Boundedness for codimension two submanifolds of quadrics

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ABSTRACT

It is proved that there are only finitely many families of codimension two subvarieties not of general type in $Q_6$.

1. Introduction

Ellingsrud and Peskine proved in [10] that smooth surfaces in $P^4$ not of general type have bounded degree. In [6] this result has been extended to any non general type codimension two submanifolds in $P^{n+2}$, of dimension $n \geq 2$.

In the same spirit Arrondo, Sols and De Cataldo proved in [1], [9] the following result

**Theorem 1.1** (Arrondo, Sols, De Cataldo)

Let $X = X_n \subset Q_{n+2}$ be a smooth variety not of general type of dimension $n$ embedded in the smooth quadric $Q_{n+2}$ of dimension $n + 2$. Let $n \geq 2$, $n \neq 4$. Then $\text{deg}(X)$ is bounded.

More precisely in [1] it is proved the case $n = 2$ while in [9] it is proved the case $n = 3$ and it is observed that the case $n \geq 5$ follows by an inequality on Chern classes along the lines of [13].

The aim of this paper is to drop the assumption $n \neq 4$ from the previous theorem. In fact we show the following

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Theorem 1.2

Let $X = X_4 \subset Q_6$ be a smooth 4-dimensional variety not of general type. Then $\deg(X)$ is bounded.

As it is well known, see Proposition 2.7, the theorem implies that there are only finitely many families of codimension two subvarieties not of general type in $Q_6$.

The paper is structured as follows. In section 2 we fix our notation and give preliminary results that will be needed later on in the paper. The sections 3 and 4 are devoted to bounding the degree of a non general type 4-fold $X \subset Q_6$.

In the last section we consider the problem of boundedness of non general type 4-folds in $P^7$. Among the log-special type 4-fold in $P^7$ (i.e., the image of the adjunction mapping has dimension less than 4) there are still two hard cases to be considered, namely the quadric bundles over surfaces and the scrolls over threefolds. We show that quadric bundles over surfaces have bounded degree with the only exception of those which lie in a 5-fold of degree 8.

The same technique can be applied to a manifold $X$ of dimension $n + 1$ embedded in $P^{2n+1}$ which is a quadric bundle over a surface. One can prove that there exists a function $F(n)$ such that $\deg(X) \leq F(n)$ or $X$ is contained in a variety of dimension $n + 2$ and degree $\left\lfloor \frac{n(n+1)(4n^2+4n+1)}{6(4n-1)} \right\rfloor$, where $[x]$ is the greatest integer less than or equal to $x$.

2. Notations and preliminaries

Throughout this article, unless otherwise specified, $X$ denotes a smooth connected projective 4-fold defined over the complex field $\mathbb{C}$, which is contained in $Q_6$. Its structure sheaf is denoted by $\mathcal{O}_X$. For any coherent sheaf $\mathcal{F}$ on $X$, $h^i(\mathcal{F})$ is the complex dimension of $H^i(X, \mathcal{F})$ and $\chi = \chi(\mathcal{O}_X) = \sum_i (-1)^i h^i(\mathcal{O}_X)$. The following notation is used:

- $X$, smooth 4-fold in $Q_6$;
- $H$ class of hyperplane section of $X$, $H = \mathcal{O}_{P^7}(1)|_X$;
- $K$ class of canonical bundle of $X$;
- $X^3$ generic 3-fold section of $X$;
- $S$ generic surface section of $X$;
- $C$ generic curve section of $X$;
- $g$ genus of $C$;
- $c_i = $ Chern classes of $X$;
- $N$ the normal bundle of $X$ in $Q_6$, $N_X/Q_6$. 
Using the self-intersection formula for the embedding of $X \subset Q_6$

$$c_2(N) = \frac{1}{2} dH^2$$  \hspace{1cm} (1)

we get the following formulae for $KH^3, K^2H^2, K^3H, K^4$ as function of $d, g, \chi(O_X), \chi(O_{X^3}), \chi(O_S)$.

$$K \cdot H^3 = 2g - 2 - 3d$$  \hspace{1cm} (2)

$$K^2 \cdot H^2 = 6\chi(O_S) - 12g + 12 + \frac{13}{2}d + \frac{1}{4}d^2$$  \hspace{1cm} (3)

$$K^3 \cdot H = -24\chi(O_{X^3}) - 48\chi(O_S) + 48g - 48 + 3d - 3d^2 + d(g - 1)$$  \hspace{1cm} (4)

$$K^4 = 120\chi(O_X) + 216\chi(O_{X^3}) + \chi(O_S) \frac{9d + 472}{2} + \frac{5d^3 + 1098d^2 - 16d(45g + 434) - 6144(g - 1)}{48}.$$  \hspace{1cm} (5)

Note that (2) follows from the adjunction formula.
To prove (3) we reason as follows. From the long exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{Q_6\mid X} \longrightarrow N \longrightarrow 0$$  \hspace{1cm} (6)

we get that

$$c_2(N) = 16H^2 + 6H \cdot K + K^2 - c_2.$$  

Such equality along with the self-intersection formula (1) gives

$$c_2 = \left(16 - \frac{1}{2}d\right)H^2 + 6H \cdot K + K^2.$$  \hspace{1cm} (7)

Hence by dotting (7) with $H^2$ we have

$$c_2 \cdot H^2 = \left(16 - \frac{1}{2}d\right)H^4 + 6H^3 \cdot K + H^2 \cdot K^2.$$  \hspace{1cm} (8)

On the other hand

$$c_2 \cdot H^2 = 12\chi(O_S) - K^2 \cdot H^2 - (12g - 12 - 11d).$$  \hspace{1cm} (9)

In fact using the following exact sequences

$$0 \longrightarrow T_{X^3} \longrightarrow T_{X\mid X^3} \longrightarrow H_{X^3} \longrightarrow 0$$
0 → $T_S → T_{X^3|S} → H_S → 0$

we get that

$$c_{2|X^3} = c_2(X^3) - K_{X^3} \cdot H_{X^3}. \quad (10)$$

Now (9) will follow if we dot (10) with $H$ and if we use the following facts:

$$c_2(X^3) \cdot H_{X^3} = c_2(S) - K_S \cdot H_S = 12\chi(O_S) - K_S^2 - (2g - 2 - d) \quad (11)$$

and

$$K_S^2 = K^2 \cdot H^2 + 8g - 8 - 8d. \quad (12)$$

Combining (8) and (9) we get (3).

In order to prove (4) we do the following. By dotting (7) with $H \cdot K$ we get

$$c_2 \cdot H \cdot K = \left(16 - \frac{1}{2}d\right)K \cdot H^3 + 6K^2 \cdot H^2 + K^3 \cdot H. \quad (13)$$

On the other hand if we dot (10) with $K$ and we use the fact that $c_2(X^3) \cdot K_{X^3} = -24\chi(O_{X^3})$ we have

$$c_2 \cdot H \cdot K = -24\chi(O_{X^3}) - 12\chi(O_S) + 8g - 8 - 6d. \quad (14)$$

Now (4) is gotten by putting together (13) and (14).

