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ON GENERIC IDENTIFIABILITY OF 3-TENSORS OF SMALL RANK*

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Abstract. We introduce an inductive method for the study of the uniqueness of decompositions of tensors, by means of tensors of rank 1. The method is based on the geometric notion of *weak defectivity*. For three-dimensional tensors of type (a, b, c), $a \leq b \leq c$, our method proves that the decomposition is unique (i.e., k-identifiability holds) for general tensors of rank k, as soon as $k \leq (a+1)(b+1)/16$. This improves considerably the known range for identifiability. The method applies also to tensor of higher dimension. For tensors of small size, we give a complete list of situations where identifiability does not hold. Among them, there are $4 \times 4 \times 4$ tensors of rank 6, an interesting case because of its connection with the study of DNA strings.

Key words. tensor decomposition, parafac, candecomp, uniqueness of decomposition, weak defectivity

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1. Introduction.

1.1. Statement of main results.

Let A, B, C be three complex vector spaces, of dimensions a, b, c respectively. A tensor $t \in A \otimes B \otimes C$ is said to have *rank* k if there is a decomposition

$$t = \sum_{i=1}^{k} u_i \otimes v_i \otimes w_i$$

with $u_i \in A, v_i \in B, w_i \in C$, and the number of summands k is minimal. Such a decomposition is said to be *unique* if for any other expression

$$t = \sum_{i=1}^{k} u'_i \otimes v'_i \otimes w'_i,$$

there is a permutation σ of $\{1, \ldots, r\}$ such that

$$u_i \otimes v_i \otimes w_i = u'_{\sigma(i)} \otimes v'_{\sigma(i)} \otimes w'_{\sigma(i)} \quad \forall i = 1, \ldots, k.$$

When t has a unique decomposition, the vectors $u_i \in A$, $v_i \in B$, $w_i \in C$ can be *identified* uniquely from t, up to scalars.

It is known that the set of tensors of rank k consists of a dense subset of an irreducible algebraic variety $S_k(Y)$, which is called the *kth secant variety* of the variety Y of tensors of rank one. The variety Y is isomorphic to the (cone over the) Segre product $\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$.

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The main result of our paper determines a bound for the rank, in terms of the dimensions of the vector spaces, which implies identifiability.

THEOREM 1.1. *Let* $a \le b \le c$. *Let* α, β *be maximal such that* $2^\alpha \le a$ *and* $2^\beta \le b$. *The general tensor* $t \in A \otimes B \otimes C$ *of rank* k *has a unique decomposition if* $k \le 2^{\alpha+\beta-2}$.

So if a, b are both powers of 2, then the general tensor of rank k has a unique decomposition if $k \le \frac{ab}{4}$. In the general case, the inequality of the theorem can be written as $k \le 2^{(\lfloor \log_2 a \rfloor + \lfloor \log_2 b \rfloor - 2)}$. Since $\frac{a+1}{2} \le 2^\alpha$ and $\frac{b+1}{2} \le 2^\beta$, one can say that the unique decomposition holds if $k \le (a+1)(b+1)/16$.

In our terminology, when the unique decomposition holds for the general tensor of rank k, we will say that the variety of tensors of rank one is k-*identifiable*.

Here the meaning of "general" is that, among tensors of rank k, the ones which do not have a unique decomposition consist of a set of zero measure, more specifically in a proper subvariety of $S_k(Y)$.

In particular, Theorem 1.1 applies to "cubic" tensors. The general tensor $t \in A \otimes A \otimes A$ of rank k has a unique decomposition if $k \le \frac{a^2}{16}$ (indeed, Theorem 1.1 provides a better bound, when a is close to a power of 2).

Our bound is log-asymptotically sharp for cubic tensors. As explained in Proposition 2.2, one cannot have a unique decomposition when the rank exceeds a value $k_{max} = k(a, b, c)$, which depends on a, b, c. Then $\sup_c \frac{k(a,b,c)}{ab}$ is finite. On the other hand, even for tensors of small size, the result is not sharp. In the first cases, with the help of a computer, we can improve Theorem 1.1.

Unique decomposition has been studied by several authors, and there is a huge amount of literature on this theme. Let us remind the reader that Strassen and Lickteig [Lick] proved that the general tensor $t \in A \otimes A \otimes A$ has rank $\lceil \frac{a^3}{3a-2} \rceil$ for $a \ne 3$ and rank 5 for $a = 3$ (indeed, the case $a = 3$ is known to be 4-*defective*, meaning that the corresponding 4th secant variety has dimension smaller than the expected one. The definition of k-*defective* is analogous). In this case, the aforementioned bound implies that if $a \ge 3$, then the general tensor of rank k can have a unique decomposition only if $k \le \lceil \frac{a^3}{3a-2} \rceil - 1$. The following theorem shows that this bound is almost always achieved for small a.

