is exact and looking at its cohomology it is easy to see that all the row must be zero (e.g. $0=h^0(\mathcal{O}^{h^n(\mathcal{F}(t))})=h^n(\mathcal{F}(t))$, $0=h^1(\Omega^{1\,h^n(\mathcal{F}(t-1))})=h^n(\mathcal{F}(t-1))$ and so on).

Then $h^n(\mathscr{F}(t)) = 0\ \forall t$, so that $\chi(\mathscr{F}(t)) = h^0(\mathscr{F}(t))$ must be a constant polynomial (of degree zero) and by [S] pag. 276 it follows dim Supp $\mathscr{F} = 0$.

Theorem 2.3. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$. Suppose that for some $t \in \mathbb{Z}$ and some $0 < j < n$*

$$H^i(\mathscr{F}(t-i-1)) = 0 \quad for\ j \le i \le n-1$$
$$H^i(\mathscr{F}(t-i+1)) = 0 \quad for\ 1 \le i \le j$$

Set $n_j = h^j(\mathscr{F}(t-j))$. Then $\mathscr{F}$ contains $\Omega^j(j-t)^{n_j}$ as direct summand.

Proof. We may suppose $t = 0$. Write the Beilinson theorem in its strong form for the sheaf $\mathscr{F}$. $\Omega^j(j)^{n_j}$ is a direct summand of $X_0^0(0)$. Exactly as in Theorem 2.1 no nonzero map $v_i^j(0,s)$ starts or ends at $\Omega^j(j)^{n_j}$, so that $\Omega^j(j)^{n_j}$ is a direct summand of $H^0(L^{\cdot}(0)) = \mathscr{F}$, as we wanted.

Corollary 2.4. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$ such that $H^i(\mathscr{F}(*)) = 0$ for $0 < i < n$ with the only exceptions $h^j(\mathscr{F}(-j)) = n_j$ for $0 < j < n$. Then $\mathscr{F} \simeq (\bigoplus_{j=1}^{n-1} \Omega^j(j)^{n_j}) \oplus$ {line bundles} $\oplus$ {a g.skyscraper sheaf}.*

3. On a theorem of M.C. Chang

M.C. Chang proves in [C2] a characterization of arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$. The main step of the proof is a cohomological characterization of bundles Ω^p ([C2] theor. 1.1) which is the following corollary 3.3. Here we generalize this characterization and we point out some sufficient condition in order that a bundle contains Ω^p as direct summand. The theorem of Chang is then extended to arbitrary coherent sheaves.

Lemma 3.1. *Let $\mathscr{F}$ be a bundle on $\mathbb{P}^n$ and let $L^{\cdot}$ be a complex with $L^s = \oplus_{m+s=t} \Omega^m(m)^{a_{t,m}}$ which is isomorphic in the derived category to the complex $0 \to \mathscr{F} \to 0$ (so that $\mathscr{F}$ is the cohomology of $L^{\cdot}$). Let $k = \min\{t > 0 \mid a_{t,t} \neq 0\}$. Suppose that the map $\mathcal{O}^{a_{0,0}} \to L^1$ is zero and that $\Omega^k(k)^q$ is a direct summand of $\mathscr{F}$. Then we can decompose $\Omega^k(k)^{a_{k,k}} = \Omega^k(k)^q \oplus \Omega^k(k)^{a_{k,k}-q}$ in such a way that $\Omega^k(k)^q \to L^1$ is zero and $L^{-1} \to \Omega^k(k)^q$ is zero, i.e. $\Omega^k(k)^q$ is a direct summand also in $L^{\cdot}$.*

Proof. By the hypothesis we have an injection $j: \Omega^k(k)^q \to \mathscr{F}$ and then a morphism in the derived category $\Omega^k(k)^q \dashrightarrow L^{\cdot}$. As $\mathrm{Ext}^i(\Omega^p(p), \Omega^q(q)) = 0$ for $i > 0$ [B] then we have a morphism ([K], lemma 1.6) ϕ

$$\begin{array}{lccccc} & 0 & \to & \Omega^k(k)^q & \to & 0 \\ (3.1) & \downarrow & & \downarrow \phi & & \downarrow \\ & L^{-1} & \xrightarrow{\alpha} & L^0 & \xrightarrow{\beta} & L^1 \end{array}$$

which induces j in cohomology. As $\mathrm{Hom}(\Omega^k(k)^q, \Omega^p(p)) = 0$ for $p > k$ [B] we have $\phi = (\phi_1, \phi_2) : \Omega^k(k)^q \to \Omega^k(k)^{a_{k,k}} \oplus \mathcal{O}^{a_{0,0}}$. As the diagram (3.1) commutes we have $\mathrm{Im}\,\phi_1 \subset \mathrm{Ker}\,\beta$. Consider now that $j = (\phi_1, \phi_2) \bmod \mathrm{Im}\,\alpha$.

Let $p : \mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha \to \Omega^k(k)^q$ the natural projection, so that we have $p \circ j = \mathrm{id}_{\Omega^k(k)_q}$. Let $\pi : \mathrm{Ker}\,\beta \to \mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha$ the canonical projection. From the hypothesis we can write $\mathrm{Ker}\,\beta = B \oplus \mathcal{O}^{a_{0,0}}$ and as $\mathrm{Hom}(\mathcal{O}, \Omega^k(k)) = 0$ we have that $p \circ \pi$ is a function ϱ of B and $\varrho \circ \phi_1 = \mathrm{Id}_{\Omega^k(k)^q}$, that implies that $\phi_1 : \Omega^k(k)^q \to \Omega^k(k)^{a_{k,k}}$ is injective, so that $\mathrm{Im}\,\phi_1$ is a direct summand of $\mathrm{Ker}\,\beta$ and we can write $\mathrm{Ker}\,\beta = B_1 \oplus \Omega^k(k)^q \oplus \mathcal{O}^{a_{0,0}}$. It remains only to prove that $\Omega^k(k)^q \cap \mathrm{Im}\,\alpha = 0$ but this is obvious because π is an isomorphism on the direct summand $\Omega^k(k)^q$.