We now prove (5). From the sequence (6) we get that

$$24H^3 = c_3 + c_2 \cdot (6H + K) - \frac{1}{2}dK \cdot H^2 \quad (15)$$

$$22H^4 = c_4 + c_3 \cdot (6H + K) - \frac{1}{2}dH^2 \cdot c_2. \quad (16)$$

Thus

$$c_3 = (3d - 72)H^3 - (52 - d)H^2 \cdot K - 12H \cdot K^2 - K^3 \quad (17)$$

$$c_4 = 454d - 26d^2 + \frac{1}{4}d^3 + (384 - 12d)H^3 \cdot K + \left(124 - \frac{3}{2}d\right)H^2 \cdot K^2 + 18H \cdot K^3 + K^4. \quad (18)$$

By Riemann Roch theorem we know that

$$-720\chi(O_X) = K^4 - 4K^2 \cdot c_2 - 3c_2^2 + K \cdot c_3 + c_4. \quad (19)$$

Combining (7), (19), (18), (17), (4), (3) and (2) we get (5).
Theorem 2.1 ([8], Theorem 5.1)

Let $C$ be an irreducible reduced curve of arithmetic genus $g$ and degree $d$ in the projective space $\mathbb{P}^{n+1}$. Assume that $C$ is not contained on any surface of degree $< s$, with $d > \left(\frac{2s}{n-1}\right)^{\sqrt{n/2}}$. Then

$$g - 1 \leq \frac{d(d-1)}{2s} + \frac{d(s-2n+1)}{2(n-1)} + \frac{(d + s - 1)(s-1)}{2s} + \frac{(d - 1)(n - 2)(s + n - 2)}{2s(n - 1)} + \frac{(s - 1)^2}{2(n - 1)}.$$ 

Proposition 2.2 ([1], Proposition 6.3)

Let $C$ be a smooth curve of degree $d$ and genus $g$ in $Q_3$ that is not contained in any surface in $Q_3$ of degree strictly less than $2k$. Then

$$g - 1 \leq \frac{d^2}{2k} + \frac{1}{2}(k - 4)d.$$ 

Theorem 2.3 (Castelnuovo bound [11])

Let $V$ be an irreducible nondegenerate variety of dimension $k$ and degree $d$ in $\mathbb{P}^n$. Put

$$M = \left\lfloor \frac{d - 1}{n - k} \right\rfloor \quad \text{and} \quad \varepsilon = d - 1 - M(n - k)$$

where $[x]$ is the greatest integer less than or equal to $x$. Then

$$p_g(V) = h^0(\tilde{V}, \Omega^k) \leq \binom{M}{k+1}(n-k) + \binom{M}{k}\varepsilon$$

where $\tilde{V}$ is a resolution of $V$ (i.e., $\tilde{V}$ is a smooth variety mapping holomorphically and birationally to $V$).

Proposition 2.4

Let $X$ be a smooth 4-fold in $Q_6$. Then

$$\chi(O_X(t)) = \frac{1}{24}dt^4 + \frac{1}{12}(2 - 2g + 3d)t^3 + \frac{1}{24}(12\chi(O_S) - 12g + 12 + 11d)t^2$$

$$+ \frac{1}{12}(12\chi(O_X) + 6\chi(O_S) - 4g + 4 + 3d)t + \chi(O_X).$$

Proof. By the Riemann-Roch theorem we have

$$\chi(O_X(t)) = \frac{1}{24}H^4t^4 + \frac{1}{12}(KH^3)t^3 + \frac{1}{24}(H^2K^2 + c_2 \cdot H^2)t^2 - \frac{1}{24}c_2 \cdot H \cdot Kt + \chi(O_X),$$

where $c_i = c_i(T_X)$. We now use (9) and (14) to get our claim. □
Proposition 2.5

Let $S \subset Q_4$ be a surface of degree $d$ contained in an irreducible threefold of degree $\sigma$, with $\sigma$ minimal. Then

$$p_g(S) = h^2(\mathcal{O}_S) \leq \frac{d^3}{24\sigma^2} + \frac{d^2(\sigma - 4)}{8\sigma} + \frac{d(2\sigma^2 - 12\sigma + 23)}{12}.$$ 

Proof. Let $C$ be the generic curve section of $S$. By [1], Proposition 6.4 for $d \gg 0$ we have

$$g - 1 \leq \frac{d^2}{4\sigma} + \frac{1}{2}(\sigma - 3)d.$$ 

We let

$$G(t) = \chi(\mathcal{O}_{Q_4}(t)) = \left(\begin{array}{c} t + 5 \\ 5 \end{array}\right) - \left(\begin{array}{c} t + 3 \\ 5 \end{array}\right)$$

and

$$\tilde{F}(t) = G(t) - G(t - \sigma) - G\left(t - \frac{d}{2\sigma}\right) + G\left(t - \sigma - \frac{d}{2\sigma}\right).$$

We set

$$P(t) = dt + \left(\begin{array}{c} -\frac{d^2}{4\sigma} + \frac{1}{2}(3 - \sigma)d \end{array}\right)$$

and

$$F(t) = \begin{cases} \tilde{F}(t) & \text{if } t \leq -1 \\ \tilde{F}(0) - 1 & \text{if } t = 0 \\ 0 & \text{if } t \geq 1 \end{cases}$$

We have the following exact sequence

$$H^1(\mathcal{O}_C(t)) \longrightarrow H^2(\mathcal{O}_{S}(t-1)) \longrightarrow H^2(\mathcal{O}_{S}(t)) \longrightarrow 0$$

from which it follows that

$$-h^2(\mathcal{O}_{S}(t)) + h^2(\mathcal{O}_{S}(t-1)) \leq h^1(\mathcal{O}_C(t))$$

$$= \begin{cases} -\chi(\mathcal{O}_C(t)) \leq -P(t) & \text{if } t \leq -1; \\ -\chi(\mathcal{O}_C) + 1 \leq -P(0) + 1 & \text{if } t = 0 \end{cases}$$

$$= F(t - 1) - F(t), \text{ for } t \leq 0.$$
The same holds for $t \geq 1$ since $h^1(O_C(t)) = 0$ and

$$F(t - 1) - F(t) = \begin{cases} \geq 0 & \text{if } t = 1 \\ 0 & \text{if } t \geq 2. \end{cases}$$

From this it follows that $F(t) - h^2(O_S(t))$ is non increasing and since for $t$ going to infinity it goes to zero it follows that $h^2(O_S(t)) \leq F(t)$ for all $t$. Thus evaluating it at $t = 0$ we have

$$F(0) = \frac{d^3}{24\sigma^2} + \frac{d^2(\sigma - 4)}{8\sigma} + \frac{d(2\sigma^2 - 12\sigma + 23)}{12}$$

and hence our claim follows. Note also that

$$F(t) - F(t - 1) = \begin{cases} dt + \left( -\frac{d^2}{4\sigma} + \frac{1}{2}(3 - \sigma)d \right) & \text{if } t \leq -1 \\ \left( -\frac{d^2}{4\sigma} + \frac{1}{2}(3 - \sigma)d \right) - 1 & \text{if } t = 0. \end{cases}$$