THEOREM 1.2. *The general tensor* $t \in A \otimes A \otimes A$ *of rank* k *has a unique decomposition if* $k \le k(a)$, *where*

a	2	3	4	5	6	7	8	9	10
$k(a)$	2	3	5	9	13	18	22	27	32

A more general list, which holds in the noncubic case, is given in section 5.

Comparing the previous table with the table of the general rank (for $a > 3$, the general rank -1 is the best possible achievement), and with Kruskal's result (see Proposition 1.5), one can appreciate the improvement.

	a	2	3	4	5	6	7	8	9	10
Gen. rank ($a \ne 3$) [Lick]	$\lceil \frac{a^3}{3a-2} \rceil$	2	4	7	10	14	19	24	30	36
Kruskal bound [K]	$\lfloor \frac{3a-2}{2} \rfloor$	2	3	5	6	8	9	11	12	14

The more evident lack of uniqueness is when $a = 4$ and $k = 6$. The case $a = 4$ is particularly interesting due to the models in phylogenetics [AR, ERSS], where a basis in $\mathbb{C}^4$ can be indexed by the nucleotids $\{A, C, G, T\}$.

Theorem 1.3. *The general tensor* $t \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ *of rank* 6 *has exactly two decompositions.*

It is interesting that the exception on uniqueness ($a = 4$) holds very close to the defective case $a = 3$. This phenomenon is quite general and it can be encountered already in the case of symmetric tensors.

In the case $c \geq (a-1)(b-1)+1$ our results become necessary and sufficient and we get the following theorem.

Theorem 1.4. *Assume* $c \geq (a-1)(b-1)+2$, $(a,b,c) \neq (2,2,3)$. *Then the general tensor of rank* k *in* $\mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ *has a unique decomposition as sum of* k *summands of rank one if and only if* $k \leq (a-1)(b-1)$. *In the case* $(a,b,c) = (2,2,3)$, *the general tensor of rank* k *has a unique decomposition as sum of* k *summands of rank one if and only if* $k \leq 2$.

1.2. A few historical remarks. In this subsection we sketch how our result fits in the literature.

The most celebrated result about uniqueness of decomposition of tensors is due to Kruskal [K]. It is often quoted in terms of Kruskal's rank. A consequence of Kruskal's criterion is the following statement, which applies to general tensors (see Corollary 3 in [AMR]).

Proposition 1.5 (Kruskal's criterion). *The general tensor* $t \in A \otimes B \otimes C$ *of rank* k *has a unique decomposition if*

$$k \leq \frac{1}{2}\left[\min(a,k) + \min(b,k) + \min(c,k) - 2\right].$$

In the cubic case, the general tensor $t \in A \otimes A \otimes A$ *of rank* k *has a unique decomposition if*

$$k \leq \frac{3a-2}{2}.$$

Kruskal's result is so important in the literature, that recently there have been published (at least!) three different proofs [Land, R, SS].

De Lathauwer [Lat] proves that the general tensor $t \in A \otimes B \otimes C$ of rank k has a unique decomposition if $k \leq c$ and $k(k-1) \leq a(a-1)b(b-1)/2$. Rhodes, in [R] addresses explicitly, as a problem at the end of the introduction, the need of sufficient conditions, stronger than Kruskal's ones, that guarantee the uniqueness of the decomposition, for general tensors. Our Theorem 1.1 gives a sufficient condition which improves Kruskal's bound for large k. For $k \leq c$ de Lathauwer's bound remains better than ours. An anonymous referee remarked that for $k > c$ our bound improves Kruskal's bound if $c + 4 \leq (a-7)(b-7)/8$. Our bound becomes considerably better than Kruskal's bound for tensors of format close to the cubic format.

Let us mention, on this subject, the work of Strassen, who gives a sufficient condition, for the identifiability, when the dimension of the largest vector space is odd (see Corollary 3.7 of [Str]). We observe that our methods have an easy, natural extension to tensors products with an arbitrary number of factors (see section 6). This extension looks difficult using Strassen's approach.

The tensor decomposition we are looking for is also called Candecomp or Parafac decomposition in the numerical literature. Among recent surveys on the topic is section 3.2 in [KB] and Landsberg's book [Land], which try to use a language understandable by both the numerical and the geometrical communities. From this point of view, one should also consider section 2 of [AMR], an interesting bridge between the two worlds.

An anonymous referee asked if the unicity results of this paper can be extended to tensors defined on the real numbers. It is important to remark that the closure of the sets of tensors of fixed rank are algebraic varieties over the complex numbers (so they are described by polynomial equations), while are semialgebraic varieties over the real numbers (so they are described by polynomial *inequalities*). It may happen that in the unique complex decomposition of a general real tensor, whose complex rank satisfies the assumptions of Theorems 1.1 or 1.2, two conjugate summands appear, so that the real rank is bigger than the complex rank. With these differences in mind, Theorems 1.1 and 1.2 still hold on the real numbers, when applied to the general real tensor of *real rank* k, with the same proof, while we do not know about Theorem 1.3.