Theorem 3.2. *Let E be a vector bundle on $\mathbb{P}^n$ and i be an integer with $1 \le i \le n-2$ such that*

(i) *$H^p(E(-j)) = 0$ for $i+1 \le p \le n-1$ and $0 \le j \le n$;*
(ii) *the maps $H^q(E|_M(-t)) \to H^q(E|_M(-t+1))$ given by multiplication by any hyperplane section are zero for any linear subspace M of codimension $0 \le m \le i-1$, $1 \le q \le i-m$ and $0 \le t \le n-m-1$; then E contains as direct summand $\Omega^q(q)^{\oplus h^q(E(-q))}$ for $1 \le q \le i$.*

Proof. We work by induction on n. We may suppose that $h^q(E(-q)) \neq 0$ for at least one $1 \le q \le i$. Note that for any hyperplane H $\min\{k \ge 2 \mid h^k(E(-k)) \neq 0\} = \min\{k \ge 1 \mid h^k(E|_H(-k)) \neq 0\} + 1$ and that $h^1(E(-1)) \neq 0$ implies $h^1(E|_H(-1)) \neq 0$. Let $q = \min\{k \ge 1 \mid h^k(E(-k)) \neq 0\}$. We write the Beilinson theorem in the strong form for E and we obtain a complex $L^\cdot$ such that $L^\cdot \sim E$ in $D^b(\mathbb{P}^n)$. As E is locally free we obtain $L^\cdot|_H \sim E|_H$ in $\mathrm{D^b}(H)$. We must consider two cases.

Case 1 $q \ge 2$. We have

$$L^0 \simeq \mathcal{O}^{h^0(E)} \oplus \Omega^q(q)^{h^q(E(-q))} \oplus \ldots,\ h^q(E(-q)) = h^{q-1}(E_H(-q+1))$$

and

$$L^0|_H = \mathcal{O}_H^{h^0(E)} \oplus \Omega_H^{q-1}(q-1)^{h^{q-1}(E_H(-q+1))} \oplus \Omega_H^q(q)^{h^q(E(-q)) + h^{q+1}(E(-q-1))}$$

By the induction hypothesis $E|_H$ contains $\Omega_H^{q-1}(q-1)^{h^{q-1}(E_H(-q+1))}$ as direct summand.

We have that $L^0|_H$ contains $\Omega_H^{q-1}(q-1)^{h^{q-1}(E_H(-q+1))}$ as direct summand and by Lemma 3.1 no arrow in $L^\cdot|_H$ starts or ends at this summand. Then consider that the maps $\Omega^q(q)^{h^q(E(-q))} \to \Omega^s(s)^{h^s(E(-s))}$ with $s < q$ restrict to

$$\Omega_H^q(q)^{h^q(E(-q))} \oplus \Omega_H^{q-1}(q-1)^{h^q(E(-q))} \to \Omega_H^s(s)^{h^s(E(-s))} \oplus \Omega_H^{s-1}(s-1)^{h^s(E(-s))}$$

so that by Lemma 1.5 we get also that no nonzero arrow in L^0 starts at $\Omega^q(q)^{h^q(E(-q))}$. Similarly no nonzero arrow ends at $\Omega^q(q)^{h^q(E(-q))}$ so that E contains $\Omega^q(q)^{h^q(E(-q))}$ as direct summand.

Case 2 $q = 1$. In this case $h^1(E(-1)) + h^2(E(-2)) = h^1(E_H(-1))$ and as above we have that $E|_H$ and $L^0|_H$ contain $\Omega_H^1(1)^{h^1(E_H(-1))}$ as direct summand. Exactly as in the case 1 we get that E contains $\Omega^1(1)^{h^1(E(-1))}$ as direct summand.

From Theorem 3.2 it follows the following characterization of Ω^p found by Chang (see [C2] Theor. 1.1 for bundles)

Corollary 3.3. *Let E be a coherent sheaf on $\mathbb{P}^n$ and let i be an integer with $1 \le i \le n-2$ such that*

(i) *$H^p(E(*)) = 0$ for $i+1 \le p \le n-1$;*
(ii) *the maps $H^q(E|_M(t)) \to H^q(E|_M(t+1))$ given by multiplication by any hyperplane section are zero for any linear subspace M of codimension $0 \le m \le i-1$, $1 \le q \le i-m$ and for any t; then E is isomorphic to $\oplus\, \Omega^{p_j}(-k_j)^{\oplus h^{p_j}(E(k_j))} \oplus \{$g.skyscraper sheaf$\}$ where $0 \le p_j \le i$ and $h^{p_j}(E(k_j))$ are the only nonzero cohomologies for $0 < p_j < n$.*

Proof. E has finite order and by Proposition 1.6 we may suppose it is locally free. Then twist E and apply Theorem 3.2.

Note also that the proof of Theorem 3.2 (case 2) gives immediately the following

Theorem 3.4. *Let E be a bundle on $\mathbb{P}^n$ such that*

(i) *the maps $H^1(E(-2)) \to H^1(E(-1))$ given by multiplication by any hyperplane section are zero*
(ii) *$h^1(E(-1)) + h^2(E(-2)) = h^1(E_H(-1))$ (this is satisfied if (i) holds with H^2 in the place of H^1)*
(iii) *for the generic hyperplane section H we have that $E|_H$ contains $\Omega_H^1(1)^{h^1(E(-1))+h^2(E(-2))}$ as direct summand.*
Then E contains $\Omega_{\mathbb{P}^n}^1(1)^{h^1(E(-1))} \oplus \Omega_{\mathbb{P}^n}^2(2)^{h^2(E(-2))}$ as direct summand.

In the same spirit of [C2] we prove the following two results that are weaker than Theorem 3.2 and Corollary 3.3 but are easier to prove and more handy for applications.

Theorem 3.5. *Let E be a coherent sheaf on $\mathbb{P}^n$ such that*
(i) *the maps $H^p(E(t)) \to H^p(E(t+1))$ given by multiplication by any hyperplane section are zero for $1 \le p \le n-1$ and $-n \le t \le 1$*
(ii) *If $H^p(E(k)) \neq 0 \neq H^q(E(h))$ and $1 \le p < q \le n-1$, $-n \le k, h \le 0$ then $(p+k) - (q+h) \neq 1$.*

Then $\oplus\, l_j \Omega^{p_j}(-k_j)$ is a direct summand of E where $h^{p_j}(E(k_j)) = l_j$ are the only nonzero cohomologies for $0 < p_j < n$, $-n \le k_j \le 0$.

Proof. Write the Beilinson theorem in the strong form for $E(p_j + k_j)$. By (i) and the proof of Lemma 1.2 we have that $v_{p_j}^{p_j}(0,1)$ and $v_{p_{j+1}}^{p_j}(0,1)$, which respectively starts and ends in $l_j \Omega^{p_j}(p_j) = X_{p_j}^{p_j}(0)$, are zero. By (ii) all other arrows which start or end in $X_{p_j}^{p_j}(0)$ are zero so that $l_j \Omega^{p_j}(p_j)$ is a direct summand of $H^0(L^{\cdot}(p_j + k_j)) = E(p_j + k_j)$.