On passing note that if $d$ is a multiple of $2\sigma$ then $\tilde{F}(t)$ is the Hilbert polynomial of the complete intersection $V_{\sigma, \frac{d}{2\sigma}}$ in $Q_4$ of hypersurfaces in $P^5$ of degree $\sigma$ and $\frac{d}{2\sigma}$ while $F(t)$ corresponds to $h^2(O_{V_{\sigma, \frac{d}{2\sigma}}}(t))$. □

**Proposition 2.6** ([9], Theorem 3.1)

Let $X^3 \subset V_{\sigma} \subset Q_5$. Then

$$-\chi(O_{X^3}) \geq \frac{1}{192\sigma^4}d^4 + \text{l.t. in } \sqrt{d}.$$  

**Proposition 2.7**

For any fixed integer $d_0$ there are only finitely many irreducible components of the Hilbert scheme of 4-folds in $Q_6$ that contain 4-folds with $d \leq d_0$.

**Proof.** By Theorem 2.3, for $d \leq d_0$, there are finitely many possible values for $g$, $p_g(S), p_g(X^3), p_g(X)$ and hence for $\chi(O_S), \chi(O_{X^3}), \chi(O_X)$ since $h^2(O_{X^3}) \leq p_g(S)$ and $h^1(O_S) = h^1(O_{X^3}) = h^1(O_X) = 0$. Thus there are only finitely many possibilities for the Hilbert polynomial

$$\chi(O_X(t)) = \frac{1}{24}dt^4 + \frac{1}{12}(2 - 2g + 3d)t^3 + \frac{1}{24}\left(12\chi(O_S) - 12g + 12 + 11d\right)t^2 + \frac{1}{12}\left(12\chi(O_{X^3}) + 6\chi(O_S) - 4g + 4 + 3d\right)t + \chi(O_X). \quad \Box$$
3. 4-Folds on a hypersurface of fixed degree

Let $X$ be a 4-fold of degree $d$ in $Q_6$ contained in an integral hypersurface $V_\sigma \in |O_{Q_6}(\sigma)|$.

**Theorem 3.1**

Let $X \subset V_\sigma \subset Q_6$ be as above. There is a polynomial $P_\sigma(t)$ of degree 10 in $\sqrt{d}$ with positive leading coefficient, such that

$$\chi(O_X) \geq P_\sigma(\sqrt{d}).$$

**Proof.** Look at the following three exact sequences

$$0 \longrightarrow O_{P^7}(t-2) \longrightarrow O_{P^7}(t) \longrightarrow O_{Q_6}(t) \longrightarrow 0$$

$$0 \longrightarrow O_{Q_6}(t-\sigma) \longrightarrow O_{Q_6}(t) \longrightarrow O_V(t) \longrightarrow 0$$

$$0 \longrightarrow I_{X,V}(t) \longrightarrow O_V(t) \longrightarrow O_X(t) \longrightarrow 0.$$

We use the first one to compute $\chi(O_{Q_6}(t))$ and the second one to compute

$$\chi(O_{Q_6}(t)) = \sigma \left(60t^5 + \frac{(6\sigma - \sigma^2)t^4}{24} + \frac{\sigma(\sigma^2 - 9\sigma + 26)t^3}{18} - \frac{\sigma(\sigma^3 - 12\sigma^2 + 52\sigma - 96)}{24}t^2 + \frac{\sigma(3\sigma^4 - 45\sigma^3 + 260\sigma^2 - 720\sigma + 949)}{180}t - \frac{\sigma^5 - 18\sigma^4 + 130\sigma^3 - 480\sigma^2 + 949\sigma - 942}{360} \right).$$

Now use Proposition 2.4, $\mu := \mu_\sigma = \frac{1}{2}d^2 + \sigma(\sigma - 3)d - 2\sigma(g - 1)$ and the third exact sequence to compute

$$\chi(I_{X,V}(t)) = \frac{\sigma}{60}t^5 + \frac{1}{24}\left((6 - \sigma)\sigma - d\right)t^4 + \left( \frac{3d^2 + 6d\sigma(\sigma - 6) - 2(3\mu - 2\sigma^2(\sigma^2 - 9\sigma + 26))}{72\sigma} \right) t^3 - \left( \frac{12\chi(O_S)\sigma - 2d^2 + d\sigma(19 - 4\sigma) + 4\mu + \sigma^2(\sigma^3 - 12\sigma^2 + 52\sigma - 96)}{24\sigma} \right) t^2$$

$$- \left( \frac{180\chi(O_S)\sigma + 360\chi(O_{X^3})\sigma - 15d^2 + 30d\sigma(4 - \sigma) + 30\mu}{360\sigma} \right) t$$

$$- \left( \frac{2\sigma^2(3\sigma^4 - 45\sigma^3 + 260\sigma^2 - 720\sigma + 949)}{360\sigma} \right) t$$

$$- \left( \frac{\sigma^5 - 18\sigma^4 + 130\sigma^3 - 480\sigma^2 + 949\sigma - 942}{360} \right) - \chi(O_X)$$

$$=: Q(t) - \chi(O_X).$$
Thus $\chi(\mathcal{O}_X) = Q(t) - \chi(\mathcal{I}_{X,V}(t))$. Define

$$t_1 := \min \left\{ t \in \mathbb{N} | \delta := 2\sigma t - d > 0, \frac{\delta^2}{2} - \mu - \delta \sigma \sigma - 3 > 0 \right\}.$$  

Then

$$\frac{d}{2\sigma} \leq t_1 \leq \frac{d}{2\sigma} + \frac{\sqrt{2d}}{2} + \sigma .$$

By plugging $t_1$ we get that

$$Q(t_1) \geq \frac{1}{60 \cdot 2^5 \sigma^4} d^5 - \frac{1}{24 \cdot 2^4 \sigma^4} d^5 - \frac{1}{24 \cdot 2^3 \sigma^4} d^5 - \frac{d}{2\sigma} \chi(\mathcal{O}_S) - \frac{d}{2\sigma} \chi(\mathcal{O}_X^3) + \text{l.t. in } \sqrt{d} .$$

We now use Proposition 2.6 and Proposition 2.5 in the above inequality to get

$$Q(t_1) \geq \frac{d^5}{\sigma^4} \left( \frac{1}{60 \cdot 2^5} \right) + \text{l.t. in } \sqrt{d}$$

and thus

$$\chi(\mathcal{O}_X) \geq Q(t_1) - \chi(\mathcal{I}_{X,V}(t_1))$$

$$\geq -\chi(\mathcal{I}_{X,V}(t_1)) + \frac{d^5}{\sigma^4} \left( \frac{1}{60 \cdot 2^5} \right) + \text{l.t. in } \sqrt{d} .$$