1.3. Outline of the proof. In a line, our technique consists of putting together the inductive approach of [AOP] with the tool of weak defectivity developed in [CC1] and [CC2].

We consider the projective space of tensors $\mathbb{P}(A \otimes B \otimes C)$. In this space, the tensors of rank one give the Segre variety $\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$.

Our geometric point of view consists of the use of the celebrated Terracini's lemma, which allows us to study the identifiability of varieties, using properties of their tangent spaces. We refer to [CC1] and [CC2] for a more precise account of the theory behind.

A variety is called *k-tangentially weakly defective* (k-twd, see Definition 2.6) if the span of the tangent spaces at k general points of X, is tangent also in some other points.

It is a consequence of Terracini's lemma that if X is k-not twd, then the general tensor of rank k has a unique decomposition.

So our aim is to prove the k-not twd of Segre varieties $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$. The proof is performed by induction, by splitting $A = A' \oplus A''$ and by specializing some points on the lower dimensional Segre varieties $\mathbb{P}(A') \times \mathbb{P}(B) \times \mathbb{P}(C)$ and $\mathbb{P}(A'') \times \mathbb{P}(B) \times \mathbb{P}(C)$. It turns out that the induction works if we prove a stronger statement, concerning the so-called (k, p, q, r)-weakly defectivity, which is defined in section 3.

1.4. Outline of this paper. In section 2 we develop the basic notations on Segre varieties and weak defectivity. At the end of this section we prove the cases $a \leq 7$ of Theorem 1.2. Section 3 contains the definition 3.1 of (k, p, q, r)-defectivity and the inductive step (Proposition 3.7). At the end of this section we prove the remaining cases of Theorem 1.2. In the section 4 we prove Theorem 1.1. In section 5 we prove Theorem 1.3 and we give other examples of small dimension. Also we expose a list of all the examples of triple Segre product that we know when the uniqueness for general tensors of a given rank does not hold. In section 6, we show an extension of the previous results to products of many factors.

2. Preliminaries on Segre varieties. Let A, B, C be complex vector spaces of dimension a, b, c, respectively. Consider the product $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$. X is naturally embedded, by means of the Segre map, into $\mathbb{P}^N$, where $N = abc - 1$.

Sometimes, when there is no need to specify the vector spaces, we will refer to the variety X also as $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$.

Glossary of algebraic geometry Let $x = (x_0, \dots, x_N)$ be a system of homogeneous coordinates in $\mathbb{P}^N$. A *projective variety* $X \subset \mathbb{P}^N$ is defined as the zero locus of a collection of homogeneous polynomials $f_1(x), \dots f_k(x)$. X is called *nondegenerate* if it is not contained in a proper linear subspace of $\mathbb{P}^N$. When $X \subset \mathbb{P}^N$ is nondegenerate, the projective space $\mathbb{P}^N$ is called the *ambient space* of X. The same equations define a cone $\hat{X} \subset \mathbb{C}^{N+1}$ and X is called the *projectification* of $\hat{X}$. X is *irreducible* when the ideal $I = (f_1, \dots f_k)$ is a prime ideal; that is if $gh \in I$, then $g \in I$ or $h \in I$. In this case the quotient ring $K[x_0, \dots, x_n]/I$ is a domain, and the trascendence degree of its field of fractions is the *dimension* of X and it is denoted by $\dim X$. The codimension of X is by definition $N - \dim X$. The *tangent space* $\mathbb{T}_y X$ at $y \in X$ is the linear subspace of $\mathbb{P}^N$ defined by the linear equations $\sum_{i=0}^N \frac{\partial f_j}{\partial x_i}(y) x_i = 0$ for $j = 1, \dots, k$. X is called *smooth* at $x \in X$ if $\dim X = \dim \mathbb{T}_x X$. Note that the cone $T_x X := \hat{\mathbb{T}_x X} \subset \mathbb{C}^{N+1}$ is a vector space of dimension $\dim \mathbb{T}_x X + 1$, which is called the *affine dimension* of X. For the Segre variety $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$, the cone $T_{u\otimes v\otimes w} X$ coincides with $A \otimes v \otimes w + u \otimes B \otimes w + u \otimes v \otimes C$. Its projectification $\mathbb{T}_{u\otimes v\otimes w} X$ is obtained identifying the vectors up to scalar multiples. A *smooth variety* is a variety which is smooth at all its points. A smooth variety of dimension n is locally homeomorphic (with the Euclidean topology) to an open subset of $\mathbb{C}^n$.

Call $S_k(X)$ the kth secant variety of X, defined as the closure of the union of linear spans of k general points in X. The following definition works for any nondegenerate projective variety X.