Corollary 3.6. *Let E be a coherent sheaf on $\mathbb{P}^n$ of order one such that if $H^p(E(k)) \neq 0 \neq H^q(E(h))$ and $1 \leq p < q \leq n-1$, then $(p+k)-(q+h) \neq 1$.*

Then we have an isomorphism $E \simeq \oplus\, l_j \Omega^{p_j}(-k_j)$ where $h^{p_j}(E(k_j)) = l_j$ are the only nonzero cohomologies for $0 < p_j < n$.

4. Sheaves on $\mathbb{P}^3$

The following theorem is a consequence of Corollary 3.3 but we prefer to give an independent easier proof:

Theorem 4.1 (see [Ho1] for bundles). *Let E be a coherent sheaf on $\mathbb{P}^2$ with $\mathrm{o}(E) \leq 1$. Then E is isomorphic to the direct sum of a g.skyscraper sheaf, of some copies of $\Omega^1(t_k)$ with $t_k \in \mathbb{Z}$ and of some line bundles.*

Proof. If $h^1(E(t-1)) \neq 0$ write the Beilinson theorem (in the strong form) for $E(t)$. Then $L^0(t) = X_0^0(t) \oplus X_1^1(t) \oplus X_2^2(t)$ and by the hypothesis no arrows $v_1^1(t,s)$ different from zero start or end at the summand $X_1^1(t) = \Omega^1(1)^{h^1(E(t-1))}$, so that the cohomology $H^0(L^\cdot(t))$ contains $\Omega^1(1)^{h^1(E(t-1))}$ as direct summand. Then $E = \Omega^1(1-t)^{h^1(E(t-1))} \oplus E'$ and we may repeat the same reasoning with E' in the place of E. Continue in this way until $h^1(E'(*)) = 0$ and apply Theorem 2.2.

Theorem 4.2. *Let E be a coherent sheaf on $\mathbb{P}^3$ indecomposable with order $\mathrm{o}(E) \leq 1$. Then E is a g.skyscraper sheaf or there exists $t_0 \in \mathbb{Z}$ such that $H^1(E(t)) = 0$ for $t \neq t_0$, $H^2(E(t)) = 0$ for $t \neq t_0 - 2$ and we have the exact sequences*

$$0 \to E(t_0) \to \Omega^2(2)^{h^2(E(t_0-2))} \oplus \mathcal{O}(-1)^{h^3(E(t_0-3))} \to \mathcal{O}^{h^1(E(t_0))} \to 0$$

$$0 \to \mathcal{O}^{h^2(E(t_0-2))} \to \mathcal{O}(1)^{h^0(E(t_0+1))} \oplus \Omega^1(2)^{h^1(E(t_0))} \to E(t_0+2) \to 0$$

In particular
- (i) *the diameter of E is less or equal than one;*
- (ii) *if $h^1(E(*)) = 0$ then E is one of the following: a g.skyscraper sheaf, a line bundle, $\Omega^2(t)$ for some $t \in \mathbb{Z}$;*
- (iii) *if $h^2(E(*)) = 0$ then E is one of the following: a g.skyscraper sheaf, a line bundle, $\Omega^1(t)$ for some $t \in \mathbb{Z}$.*

Proof. By Proposition 1.6 we may suppose that E is a vector bundle.

If $h^1(E(*)) = 0$ we may suppose that $h^2(E(-2)) \neq 0$. Then no arrows $v_2^2(0,s)$ different from zero start or end at the summand $X_2^2(0) = \Omega^2(2)^{h^2(E(-2))}$, so that the cohomology $H^0(L^\cdot(0))$ contains $\Omega^2(2)^{h^2(E(-2))}$ as direct summand and we obtain $E = \Omega^2(2)$. In an analogous way we can consider the case $h^2(E(*)) = 0$.

Thus we may suppose $t_0 = \sup\{t \in \mathbb{Z} \mid h^1(E(t)) \neq 0\} \in \mathbb{Z}$. We obtain $h^2(E(t)) = 0$ for $t \geq t_0 - 1$ otherwise always from the Beilinson theorem a twist of E would be isomorphic to Ω^2 and $h^1(E(*)) = 0$ against our assumption. We write the Beilinson theorem (in the strong form) for $E(t_0+3)$. We get the complex

$$0 \to X_3^0(t_0+3) \to X_3^1(t_0+3) \oplus X_2^0(t_0+3) \to X_1^0(t_0+3)$$

$$\xrightarrow{(0, v_1^0(t_0+3,1))} X_3^3(t_0+3) \oplus X_0^0(t_0+3)$$

$$\xrightarrow{v_3^3(t_0+3,1)+0} X_2^3(t_0+3) \to X_1^3(t_0+3) \to X_0^3(t_0+3) \to 0$$

so that $E(t_0+3) = [\mathrm{Ker}(v_3^3(t_0+3,1))] \oplus [X_0^0(t_0+3)/\mathrm{Im}(v_1^0(t_0+3,1))]$.

We want to show that $\mathrm{Ker}(v_3^3(t_0+3,1)) = 0$. If on the contrary $E(t_0+3) = \mathrm{Ker}(v_3^3(t_0+3,1))$ and $X_0^0(t_0+3)/\mathrm{Im}(v_1^0(t_0+3,1)) = 0$ we have the exact sequence

$$0 \to X_3^0(t_0+3) \to X_3^1(t_0+3) \oplus X_2^0(t_0+3) \to$$

$$\to X_1^0(t_0+3) \to X_0^0(t_0+3) \to 0$$

Looking at its cohomology we get first $h^0(E(t_0+3)) = 0$, then $h^0(E(t_0+2)) = h^0(E(t_0+1)) = h^0(E(t_0)) = 0$ and at last $h^1(E(t_0)) = 0$ which contradicts our assumption. Thus $\mathrm{Ker}(v_3^3(t_0+3,1)) = 0$ and we have the exact sequence

$$0 \to X_3^3(t_0+3) \to X_2^3(t_0+3) \to X_1^3(t_0+3) \to X_0^3(t_0+3) \to 0.$$

Looking at its cohomology as before we get $X_3^3(t_0+3) = X_3^2(t_0+3) = X_3^1(t_0+3) = X_3^0(t_0+3) = 0$.