Moreover

$$-\chi(\mathcal{I}_{X,V}(t_1)) \geq -h^0(\mathcal{I}_{X,V}(t_1)) - h^2(\mathcal{I}_{X,V}(t_1)) - h^4(\mathcal{I}_{X,V}(t_1)) .$$

It will be enough to bound from above $h^2(\mathcal{I}_{X,V}(t_1))$ since $h^0(\mathcal{I}_{X,V}(t_1))$ and $h^4(\mathcal{I}_{X,V}(t_1))$ have been bounded in ([9], Lemma 3.3). In order to bound $h^2(\mathcal{I}_{X,V}(t_1))$ we consider the following exact sequences:

$$0 \rightarrow \mathcal{O}_{Q_2}(-\sigma) \rightarrow \mathcal{I}_{X^3,Q_3} \rightarrow \mathcal{I}_{X^3,V^4} \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_{P^6}(-2) \rightarrow \mathcal{I}_{X^3,P^6} \rightarrow \mathcal{I}_{X^3,Q_5} \rightarrow 0 .$$

By [2], Theorem 3.12 (b)

$$H^1(\mathcal{I}_{X^3,P^6}(t)) = 0 \quad \text{for} \quad t \geq 4d - 7 .$$
The latter along with the above sequences give

$$H^1(I_{X^3,V^4}(t)) = 0 \text{ for } t \geq 4d - 7.$$ 

From

$$0 \to I_{X,V}(k - 1) \to I_{X,V}(k) \to I_{X^3,V^4}(k) \to 0$$

it follows that

$$H^2(I_{X,V}(t)) = 0 \text{ for } t \geq 4d - 8.$$ 

Moreover by ([9], Lemma 3.3) we have that

$$h^2(I_{X,V}(t_1)) \leq \sum_{k = t_1 + 1}^{4d - 7} h^1(I_{X^3,V^4}(k)) \leq (4d - 7)Ad^2 + \ldots \leq 4Ad^2 + \ldots.$$ 

Hence

$$\chi(O_X) \geq Q(t_1) - \chi(I_{X,V}(t_1)) \geq Ad^3 + \text{ l.t. in } \sqrt{d}$$ 

which gives our claim. □

**Corollary 3.2**

Let $X \subset V_\sigma \subset Q_6$ be as above. Assume that $X$ is not of general type. Then there exists $d_0$ such that $\deg(X) \leq d_0$.

**Proof.** Consider the following exact sequence

$$0 \to K_X(-1) \to K_X \to K_{X^3}(-1) \to 0.$$ 

Since $X$ is not of general type we have $h^0(K_X(-1)) = 0$ and thus

$$p_g(X) = h^0(K_X) \leq h^0(K_{X^3}(-1)) \leq h^0(K_{X^3}) = p_g(X^3). \quad (21)$$

This along with Harris bound give

$$\chi(O_X) = 1 + h^2(O_X) - h^3(O_X) + p_g(X) \leq 1 + h^2(O_X) + p_g(X^3)$$

$$\leq 1 + p_g(S) + p_g(X^3) \leq \frac{1}{216} d^4 + \text{ l.t. in } \sqrt{d}. \quad (22)$$

On the other hand by the proof of Theorem 3.1 we obtain that

$$\chi(O_X) \geq \frac{1}{1920\sigma^4} d^5 + \text{ l.t. in } \sqrt{d}. \quad (23)$$

The boundedness of $d$ will now follow from (22) and (23). Hence we get our claim. □
4. Boundedness

Proposition 4.1
Let $X$ be a smooth 4-fold in $Q_6$. Denote $\chi(\mathcal{O}_S)$, $\chi(\mathcal{O}_{X^3})$, $\chi(\mathcal{O}_X)$, by $s$, $x$ and $v$ respectively. Then

a) $s \leq \frac{2}{3} \frac{(g-1)^2}{d} + \frac{5}{3} (g - 1) - \frac{1}{24} d^2 + \frac{5}{12} d.$

b) $-24x(2g - 2 - 3d) \leq 36s^2 + 3s(d^2 - 22d - 16(g - 1)) + \frac{1}{16} (d^4 - 92d^3 + 4d^2(12g + 193) + 32d(1-g)(g+8) + 768(g - 1)^2).$

If $X$ is not of general type then

c) $v \leq 2s - x.$

Proof. By the generalized Hodge index theorem we know that

\[ (K^2 \cdot H^2)H^4 \leq (K \cdot H^3)^2 \] (24)

\[ (K \cdot H^3)(K^3 \cdot H) \leq (K^2 \cdot H^2)^2. \] (25)

We observe that (2), (3) and (24) give a) while (2), (3), (4) and (25) give b). In order to prove c) we note that (21) along with the Lefschetz theorem give $h^2(\mathcal{O}_X) \leq h^2(\mathcal{O}_S).$ Moreover being $h^1(\mathcal{O}_X) = 0$ it follows that $\chi(\mathcal{O}_X) \leq \chi(\mathcal{O}_S) + p_g(X^3) = 2\chi(\mathcal{O}_S) - \chi(\mathcal{O}_{X^3})$, which is our claim. \(\square\)

Proposition 4.2
Let $X$ be a smooth 4-fold in $Q_6$. Then

a) $24\chi(\mathcal{O}_S) \geq d^2 - 2d - 24(g - 1).$

b) $240\chi(\mathcal{O}_{X^3}) \leq 120\chi(\mathcal{O}_X) + \frac{1}{2} (280 - 9d)\chi(\mathcal{O}_S) - \frac{1}{48} (d^3 - 6d^2 + 16d(12g - 13) - 960(g - 1)).$

Proof. Since $N(-1)$ is globally generated, the Segre classes satisfy:

\[ s_2 \left( N(-1) \right) \cdot H^2 \geq 0, \quad s_4 \left( N(-1) \right) \geq 0. \]

Recall that

\[ s_2 = c_1^2 - c_2 \]
\[ s_4 = c_1^4 + c_2^2 - 3c_1^2. \]

Moreover

\[ c_1 \left( N(-1) \right) = K + 4H \]
c_2(N(-1)) = \frac{1}{2}d - 5)H^2 - H \cdot K. Hence by (2), (3), (4), (5)

0 \leq s_2(N(-1)) \cdot H^2 = K^2 \cdot H^2 + 16d + 9K \cdot H^3 - \left(\frac{1}{2}d - 5\right)H^4

= 6\chi(O_S) + 6(g - 1) + \frac{1}{2}d - \frac{1}{4}d^2 \tag{26}

0 \leq s_4(N(-1)) = (434 - 13d)K \cdot H^3 + \left(136 - \frac{3}{2}d\right)K^2 \cdot H^2 + 19K^3 \cdot H

+ K^4 + 521d - 29d^2 + \frac{1}{4}d^3 = -240\chi(O_X^3) + 120\chi(O_X)

+ \frac{280 - 9d}{2}\chi(O_S) - \frac{d^3 - 6d^2 + 16d(12g - 13) - 960(g - 1)}{48} \tag{27}

\square

Theorem 4.3

There are only finitely many irreducible components of the Hilbert scheme of smooth 4-folds in \(Q_6\) that are not of general type.

Proof. Let \(X\) be a smooth 4-fold in \(Q_6\) that is not of general type. By Proposition 2.7 it is enough to bound \(d = \deg X\). We will do so by considering separately the cases \(2g - 2 - 3d \leq 0\) and \(2g - 2 - 3d > 0\).

Assume that \(2g - 2 - 3d \leq 0\), i.e. \(g - 1 \leq \frac{3}{2}d\). Using Proposition 4.1 along with a) in Proposition 4.2 we get

\[0 \leq -\frac{1}{2}d^2 + 6d.\]

Hence \(d\) is bounded in this case.

Assume now that \(2g - 2 - 3d > 0\). Using b) in Proposition 4.2 along with c) and b) in Proposition 4.1 we get

\[0 \leq \frac{540}{2g - 2 - 3d}s^2 + \frac{117d^2 - 6d(3g + 707) + 80(g - 1)}{2(2g - 2 - 3d)}s + \frac{d^4}{2g - 2 - 3d}

+ \frac{-d^3(g + 2078) + 6d^2(229g + 2842) + 304d(1 - g)(3g + 23) + 1824(g - 1)^2}{24(2g - 2 - 3d)}.

Solving the above inequality with respect to \(s\) we see that either

\[s \geq \frac{b + \sqrt{L}}{2160} \geq \frac{b}{2160} \tag{28}\]

or

\[s \leq \frac{b - \sqrt{L}}{2160} \tag{29}\]
where $b = -117d^2 + 6d(3g + 707) - 80(g - 1)$ and

$L = 5049d^3 - 36d^3(107g + 6793) + 36d^2(9g^2 - 8978g + 328809) + 960d(g - 1)(339g + 1915) - 6560000(g - 1)^2$.

If (28) holds then combining it with Proposition 4.1 we get

$$0 \leq \frac{d^2}{80} - \frac{d(3g + 557)}{360} + \frac{2(g - 1)^2}{3d} - \frac{46}{27}(g - 1).$$

(30)

If (29) holds then such inequality along with a) in Proposition 4.2 gives

$$0 \leq \frac{7}{864}d^4 - \frac{d^3(5g + 2203)}{6480} + \frac{d^2(595 - 263g)}{3240} + \frac{d(1 - g)(87g - 5749)}{1620} + \frac{7}{3}(g - 1)^2.$$

(31)

Fix a positive integer $k$ and let $d > 2k^2$. Assume that $X$ does not lie on any hypersurface of $Q_6$ of degree strictly less than $2k$. Then by a well known theorem of Roth $C$ does not lie on any hypersurface of $Q_6$ of degree strictly less than $2k$. Hence by Proposition 2.2 the genus of $C$ satisfies

$$g - 1 \leq \frac{d^2}{2k} + \frac{1}{2}(k - 4)d.$$

(32)

Rewriting (30) in the following way

$$0 \leq (g - 1)\left(\frac{2}{3d}(g - 1) - \frac{d}{120} + \frac{46}{27}\right) - \frac{7}{4}d + \frac{d^2}{80}$$

and using (32) we get

$$(g - 1) \leq \frac{3k}{2(k - 40)}d + \text{l.t. in } d.$$

(33)

In the case (31) a similar reasoning yields

$$(g - 1) \leq \frac{21}{2}d + \text{l.t. in } d.$$

(34)

The following inequality, gotten by combining a) in Proposition 4.1 and a) in Proposition 4.2 will be needed:

$$0 \leq -\frac{1}{12}d^2 + \frac{1}{2}d + (g - 1)\left(\frac{2}{3d}(g - 1) + \frac{8}{3}\right).$$

(35)
Plugging (33) and (32) in (35) gives
\[ 0 \leq d^2 \left( \frac{1}{2(k-40)} - \frac{1}{12} \right) + \text{t.l. in } d. \] (36)

Similarly, plugging (34) and (32) in (35) gives
\[ 0 \leq d^2 \left( \frac{7}{2k} - \frac{1}{12} \right) + \text{t.l. in } d. \] (37)

The coefficient of \( d^2 \), both in (36) and (37) is negative for \( k=47 \). Hence \( d \) is bounded from above if \( X \) is not in a hypersurface of degree strictly less than \( 2 \cdot 47 \). If \( X \) is not of general type and is contained in a hypersurface of degree less than or equal to \( 2 \cdot 47 \), then by Corollary 3.2 there exists \( d_0 \) such that \( \deg(X) \leq d_0 \). Hence the theorem is proved. \( \Box \)

5. Quadratic bundles over surfaces in \( \mathbb{P}^7 \)

Throughout this section \( X \) will denote a smooth 4-fold of degree \( d \) in \( \mathbb{P}^7 \) which is a quadratic bundle over a surface. We will show that either its degree \( d \) is bounded or \( X \) is contained in a 5-fold of degree 8.

For 4-folds in \( \mathbb{P}^7 \) by the self-intersection formula we have:
\[ c_3(N_{X/\mathbb{P}^7}) = dH^3. \]

5.1 From the exact sequence
\[ 0 \rightarrow T_X \rightarrow T_{\mathbb{P}^7} \vert_X \rightarrow N_{X/\mathbb{P}^7} \rightarrow 0 \]
we get that
\[ c_3(X) = (56 - d)H^3 - 28H^2 \cdot c_1(X) + 8H \cdot \left( c_1^2(X) - c_2(X) \right) - c_3^1(X) + 2c_1(X)c_2(X) \]
\[ c_4(X) = 70d - c_1(X)dH^3 - c_2(X) \left( 28H^2 - 8c_1(X)H + c_1^2(X) - c_2(X) \right) - c_3(X) \left( 8H - c_1(X) \right). \]

Definition 5.2. A 4-fold \( X \) is called a geometric quadratic bundle if there exists a morphism \( p : X \rightarrow B \) onto a normal surface \( B \) such that every fibre \( p^{-1}(b) \) is isomorphic to a quadric. A 4-fold \( X \) is a quadratic bundle in the adjunction theoretic sense if there exists a morphism \( p : X \rightarrow B \) onto a normal surface \( B \) and an ample Cartier divisor \( L \) on \( B \) such \( p^*L = K + 2H \).
The following proposition relates the two notions.

**Proposition 5.1**

Let $X$ be a quadric bundle in the adjunction theoretic sense. Then $X$ is a geometric quadric bundle. Moreover the base $B$ is smooth.

**Proof.** By ([3], Theorem 2.3) we know that $p$ is equidimensional, being dim $X = 4$. Moreover by ([5], Theorem 8.2) the base $B$ is smooth. □

From now on our notations and arguments follow closely the ones in [7].

**5.3 Notations**

Let $p : X → B$ be a geometric quadric bundle in $\mathbb{P}^7$. We have a natural morphism $f : B → Gr(\mathbb{P}^3, \mathbb{P}^7)$.