Now we write the Beilinson theorem for $E(t_0+2)$ and in the same way we obtain $X_3^3(t_0+2) = 0$. Again writing the Beilinson theorem for $E(t_0+1)$ we obtain $X_3^3(t_0+1) = 0$. At last we write the Beilinson theorem for $E(t_0)$. We get the complex

$$\to X_3^2(t_0) \oplus X_2^1(t_0) \oplus X_1^0(t_0) \xrightarrow{\alpha}$$

$$\xrightarrow{\alpha} X_3^3(t_0) \oplus X_2^2(t_0) \oplus X_1^1(t_0) \oplus X_0^0(t_0) \xrightarrow{\beta} X_0^1(t_0) \to 0$$

where $E(t_0) = \mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha$.

We have $\beta = v_3^3(t_0,3) + v_2^2(t_0,2) + 0 + 0$ so that $\mathrm{Ker}\,\beta = \mathrm{Ker}[v_3^3(t_0,3) + v_2^2(t_0,2)] \oplus X_1^1(t_0) \oplus X_0^0(t_0)$. We have $\alpha = (0, 0, v_3^2(t_0,2), v_3^2(t_0,3) + v_2^1(t_0,2) + v_1^0(t_0,1))$ so that $\mathrm{Im}\,\alpha = (0,0,*)$ and $\mathrm{Ker}\,\beta/\mathrm{Im}\,\alpha$ contains $\mathrm{Ker}[v_3^3(t_0,3) + v_2^2(t_0,2)]$ as direct summand. The sheaf $\mathrm{Ker}[v_3^3(t_0,3) + v_2^2(t_0,2)]$ cannot be zero, otherwise we have $X_3^3(t_0) \oplus X_2^2(t_0) = X_0^1(t_0) \neq 0$ that means that a sum of $\mathcal{O}(-1)$ and $\Omega^2(2)$ is trivial. Then $E(t_0) = \mathrm{Ker}[v_3^3(t_0,3) + v_2^2(t_0,2)]$ and we have the sequence

$$0 \to E(t_0) \to X_3^3(t_0) \oplus X_2^2(t_0) \to X_0^1(t_0) \to 0$$

that is exactly the first sequence stated in the theorem, as we wanted. From this sequence it follows also $h^1(E(t)) = 0$ for $t < t_0$ and $h^2(E(t)) = 0$ for $t < t_0 - 2$ so that $h^1(E(t_0))$ and $h^2(E(t_0-2))$ are the only possible nonzero intermediate cohomologies. Note also that $h^0(E(t)) = 0$ for $t \le t_0$. Now writing again the Beilinson theorem for $E(t_0+1)$ we get the second sequence stated in the theorem. This concludes the proof.

Corollary 4.3. *Let E be an indecomposable sheaf of order* 1 *on* $\mathbb{P}^3$ *with* $h^1(E) \neq 0$ *or* $h^2(E(-2)) \neq 0$. *We have* $4[h^1(E) - h^2(E(-2))] = h^3(E(-3)) - h^0(E(1))$.

Proof. From Theorem 4.2 we have rank $E = 3h^2(E(-2)) + h^3(E(-3)) - h^1(E) = h^0(E(1)) + 3h^1(E) - h^2(E(-2))$.

Remark 4.4. There are many vector bundles on $\mathbb{P}^3$ of order 1. For example one can consider the extensions $0 \to \Omega^1 \to ? \to N \to 0$ or $0 \to N \to ? \to N \to 0$ where N is a nullcorrelation bundle.

Proposition 4.5 (see [El] for bundles). *Let E be a torsion free sheaf on* $\mathbb{P}^3$ *of order* 1 *and rank* 2. *Then E is isomorphic to a sum of two line bundles or to a twist of a nullcorrelation bundle.*

Proof. From Theorem 4.2 we may suppose that E is a vector bundle so that by Serre duality we have (up to twist E) $h^1(E) = h^2(E(-2))$. Then from the sequence of Theorem 4.2 we obtain $3h^1(E) + h^3(E(-3)) = h^1(E) + 2$ so that $2h^1(E) = 2 - h^3(E(-3))$. The only possibilities are $h^3(E(-3)) = 0$ or 2.

If $h^3(E(-3)) = 0$ then $h^1(E) = 1$ and E is a nullcorrelation bundle.

If $h^3(E(-3)) = 2$ then $h^1(E) = 0$ that means $h^1(E(*)) = h^2(E(*)) = 0$ and the result follows from the Horrocks criterion.

5. The bundles ψ_i on Q_n

Let Q_n be the smooth quadric in $\mathbb{P}^{n+1}$. Kapranov defined in [K] the bundles ψ_i to construct a resolution of the diagonal in $Q_n \times Q_n$, with the aim to give a description of the derived category $D^b(\mathrm{Coh}(Q_n))$. The bundles ψ_i are simple (i.e. $h^0(\psi_i \otimes \psi_i^*) = 1$), then indecomposable and are homogeneous (but reducible, a filtration of $\psi_1 = \Omega^1_{\mathbb{P}^{n+1}}(1)|_{Q_n}$ with irreducible quotients is $0 \subset \mathcal{O}(-1) \subset \psi_1$, $\psi_1/\mathcal{O}(-1) \simeq TQ_n(-1)$). First we will give an elementary definition of the bundles ψ_i and we show as from this definition it is possible to prove many elementary properties of ψ_i. For the reader interested in the original definition via graded Clifford algebras we refer to [K].

From now on we set $\Omega^p = \Omega^p_{\mathbb{P}^{n+1}}$. We want to define inductively ψ_i from the sequence

(5.1) $$0 \to \Omega^i(i)|_{Q_n} \to \psi_i \to \psi_{i-2} \to 0$$

We set $\psi_0 := \mathcal{O}$ $\psi_1 := \Omega^1(1)|_{Q_n}$.