Let $S$ be a generic surface section of $X$. Then $p : S → B$ is finite 2:1. Let $2R ⊂ B$ be the ramification divisor of $p : S → B$.

We set $p_1O_X(1) = E$, a rank 4 vector bundle over $B$. We have $E = f^*(U^\vee)$, where $U$ is the universal bundle of $Gr(\mathbb{P}^3, \mathbb{P}^7)$, in particular $\det E = f^*(2 \wedge U^\vee)$ is ample. Note that $W := P(E)$ is a $\mathbb{P}^3$-bundle in the natural incidence variety $\mathbb{P}^7 × Gr(\mathbb{P}^3, \mathbb{P}^7)$ whose projection $\pi$ into $\mathbb{P}^7$ is the variety $V$ given by the union of all the 3-planes containing the quadrics of $X$.

Moreover $\pi^{-1}(X) = \tilde{X}$ is smooth and isomorphic to $X$. We denote the natural projection of $W$ onto $B$ also by $p$ and by $H$ the divisor on $W$ corresponding to $O_W(1)$. Hence $\tilde{X} = 2H - p^*L$ for some divisor $L$ on $B$.

The divisor $D ⊂ B$ corresponding to points whose fibres are singular quadrics, is called the discriminant divisor. Moreover $D = c_1(E) - c_1(L ⊗ E^\vee) = 2c_1(E) - 4L$.

In fact $\tilde{X}$ determines a section of $S^2E ⊗ E^\vee$, hence a morphism $\phi : L ⊗ E^\vee → E$. $D$ is given by the equation det $\phi = 0$. Thus our claim.

In order to bound $d$ we need several preliminary computations.

**Proposition 5.2**

\[
c_1(W) = 4H - p^*c_1(E) + p^*c_1(B)
\]

\[
c_2(W) = 6H^2 + H \cdot \left(4p^*c_1(B) - 3p^*c_1(E)\right) + p^*c_2(B) - p^*c_1(E) \cdot p^*c_1(B) + p^*c_2(E)
\]

\[
c_3(W) = 4H^3 + H^2 \cdot \left(6p^*c_1(B) - 3p^*c_1(E)\right) + H \cdot \left(2p^*c_2(B) + 4p^*c_2(B) - 3p^*c_1(E) \cdot p^*c_1(B)\right)
\]

\[
c_4(W) = 4H^3 \cdot p^*c_1(B) + H^2 \cdot \left(6p^*c_2(B) - 3p^*c_1(E) \cdot p^*c_1(B)\right)
\]

$H^4 - H^3 \cdot p^*c_1(E) + H^2 \cdot p^*c_2(E) = 0$.
Proposition 5.4

The Chern polynomial of $p^*E^\vee \otimes \mathcal{O}_W(1)$ is

$$1 + c_1(p^*E^\vee \otimes \mathcal{O}_W(1)) t + c_2(p^*E^\vee \otimes \mathcal{O}_W(1)) t^2 + c_3(p^*E^\vee \otimes \mathcal{O}_W(1)) t^3 = 1 + \left[ 4H - p^*c_1(E) \right] t + \left[ 6H^2 - 3p^*c_1(E) \cdot H + p^*c_2(E) \right] t^2 + \left[ 4H^3 - 3p^*c_1(E) \cdot H^2 + 2p^*c_2(E) \cdot H \right] t^3. \]

On the other hand $\text{ch}(T_W) = \text{ch}(p^*E^\vee \otimes \mathcal{O}_W(1)) \cdot \text{ch}(p^*T_B)$ hence we get that

$$1 + c_1(W) t + c_2(W) t^2 + c_3(W) t^3 + c_4(W) t^4 + c_5(W) t^5 = \left\{ 1 + \left[ 4H - p^*c_1(E) \right] t + \left[ 6H^2 - 3p^*c_1(E) \cdot H + p^*c_2(E) \right] t^2 + \left[ 4H^3 - 3p^*c_1(E) \cdot H^2 + 2p^*c_2(E) \cdot H \right] t^3 \right\} \left\{ 1 + p^*c_1(B) t + p^*c_2(B) t^2 \right\}.$$ Expanding the right hand side we get the first four equations. The last one is the Wu-Chern equation on $W = P(E)$, that is, $c_4(p^*E^\vee \otimes \mathcal{O}_W(1)) = 0.$ \hfill $\Box$

Lemma 5.3

$$c_1(E) = 2R - \frac{D}{2}, \quad L = R - \frac{D}{2}.$$ \hfill $\square$

Proof. We identify $X$ and $\tilde{X}$. We have $K_S = p^*(K_B + R)$, hence by the adjunction formula $K_X = -2H + p^*R + p^*K_B$. From Proposition 5.2 $K_W = -4H + p^*c_1(E) - p^*c_1(B)$. Putting this together with the adjunction formula $K_X = K_W|_X + 2H - p^*L$ gives $-p^*L + p^*c_1(E) = p^*R$, that is $c_1(E) = L + R$. Substituting this in $c_1(E) = \frac{D+4L}{2}$ we get $L = R - \frac{D}{2}$ and hence $c_1(E) = 2R - \frac{D}{2}$. \hfill $\square$

Proposition 5.4

$$c_1(X) = 2H - p^*K_B - p^*R$$

$$c_2(X) = 2H^2 + H \cdot \left( -p^*c_1(E) + 2p^*c_1(B) \right) + p^*R^2 - p^*c_1(E) \cdot p^*R - p^*c_1(B) \cdot p^*R + p^*c_2(B) + p^*c_2(E)$$

$$c_3(X) = H^2 \cdot \left( 2p^*c_1(B) + p^*c_1(E) - 2p^*R \right) + H \cdot \left( 2p^*c_2(B) - p^*c_1(B) \cdot p^*c_1(E) - 2p^*R^2 + 3p^*c_1(E) \cdot c_2^2(E) \right)$$

$$c_4(X) = H^3 \cdot \left( -2p^*c_1(E) + 4p^*R \right) + H^2 \cdot \left( 2p^*c_2(B) + p^*c_1(B) \cdot p^*c_1(E) + 6p^*R^2 + 3p^*c_1^2(E) - 9p^*R \cdot p^*c_1(E) - 2p^*R \cdot p^*c_1(B) \right).$$

Proof. The following sequence

$$0 \rightarrow T_X \rightarrow T_W|_X \rightarrow \mathcal{O}_X \left( 2H + p^*\left( R - c_1(E) \right) \right) \rightarrow 0$$

along with Proposition 5.2 gives the proof. \hfill $\square$
Lemma 5.5

Let $Z, Z'$ be arbitrary divisors on $B$. Then

i) $H^2 \cdot p^* Z \cdot p^* Z' = 2 \cdot Z \cdot Z'$

ii) $H^3 \cdot p^* Z = \left(c_1(E) + R\right) \cdot Z = \left(3R - \frac{D^2}{2}\right) \cdot Z$.

Proof. i) follows from the fact that the fibres of $X$ over $B$ are quadrics. As for ii) note that $H^3 \cdot p^* Z$ is equal to the intersection product in $W$

$$H^3 \cdot p^* Z \cdot \left(2H + p^* R - p^* c_1(E)\right).$$

Intersecting the Wu-Chern equation with $p^* Z$ we get that

$$H^4 \cdot p^* Z - H^3 \cdot p^* Z \cdot p^* c_1(E) = 0.$$

Hence

$$H^3 \cdot p^* Z \cdot (2H + p^* R - p^* c_1(E)) = 2H^3 \cdot p^* Z + H^3 \cdot p^* Z \cdot p^* R - H^3 \cdot p^* Z \cdot p^* c_1(E) = 2H^3 \cdot p^* Z \cdot p^* c_1(E) + H^3 \cdot p^* R \cdot p^* Z - H^3 \cdot p^* Z \cdot p^* c_1(E) = (c_1(E) + R) \cdot Z = (3R - \frac{D^2}{2}) \cdot Z.$$

Proposition 5.6

The surface $f(B) in Gr(P^3, P^7)$ has bidegree $(\delta, c_2(E))$ and it holds

$$\delta = \deg V = \frac{d - R \cdot c_1(E) + c_2^2(E)}{2}, \quad c_2(E) = \frac{-d + R \cdot c_1(E) + c_2^2(E)}{2}.$$

Proof. We intersect the Wu-Chern equation in Proposition 5.2 with $H$ and we get

$$\delta - H^4 \cdot c_1(E) + H^3 \cdot c_2(E) = 0.$$

Now cut the equation $X = 2H + p^* R - p^* c_1(E)$ with $H^4$ and we obtain

$$d = 2\delta + H^3 \cdot p^* R \cdot p^* c_1(E) - H^3 \cdot p^* c_1^2(E).$$

From these two equalities and Lemma 5.5 we get our claim. □
Proposition 5.7

c_1(X) = 2H + p^* c_1(B) - p^* R

c_2(X) = 2H^2 + H \cdot \left(2p^* c_1(B) - 2p^* R + \frac{1}{2} p^* D\right) - p^* R^2 + p^* c_2(E) - p^* c_1(B) \cdot p^* R

+ p^* c_2(B) + \frac{1}{2} p^* D \cdot p^* R

c_3(X) = H^2 \cdot \left(-\frac{1}{2} p^* D + 2p^* c_1(B)\right) + H \cdot \left(\frac{1}{2} p^* c_1(B) \cdot p^* D - 2p^* c_1(B) \cdot p^* R

+ 2p^* c_2(B) - \frac{1}{4} p^* D^2 + \frac{1}{2} p^* D \cdot p^* R\right)

c_4(X) = H^3 \cdot p^* D + H^2 \cdot \left(-\frac{1}{2} p^* c_1(B) \cdot p^* D + 2p^* c_2(B) + \frac{3}{4} p^* D^2 - \frac{3}{2} p^* D \cdot p^* R\right).

Proof. Using Proposition 5.3-Proposition 5.6 and easy computations give the formulas for $c_2(X)$, $c_3(X)$, $c_4(X)$. □

Proposition 5.8. Set

$$P(d) = \frac{1}{9d^3 - 50d^2 - 10949d + 169120}, \quad x = K_B^2, \quad y = D \cdot R.$$ The following hold:

$$R^2 = -\frac{1}{2} P(d) \left(-192864x + 2842d^3 + 332024d - 70224y - 53900d^2 - 15yd^2

- 49d^4 + 4974yd - 3yd^3 + 9016dx\right)$$

$$D^2 = P(d) \left(-882d^4 + 54y^3 + 54684d^3 - 870yd^2 - 1100736d^2 + 190512dx

- 39792yd + 7112448d - 3035648x + 709632y\right)$$

$$c_2(B) = \frac{1}{16} P(d) \left(9d^5 - 328d^4 + 144d^3 x + 3yd^3 + 3036d^3 - 173yd^2 - 37416d^2

+ 712d^2 x + 2608yd - 102048dx + 728896d + 675584x - 8576y\right)$$

$$K_B \cdot R = -\frac{1}{8} P(d) \left(-175392dx + 1722d^4 + 696696d^2 - 22686yd - 52332d^3

- 11yd^3 - 21d^5 + 879yd^2 + 3864d^2 x + 1983744x - 3415104d + 190784y\right)$$

$$K_B \cdot D = -\frac{1}{4} P(d) \left(-63d^5 + 5292d^4 - 164556d^3 - 33yd^3 + 2237760d^2 + 13608d^2 x

+ 2703yd^2 - 11176704d - 71880yd - 516208dx + 624384y + 4770304x\right).$$
Proof. The idea will be to solve a linear system of five equations in the unknowns $R^2$, $D^2$, $c_2(B)$, $K_B \cdot R$, $K_B \cdot D$, with coefficients rational functions of $d$. In fact from 5.1 we have

$$c_3(X) = (56 - d)H^3 - 28H^2 \cdot c_1(X) + 8H \cdot (c_1^2(X) - c_2(X)) - c_1^3(X) + 2c_1(X)c_2(X).$$

Substituting the values of $c_1(X)$, $c_2(X)$, $c_3(X)$ of Proposition 5.7 we get

$$(d - 16)H^3 + \left(-12p^*R + \frac{3p^*D}{2} + 14p^*c_1(B)\right) \cdot H^2 + \left(-6p^*c_1(B)^2 - 10p^*R^2 + 6p^*c_1(B) \cdot p^*D \right) \cdot H = 0.$$ 

Cutting respectively with $H$, $p^*R$, $p^*K_B$, $p^*D$ we get four equations. For instance if we cut with $p^*D$ we obtain:

$$(d - 16)H^3 \cdot p^*D + H^2 \cdot \left(-12p^*R \cdot p^*(D) + \frac{3}{2} p^*D \cdot p^*D + 14p^*c_1(B) \cdot p^*D\right) = 0.$$ 

Using now lemma 5.3 and lemma 5.5 we get

$$(d - 16)\left(3R - \frac{D}{2}\right) \cdot D - 24R \cdot D + 3D \cdot D + 28c_1(B) \cdot D = 0$$

and simplifying

$$-72R \cdot D + 11D^2 + 3dR \cdot D - \frac{dD^2}{2} - 28K_B \cdot D = 0.$$ 

The fifth equation is gotten by substituting the values of Proposition 5.7 in the second formula of 5.1:

$$c_4(X) = 70d - c_1(X)dH^3 - c_2(X) \left(28H^2 - 8c_1(X)H + c_1^2(X) - c_2(X)\right) - c_3(X) \left(8H - c_1(X)\right).$$

Solving such system we get the claim. □
Proposition 5.9

\[ c_2(E) = \frac{1}{4} P(d) \left( (41160d - 360640)x + (-95d^2 + 5005d - 69440)y - 165d^4 \\
+ 10390d^3 - 205070d^2 + 1225840d \right) \]

\[ g - 1 = \frac{1}{8} P(d) \left( (1008d^2 - 49112d + 566720)x + (223d^2 - 9857d + 109120)y \\
+ 393d^4 - 21032d^3 + 353440d^2 - 1781360d \right). \]

Proof. Using Proposition 5.6 and Lemma 5.3 we get

\[ c_2(E) = \frac{1}{2} \left( 6R^2 - \frac{5D \cdot R}{2} + \frac{D^2}{4} - d \right). \]

Moreover from the adjunction formula and Lemma 5.5 ii) we obtain

\[ g - 1 = \frac{1}{2} d + \frac{1}{2} H^3(p^*K_B + p^*R) = \frac{1}{2} d + \frac{1}{2} (p^*K_B + p^*R) \left( 3R - \frac{D}{2} \right) \]

\[ = \frac{1}{2} d + \frac{3}{2} K_B \cdot R - \frac{1}{4} K_B \cdot D + \frac{3}{2} R^2 - \frac{1}{4} D \cdot R. \]

Substituting the values of Proposition 5.8 and simplifying we get the assertions. □

We need a Roth type result.