Lemma 5.1. *Let* $n + 1 \geq q \geq p + 2 \geq 2$. *Then*

$$H^i(\mathbb{P}^{n+1}, \Omega^p(p)^* \otimes \Omega^q(q)) = 0 \;\forall i$$

$$H^i(\mathbb{P}^{n+1}, \Omega^p(p)^* \otimes \Omega^q(q-2)) = \begin{cases} 0 \text{ for } i \neq 2 \text{ or } \; q \neq p+2 \\ \mathbb{C} \text{ for } i = 2 \text{ and } q = p+2. \end{cases}$$

Proof. The first part is [B] lemma 2. We shall see as both the statements follow from Bott theorem [Bo]. Let $\Delta = \{\alpha_1, \ldots, \alpha_{n+1}\}$ be a fundamental system of roots for $\mathrm{SL}(n+2)$, $(,)$ be the Killing form and $\lambda_1, \ldots, \lambda_{n+1}$ be the fundamental weights with respect to Δ. $\Omega^1(1)^*$ is a homogeneous bundle on $\mathbb{P}^{n+1} \simeq \mathrm{SL}(n+2)/\mathrm{P}(\alpha_{n+1})$ with maximal weight λ_1. $\mathcal{O}(1)$ has maximal weight λ_{n+1} and we have the isomorphism $\Omega^q(q) \simeq [\Omega^{n-q+1}(n-q+1)]^* \otimes \mathcal{O}(-1)$. By the Littlewood-Richardson rule [LR] the tensor product $\Omega^p(p)^* \otimes \Omega^q(q)$ decomposes as a direct sum where the summands are irreducible with maximal weights (if for example $p \geq n-q+1$) $\lambda_{n-q-j+1} + \lambda_{p+j} - \lambda_{n+1}$ for $j = 0, \ldots, n-q+1$. Summing up $\delta = \sum \lambda_i$ we get singular weights and this proves (i) by Bott theorem.

For $\Omega^p(p)^* \otimes \Omega^q(q-2)$ we have to sum $\delta - 2\lambda_{n+1}$ and we get only one regular weight in the case $q = p+2$, $j = n-q+1$ that is $\lambda_1 + \ldots \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n - 2\lambda_{n+1}$ which is regular of index 2. In fact $(\lambda_1 + \ldots \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n - 2\lambda_{n+1}, \alpha) < 0$ for $\alpha \in \Phi^+$ only for $\alpha = \alpha_{n+1}$, $\alpha_n + \alpha_{n+1}$. Under the action of the Weyl group $\lambda_1 + \ldots \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n - 2\lambda_{n+1}$ is congruent to δ and this proves (ii) by Bott theorem again.

Corollary 5.2. *Let $n+1 \geq q \geq p+2 \geq 2$. Then*

$$H^i(Q_n, \Omega^p(p)^*|_{Q_n} \otimes \Omega^q(q)|_{Q_n}) = \begin{cases} 0 \text{ for } i \neq 1 \text{ or } q \neq p+2 \\ \mathbb{C} \text{ for } i = 1 \text{ and } q = p+2. \end{cases}$$

Proof. We consider the sequence

$$0 \to \Omega^p(p)^* \otimes \Omega^q(q-2) \to \Omega^p(p)^* \otimes \Omega^q(q) \to \Omega^p(p)^*|_{Q_n} \otimes \Omega^q(q)|_{Q_n} \to 0$$

and the previous theorem.

Theorem 5.3 (and definition of ψ_i). *Suppose we have already defined the bundles ψ_i for $i \leq i_0 \leq n-1$ by means of sequence (5.1). Then if $n+1 \geq q \geq i_0+2$ we have*

$$H^s(\Omega^q(q)|_{Q_n} \otimes \psi_{i_0}^*) = \begin{cases} 0 \text{ for } s \neq 1 \text{ or } q \neq i_0+2 \\ \mathbb{C} \text{ for } s = 1 \text{ and } q = i_0+2 \end{cases}$$

and in particular $\mathrm{Ext}^1(\psi_{i_0}, \Omega^{i_0+2}(i_0+2)|_{Q_n}) = \mathbb{C}$ *so that we may define ψ_{i_0+2} as the unique nonsplitting extension in the sequence* $0 \to \Omega^{i_0+2}(i_0+2)|_{Q_n} \to \psi_{i_0+2} \to \psi_{i_0} \to 0$.

As $\Omega^i = 0$ for $i \geq n+2$ the sequence above defines in a natural way ψ_i for all $i \in \mathbb{N}$.

Proof. We use induction, the previous corollary and the sequence

$$0 \to \psi_{i_0-2}^* \to \psi_{i_0}^* \to \Omega^{i_0}(i_0)|_{Q_n}{}^* \to 0$$

From the definition itself, it is easy to check some first properties of the bundles ψ_i.

Theorem 5.4. *Let ψ_i be the bundles on Q_n defined as above. Then*

(i) *for $i \geq n$ we have $\psi_i \simeq \psi_{i+2}$*

(ii) *rank* $\psi_i = \sum_{j=0}^{i} \binom{n}{j}$, *in particular for* $i \geq n$ *rank* $\psi_i = 2^n$

(iii) $h^0(\psi_i) = 0$ *for* $i \geq 1$

$$h^0(\psi_i(1)) = \sum_{j=0}^{i+1} \binom{n+1}{j}$$

(iv) $c_1(\psi_i) = -\sum_{j=0}^{i-1} \binom{n-1}{j}$

(v) ψ_i *are homogeneous*

(vi) *if* r *is any line in* Q_n *then* $\psi_i|_r \simeq \mathcal{O}(-1)^{\sum_{j=0}^{i-1}\binom{n-1}{j}} \oplus \mathcal{O}^{\sum_{j=0}^{i}\binom{n-1}{j}}$.

Proof. (i) is true because $\Omega^i = 0$ for $i \geq n+2$. (ii)–(iv) are proved by induction. In order to prove (v) let f be an automorphism in the connected component of the identity of $\mathrm{Aut}(Q_n)$. We have $f^*\Omega^i(i)|_{Q_n} \simeq \Omega^i(i)|_{Q_n}$ because f extends to an automorphism of $\mathbb{P}^{n+1}$ and $f^*\psi_{i-2} \simeq \psi_{i-2}$ by induction. Then as the extension in the sequence defining ψ_i is unique we must have $f^*\psi_i \simeq \psi_i$ as we wanted. To prove (vi) consider that

$$\Omega^i(i)|_r \simeq \bigwedge^i (\Omega^1(1)|_r) \simeq \bigwedge^i [\mathcal{O}^n \oplus \mathcal{O}(-1)] \simeq \mathcal{O}^{\binom{n}{i}} \oplus \mathcal{O}(-1)^{\binom{n}{i-1}}$$

so that it is easy to prove that $\psi_i|_r$ may have as direct summands only $\mathcal{O}$ and $\mathcal{O}(-1)$. Now (vi) follows from (iv) and (ii).