Proposition 5.10

Let \( X \) be a codimension 3 integral subvariety of \( \mathbb{P}^n \) of degree \( d \). If the generic section \( C = X \cap \mathbb{P}^4 \) with a linear \( \mathbb{P}^4 \) is contained in a surface \( S_\sigma \subset \mathbb{P}^4 \) of degree \( \sigma \) with \( \sigma^2 \leq d \) then \( X \) itself is contained in a codimension 2 subvariety \( V_\sigma \subset \mathbb{P}^n \) of degree \( \sigma \).

Proof. We first check that the generic section \( S = X \cap \mathbb{P}^5 \) with a linear \( \mathbb{P}^5 \) is contained in a 3-fold \( Y_\sigma \subset \mathbb{P}^5 \) of degree \( \sigma \). By the assumptions, the surface \( S_\sigma \) is unique. On the contrary, suppose there are two such surfaces \( S'_\sigma \) and \( S''_\sigma \). There exists a linear projection \( \pi \) from a point \( p \in \mathbb{P}^4 \) on a hyperplane \( \mathbb{P}^3 \) such that \( \pi(S'_\sigma) \) and \( \pi(S''_\sigma) \) are two irreducible distinct surfaces of degree \( \sigma \) containing \( \pi(C) \) against Bezout theorem. We get a rational map from \( (\mathbb{P}^5)^\vee \) in the Hilbert scheme of degree \( \sigma \) surfaces of \( \mathbb{P}^4 \). It follows that the closure of the union of all surfaces \( S_\sigma \) is the 3-fold \( Y_\sigma \) we looked for.

By iterating this process we get the thesis. □
Proposition 5.11

Let $X$ be a quadric bundle in $\mathbb{P}^7$. Then $d \leq 2963$ or $X$ is contained in a 5-fold of degree 8.

Proof. We consider the possible values of $x$ and $y$ compatible with the following three inequalities:

$D \cdot R \geq 0$, (Lemma 4.15 in [7])
$c_2(E) \geq 0$,
$c_1(E) \cdot D \geq 0$.

Using Lemma 5.3, Proposition 5.8, Proposition 5.9 and easy computations we have that the three above inequalities correspond respectively to the following

\begin{align*}
y \geq 0 \\
(41160d - 360640) x + (-95d^2 + 5005d - 69440) y - 165d^4 + 10390d^3 - 205070d^2 \\
+ 1225840d \geq 0
\end{align*}

\begin{align*}
- (190512d - 3035648) x - (18d^3 - 670d^2 + 4004d + 33152) y + 882d^4 - 54684d^3 \\
+ 1100736d^2 - 7112448d \geq 0.
\end{align*}

These inequalities for $d \geq 26$ bound the inside of a triangle whose vertices are:

\begin{align*}
A_d &= \left( \frac{d \left( 33d^3 - 2078d^2 + 41014d - 245168 \right)}{8232d - 72128}, 0 \right) \\
B_d &= \left( \frac{9d \left( d^3 - 62d^2 + 1248d - 8064 \right)}{1944d - 30976}, 0 \right) \\
C_d &= \left( \frac{(33d^3 - 2192d^2 + 46900d - 316064) d}{8232d - 131712}, \frac{2d \left( 69d - 872 \right)}{21d - 336} \right).
\end{align*}

The minimum and the maximum of $g - 1$ considered as a function in $x$ and $y$ with $d$ fixed (see Proposition 5.9) have to be attained in one of the vertices. Substituting the coordinates of $A_d, B_d, C_d$ in the expression of $g - 1$ we get respectively

\begin{align*}
g - 1 &= \frac{(33d^2 - 293d - 552) d}{588d - 5152}, \frac{(63d^2 - 1320d + 5192) d}{972d - 15488}, \frac{(33d^2 - 407d - 1008) d}{588d - 9408}.
\end{align*}

It is a straightforward computation to see that for $d \geq 65$ we have

\begin{align*}
\frac{(33d^2 - 293d - 552) d}{588d - 5152} \leq g - 1 \leq \frac{(63d^2 - 1320d + 5192) d}{972d - 15488}.
\end{align*}
We now distinguish two cases. Suppose first that the curve section $C$ is not contained in any surface of degree 8. Then from Theorem 2.1 and (38) we have

$$\frac{(33d^2 - 293d - 552)}{588d - 5152} \leq \frac{d^2}{18} + \frac{5}{3}d + \frac{347}{18},$$

that is $3d^3 - 8881d^2 - 29706d + 893872 \leq 0$, which implies $d \leq 2963$.

If $C$ is contained in an octic, then from Proposition 5.10, it follows that also $X$ is contained in an octic. □

Remark 5.12. We remark that a similar reasoning gives an analogous result for manifolds $X$ of dimension $n + 1$ and degree $d$ embedded in $\mathbb{P}^{2n+1}$ which is a quadric bundle over a surface. More precisely we have the following

**Proposition 5.13**

Let $X$ be a manifold of dimension $n + 1$ and degree $d$ embedded in $\mathbb{P}^{2n+1}$ which is a quadric bundle over a surface. Then there exists a function $F(n)$ such that $d \leq F(n)$ or $X$ is contained in a variety of dimension $n + 2$ and degree $\left\lfloor \frac{n(n+1)(4n^2+4n+1)}{6(4n-1)} \right\rfloor$, where $[x]$ is the greatest integer less than or equal to $x$.

**Proof.** Similarly as in proposition 5.11, we get that

$$\frac{3(4n - 1)}{n(n + 1)(4n^2 + 4n + 1)}d^2 + O(d) \leq g - 1 \leq \frac{3(2n + 1)}{(n + 1)(2n^3 + 2n^2 + 2n + 3)}d^2 + O(d).$$

The details are left to the reader. We now use Theorem 2.1 and Proposition 5.10 to see that there exists a function $F(n)$ such that $d \leq F(n)$ or $X$ is contained in a variety of dimension $n + 2$ and degree $\left\lfloor \frac{n(n+1)(4n^2+4n+1)}{6(4n-1)} \right\rfloor$. □

**References**