In order to state the Kapranov theorem we have to recall the definition of the spinor bundles (see [K], [O2]). For n odd, $n = 2m+1$ consider the varieties

$$\mathbb{F}(0,m,Q_n) = \{(p,\mathbb{P}^m) \in \mathbb{F}(0,m,\mathbb{P}^{n+1}) \mid p \in \mathbb{P}^m \subset Q_n\}$$

and

$$\mathrm{Gr}(m,Q_n) = \{\mathbb{P}^m \in \mathrm{Gr}(m,\mathbb{P}^{n+1}) \mid \mathbb{P}^m \subset Q_n\} = S_{m+1}$$

that is called a spinor variety.

We have $\mathrm{Pic}(S_{m+1}) = \mathbb{Z}$, and the generator $\mathcal{O}(1)$ of $\mathrm{Pic}[\mathrm{Gr}(m,\mathbb{P}^{n+1})]$ restricts to twice the generator of $\mathrm{Pic}(S_{m+1})$. We have the natural projections:

$$Q_n \xleftarrow{\alpha} \mathbb{F}(0,m,Q_n) \xrightarrow{\beta} \mathrm{Gr}(m,Q_n)$$

We define $S := [\alpha_*\beta^*\mathcal{O}(1)]^*$ and we call it the spinor bundle on Q_n. For n even we have that $\mathrm{Gr}(\frac{n}{2},Q_n)$ contains two connected components, each one isomorphic to $S_{\frac{n}{2}}$. Then we obtain in the same way two spinor bundles S', S'' on Q_n. In any case the rank of a spinor bundle on Q_n is $2^{[\frac{n-1}{2}]}$.

We shall use the following exact sequences on Q_n ([O2], theor. 2.8):
if $n = 2m+1$ (odd)

(5.1a) $0 \to S \to \mathcal{O}_{Q_n}^{\oplus 2^{m+1}} \to S(1) \to 0$

if $n = 2m$ (even)

(5.1b) $0 \to S' \to \mathcal{O}_{Q_n}^{\oplus 2^m} \to S''(1) \to 0$

(5.1c) $0 \to S'' \to \mathcal{O}_{Q_n}^{\oplus 2^m} \to S'(1) \to 0$

We shall need also the following isomorphisms

$$n = 2m+1 \Rightarrow S^* \simeq S(1)$$
$$n = 4m \Rightarrow S'^* \simeq S'(1) \text{ and } S''^* \simeq S''(1)$$
$$n = 4m+2 \Rightarrow S'^* \simeq S''(1) \text{ and } S''^* \simeq S'(1)$$

Theorem 5.5 (Kapranov) [K]. *On $Q_n \times Q_n$ we have the following resolution of the diagonal Δ*

$$0 \to K \to \psi_{n-1} \boxtimes \mathcal{O}(1-n) \to \ldots \to \psi_1 \boxtimes \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_\Delta \to 0$$

where K is isomorphic

$$\text{to } \begin{cases} S \boxtimes S^*(-n) & \text{for } n \text{ odd} \\ [S' \boxtimes S'^*(-n)] \oplus [S'' \boxtimes S''^*(-n)] & \text{for } n \text{ even (see [O2]).} \end{cases}$$

Let $\mathcal{F}$ be a coherent sheaf on Q_n. The resolution gives a spectral sequence $E_r^{p,q}$ with E_1-term $E_1^{p,q} = H^q(Q_n, \mathcal{F}(p)) \otimes \psi_{-p}$ for $p > -n$

and

$$E_1^{-n,q} = \begin{cases} H^q(\mathcal{F} \otimes S^*(-n)) \otimes S & \text{for } n \text{ odd} \\ [H^q(\mathcal{F} \otimes S'^*(-n)) \otimes S'] \oplus [H^q(\mathcal{F} \otimes S''^*(-n)) \otimes S''] & \text{for } n \text{ even} \end{cases}$$

such that $E_\infty^{p,q} = 0$ for $p + q \neq 0$ and $\bigoplus_{p=0}^{n} E_\infty^{-p,p}$ is the associated graded sheaf of a filtration of $\mathcal{F}$.

Furthermore, given any $\mathcal{F}, \mathcal{G} \in \{$spinor bundles$, \psi_{n-2}, \ldots, \psi_1, \psi_0\}$ we have $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ *for $i > 0$.*

It is easy to check, using the spectral sequence above for $\mathcal{F} = \Omega^i_{\mathbb{P}^{n+1}}(i)|_{Q_n}$ that the bundles ψ_i defined by Kapranov in [K] fit together into the exact sequences

$$0 \to \Omega^i_{\mathbb{P}^{n+1}}(i)|_{Q_n} \to \psi_i \to \psi_{i-2} \to 0$$

and then coincide with the ψ_i defined by Theorem 5.3.

Kapranov proves also the following form that will be used only in Theorems 6.3 and 6.7.

5.6 Kapranov theorem (strong form) [K]. *Let $X = Q_n$, denote by $p, q : X \times X \to X$ the two projections and by Δ the diagonal in $X \times X$. For $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$ let us put $\mathcal{F} \boxtimes \mathcal{G} := p^*\mathcal{F} \otimes q^*\mathcal{G}$. Let $\mathcal{F} \in \mathrm{Coh}(X)$, $t \in \mathbb{Z}$. Then there exists a complex of vector bundles $L^\cdot(t)$ on X such that:*

1) $H^k(L^\cdot(t)) = \begin{cases} \mathscr{F}(t) & \text{if } k=0 \\ 0 & \text{if } k \neq 0; \end{cases}$

2) $L^k(t) = \bigoplus_{j+k=i} X_j^i(t),\ X_j^i(t) = \psi_j^{\oplus h^i(\mathscr{F}(t-j))}$ *for* $j<n$,

$$X_n^i(t) = \begin{cases} S^{h^i(\mathscr{F}\otimes S^*(-n))} & \text{for } n \text{ odd} \\ S'^{h^i(\mathscr{F}\otimes S'^*(-n))} \oplus S''^{h^i(\mathscr{F}\otimes S''^*(-n))} & \text{for } n \text{ even} \end{cases}$$

and all other $X_j^i(t)$ *are zero;*

3) a) *the maps* $v_j^i(t,s) : X_j^i(t) \to X_{j-s}^{i-s+1}(t)$ $(s \in \mathbb{Z})$ *induced by the differentials* $L^k \to L^{k+1}$ *are zero for* $s \leq 0$

b) *the maps* $v_j^i(t,1)$ *agree with the natural maps* $R^i p_*(\psi_j \boxtimes \mathscr{F}(t-j)) \to R^i p_*(\psi_{j-1} \boxtimes \mathscr{F}(t-j+1))$ *coming from the exact sequence (tensored by* $\mathcal{O}(t)$*)*

$$0 \to K \to \psi_{n-1} \boxtimes \mathscr{F}(1-n) \to \dots \to \psi_1 \boxtimes \mathscr{F}(-1) \to q^*\mathscr{F} \to q^*\mathscr{F}|_\Delta \to 0$$

where K *is as in Theorem 5.5 tensored by* $q^*\mathscr{F}$.

6. Sheaves on Q_n

Theorem 6.1. *Let* $\mathscr{F}$ *be a coherent sheaf on* Q_n. *Suppose that for some* $t \in \mathbb{Z}$

$$h^i(\mathscr{F}(t-i-1)) = 0 \text{ for } 0 \leq i \leq n-2,$$
$$h^{n-1}(\mathscr{F} \otimes S(t-n+1)) = 0 \text{ for any } S \text{ spinor bundle.}$$

Then $\mathscr{F}$ *has* $\mathcal{O}(-t)^{h^0(\mathscr{F}(t))}$ *as direct summand.*

Proof. We may suppose $t=0$. Write the Kapranov theorem 5.6 for the sheaf $\mathscr{F}$. The reasoning is the same as in Theorem 2.1.

Corollary 6.2. *Let* $\mathscr{F}$ *be a coherent sheaf on* Q_n *such that*

$$h^i(\mathscr{F}(*)) = 0 \text{ for } 0 < i < n-1$$
$$h^{n-1}(\mathscr{F} \otimes S(*)) = 0 \text{ for any } S \text{ spinor bundle.}$$

Then $\mathscr{F}$ *splits as a direct sum of line bundles and a g.skyscraper sheaf.*

Proof. By the hypothesis and (5.2) we get that also $h^{n-1}(\mathscr{F}(*)) = 0$. Then the order of $\mathscr{F}$ is finite and by Proposition 1.6 we may suppose that $\mathscr{F}$ is locally free and indecomposable. Twisting we may suppose that $h^0(\mathscr{F}(-1)) = 0$ and $h^0(\mathscr{F}) \neq 0$, so that the result follows from Theorem 6.1.

Theorem 6.3. *Let* $\mathscr{F}$ *be a coherent sheaf on* Q_n. *Suppose that for some* $t \in \mathbb{Z}$ *and some* $0 < j < n$

$$H^i(\mathscr{F}(t-i-1)) = 0 \text{ for } j \leq i \leq n-2$$
$$H^i(\mathscr{F}(t-i+1)) = 0 \text{ for } 1 \leq i \leq j$$
$$H^{n-1}(\mathscr{F} \otimes S(t-n+1)) = 0 \text{ for any spinor bundle } S.$$

Set $n_j = h^j(\mathscr{F}(t-j))$. Then $\mathscr{F}$ contains $\psi_j^{n_j}$ as direct summand.

Proof. We may suppose $t = 0$. Write the Kapranov theorem 5.6 for $\mathscr{F}$. We have $L^0(0) = \psi_j^{n_j} \oplus X_n^n(0)$ and by the hypothesis no arrows $v_j^i(0,s)$ different from zero start or end at the summand $\psi_j^{n_j}$, so that the cohomology $H^0(L^\cdot(0))$ contains $\psi_j^{n_j}$ as direct summand.

Remark 6.4. The bundle $\psi_1^* = \mathscr{F}$ satisfies $h^i(\mathscr{F}(j)) = 0$ for $0 < i < n$, $-n+1 \le j \le 0$ with the only exception $h^{n-1}(\mathscr{F}(-n+1) = 1$ but $h^{n-1}(\mathscr{F} \otimes S(-n+1)) \neq 0$. This shows that the hypothesis $h^{n-1} = 0$ in Theorem 6.3 is necessary.

Corollary 6.5. *Let $\mathscr{F}$ be a coherent sheaf on Q_n such that $h^i(\mathscr{F}(*)) = 0$ for $0 < i < n$, with the only exceptions $h^j(\mathscr{F}(-j)) = n_j$ for $1 \le j \le n-1$. Let $h^{n-1}(\mathscr{F} \otimes S(*)) = 0$ for any S spinor bundle. Then $\mathscr{F} \simeq \bigoplus_{j=1}^{n-1} \psi_j^{n_j} \oplus \{$line bundles$\} \oplus \{$g.skyscraper sheaf$\}$*

Proof. From Corollary 6.2 and Theorem 6.3.

Example 6.6. An example of an indecomposable bundle F on Q_3 satisfying $H^i(F(t)) = 0$ for $1 \le i \le 2$, $\forall t \in \mathbb{Z}$ with the only exception $H^2(F(-2)) = \mathbb{C}$ such that $F \neq \psi_1^*$, $F \neq \psi_2$. Define first a bundle E of rank 6 on Q_3 as a nonsplitting extension:

$$(6.1) \qquad 0 \to S \to E \to \psi_1^* \to 0$$

In fact $\mathrm{Ext}^1(\psi_1^*, S) = H^1(\psi_1 \otimes S) = \mathbb{C}^4$. It is easy to check that $H^i(E(t)) = 0$ for $1 \le i \le 2$, $\forall t \in \mathbb{Z}$ with the only exception $H^2(E(-2)) = \mathbb{C}$. Then the only possible direct summands of E among the $\psi_i(t)$ and $\psi_i^*(t)$ would be ψ_2 or ψ_1^*. ψ_2 is not a direct summand of E because rank $\psi_2 = 7$. ψ_1^* has rank 4 and is not a direct summand of E because each endomorphism of $\psi_1^* \simeq T\mathbb{P}^4(-1)|_{Q_3}$ is invertible or is zero so that the morphism $E \to \psi_1^*$ splits the sequence (6.1) or cannot be surjective (again by rank reasons). We are not sure that E is indecomposable, but there exists a direct summand F of E with the required properties.

Theorem 6.7. *Let $\mathscr{F}$ be a coherent sheaf on Q_n. Suppose that for some $t \in \mathbb{Z}$*

$$h^i(\mathscr{F}(t-i-1)) = 0 \text{ for } 0 \le i \le n-2.$$

If $H^{n-1}(\mathscr{F} \otimes S(t-n+1)) = 0$ then $\mathscr{F}$ has $\mathcal{O}(-t)^{h^0(\mathscr{F}(t))}$ as direct summand.
If $H^{n-1}(\mathscr{F}(t-n)) = 0$ then
for n odd $\mathscr{F}$ has $S^*(-t)^{h^{n-1}(\mathscr{F} \otimes S(t-n+1))}$ as direct summand
for n even $\mathscr{F}$ has $S'^*(-t)^{h^{n-1}(\mathscr{F} \otimes S''(t-n+1))} \oplus S''^*(-t)^{h^{n-1}(\mathscr{F} \otimes S'(t-n+1))}$ as direct summand.

Proof. We may suppose $t = 0$. We suppose for simplicity n odd (for n even the proof is the same with slight variations). The Kapranov theorem 5.6 gives the exact sequence

$$0 \to S^{h^{n-1}(\mathscr{F} \otimes S(-n+1))} \to \mathcal{O}^{h^0(\mathscr{F})} \to \mathscr{G} \to 0$$

where the sheaf $\mathscr{G}$ is a direct summand of $\mathscr{F}$ and $h^{n-1}(\mathscr{F} \otimes S(-n+1)) = h^{n-1}(\mathscr{G} \otimes S(-n+1))$, $h^0(\mathscr{F}) = h^0(\mathscr{G})$. If $H^{n-1}(\mathscr{F} \otimes S(-n+1)) = 0$ this sequence gives the result. So we may suppose $a := h^{n-1}(\mathscr{F} \otimes S(-n+1)) \neq 0$. It suffices to prove that $\mathscr{G}$ contains S^* as direct summand. We set $N = 2^{\frac{n+1}{2}}$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \to & S^a & \xrightarrow{\alpha} & \mathcal{O}^{h^0(\mathscr{G})} & \xrightarrow{\beta} & \mathscr{G} & \to & 0 \\
 & & \downarrow p & & \downarrow t & & \downarrow z & & \\
0 & \to & S & \xrightarrow{\gamma} & \mathcal{O}^N & \xrightarrow{\delta} & S^* & \to & 0
\end{array}$$

where the second row is (5.2a), p is the natural projection on the first summand, t exists for the vanishing of $\mathrm{Ext}^1(\mathscr{G}, \mathcal{O}^N) \subset \mathrm{Ext}^1(\mathscr{F}, \mathcal{O}^N) = H^{n-1}(\mathscr{F}(-n))^N$ and the existence of z is obvious.

In the same way (using $h^1(S) = 0$, [O2] th. 2.3) we can construct the diagram

$$\begin{array}{ccccccccc}
0 & \to & S^a & \xrightarrow{\alpha} & \mathcal{O}^{h^0(\mathscr{G})} & \xrightarrow{\beta} & \mathscr{G} & \to & 0 \\
 & & \uparrow i & & \uparrow t' & & \uparrow z' & & \\
0 & \to & S & \xrightarrow{\gamma} & \mathcal{O}^N & \xrightarrow{\delta} & S^* & \to & 0
\end{array}$$

where i is the natural immersion into the first summand such that $p \circ i$ is the identity map.

We have $tt'\gamma = t\alpha i = \gamma p i = \gamma$, then tt' is the identity on $\mathrm{Im}\,\gamma$, so that $tt' \neq 0$. But tt' is defined by a constant matrix, so that there is a trivial line bundle $L \subset \mathcal{O}^N$ such that $tt'|_L$ is an automorphism of L. L is not contained in $S \simeq \mathrm{Ker}\,\delta$ because $h^0(S) = 0$, then there exists $l \in L$ such that $\delta(l) \neq 0$. Let $l' \in L$ be such that $tt'(l') = l$. We get $zz'\delta(l') = z\beta t'(l') = \delta tt'(l') = \delta(l) \neq 0$. Then $zz' \neq 0$ and as S^* is simple ([O2], th. 2.1) zz' must be the identity. This shows that S^* is a direct summand of $\mathscr{G}$ as we wanted.

Corollary 6.8 (see [BGS], [Kn] for vector bundles and [O1], [O3] for related results). *Let $\mathscr{F}$ be a coherent sheaf on Q_n such that $h^i(\mathscr{F}(*)) = 0$ for $0 < i < n$. Then $\mathscr{F} \simeq$ {line bundles} $\oplus$ {spinor bundles twisted} $\oplus$ {g.skyscraper sheaf}.*

We remark that if $\mathscr{F}$ is a spinor bundle then $h^{n-1}(\mathscr{F} \otimes S(*)) \neq 0$ for some S spinor bundle. For this reason spinor bundles did not occur in Corollary 6.2.

Theorem 6.9 (Kapranov [K]). *On Q_n we have*
for n odd

$$\psi_n = \psi_{n+1} = S^{\oplus 2^{\frac{n+1}{2}}}$$

for n even

$$\psi_n = \psi_{n+1} = S'^{\oplus 2^{\frac{n}{2}}} \oplus S''^{\oplus 2^{\frac{n}{2}}}.$$

Corollary 6.10. *Let Q_{n-1} be a smooth hyperplane section of Q_n. Then $\psi_i|_{Q_{n-1}} \simeq \psi_i \oplus \psi_{i-1}|_{Q_{n-1}}$ for $i \leq n-1$. Furthermore, for $1 < i < n-1$ ψ_i does not extend as vector bundle to Q_{n+1} and neither to $\mathbb{P}^{n+1}$. The bundles ψ_1 and ψ_i for $i \geq n-1$ extend to Q_{n+1}.*

Proof. The statement about the restriction is immediate from Corollary 6.5. If E is a bundle on Q_{n+1} (or on $\mathbb{P}^{n+1}$) such that $E|_{Q_n} \simeq \psi_i$ for $1 < i < n-1$ then the intermediate cohomology of E vanishes (because $h^j(E(t)) \to h^j(E(t+2))$ are always surjective or injective $\forall\, t$ for $1 \leq j \leq n$) and by Corollary 6.7 this is a contradiction. As $\psi_1 = \Omega^1_{\mathbb{P}^{n+1}}(1)|_{Q_n}$ we have the sequence $0 \to \psi_1 \to \mathcal{O}^{n+2} \to \mathcal{O}(1) \to 0$ that extend obviously to Q_{n+1}. The bundles ψ_i for $i \geq n$ extend trivially to Q_{n+1}, by Theorem 6.9 and the restriction behaviour of the spinor bundles ([O2], theor. 1.4). From the sequence $0 \to \mathcal{O}(-1) \to \psi_{n+1} \to \psi_{n-1} \to 0$ it can be shown that ψ_{n-1} extends to Q_{n+1} as the morphism $\mathcal{O}(-1) \to \psi_{n+1}$ extends.

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