DEFINITION 2.1. *X is called k-identifiable if a general element in $S_k(X)$ has a unique expression as the sum of k elements in X.*

From the tensorial point of view, this means that a general tensor of type $a \times b \times c$ and rank k can be written uniquely (up to scalar multiplication) as a sum of k decomposable tensors.

PROPOSITION 2.2. *There is a maximal rank for which the k-identifiability of tensors is possible, namely*

$$k_{max} = \left\lfloor \frac{N+1}{\dim(X)+1} \right\rfloor = \left\lfloor \frac{abc}{a+b+c-2} \right\rfloor.$$

Moreover $\lfloor \frac{ab}{3} \rfloor \le k_{max} \le ab$.

Proof. For $k > k_{max}$, the abstract secant variety

$$\{(x_1, \dots, x_k, u) \in X^k \times \mathbb{P}^N : u \in < x_1, \dots, x_k >\}$$

has dimension bigger than N, so that necessarily the general $u \in S_k(X)$ belongs to infinitely many k-secant spaces. The inequality follows from

$$\frac{ab}{3} \le \frac{abc}{a+b+c-2} = \frac{ab}{\frac{a+b-2}{c}+1} \le ab. \qquad \square$$

Our theoretical starting point is a criterion for k-identifiability, which follows from Terracini's lemma and which we will use under the following form (see, e.g., [CC1], the tangent spaces to X have been described in the glossary in section 2).

LEMMA 2.3 (Terracini). *Let X be an irreducible variety and consider a general point $u \in S_k(X)$. If u belongs to the span of points $x_1, \dots, x_k \in X$, then the tangent space to $S_k(X)$ at u is the span of the tangent spaces to X at the points $x_1, \dots, x_k$.*

Our criterion is the following.

PROPOSITION 2.4. *Let $X \subset \mathbb{P}^N$ be a nondegenerate, irreducible variety of dimension n. Consider the following statements:*

(i) *X is k-identifiable*

(ii) *Given k general points $x_1, \dots, x_k \in X$, the span $< \mathbb{T}_{x_1}X, \dots, \mathbb{T}_{x_k}X >$ contains $\mathbb{T}_x X$ only if $x = x_i$ for some $i = 1, \dots, k$.*

(iii) *There exists a set of k particular points $x_1, \dots, x_k \in X$, such that the span $< \mathbb{T}_{x_1}X, \dots, \mathbb{T}_{x_k}X >$ contains $\mathbb{T}_x X$ only if $x = x_i$ for some $i = 1, \dots, k$.*

Then we have (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Proof. (iii) $\Longrightarrow$ (ii) follows at once by semicontinuity.

Let us prove that (ii) $\Longrightarrow$ (i). Take a general point $u \in S_k(X)$ and assume that u belongs to the span of points $x_1, \dots, x_k \in X$. By the generality of u, we may assume that $x_1, \dots, x_k$ are general points of X. If u also belongs to the span of points $y_1, \dots, y_k \in X$, with at least one of them, say y_1, not among the x_i's, then, by Terracini's lemma, the span of the tangent spaces to X at the points x_i's, which is the tangent space to $S_k(X)$ at u, also contains the tangent space to X at y_1. This contradicts (ii). ∎

Condition (ii) of Proposition 2.4 is strongly related with the notion of k-weak defectivity.

In [CC1], Ciliberto and the first author give the following definition: a variety X is *k-weakly defective* if the general hyperplane which is tangent to X at k general points $x_1, \dots, x_k$, is also tangent in some other point $y \neq x_1, \dots, x_k$.

It is clear that a variety which does not satisfy condition (ii) of Proposition 2.4, is also k-weakly defective. However, the converse does not hold.

Example 2.5. Consider the Segre product $X = \mathbb{P}^1 \times \mathbb{P}^2$. It is classical (see, e.g., Zak's theorem on tangencies in [Z]) that the tangent space at one point to a smooth variety is not tangent elsewhere.

On the other hand, a general hyperplane tangent to X at one point is also tangent along a line. Indeed, it is well known that the dual variety of X is not a hypersurface (see [E]). Thus X is 1-weakly defective.

For maintaining the consistency with all the previous notation in this subject, we dare propose the following.

DEFINITION 2.6. *If X satisfies condition* (ii) *of Proposition* 2.4*, we will say that X is* k-not tangentially weakly defective*. Otherwise, we say that X is* k-tangentially weakly defective *(k-twd, for short).*

We understand that the notation is becoming odd. However, the increasing number of definitions is a phenomenon which also occurs in the study of *contact loci*, which seems however helpful for applications to the geometry of secant varieties (see, e.g., [CC3]).

Weak defectivity has been intensively studied in [CC1]. Notice that when X is k weakly defective, a general hyperplane tangent to X at general points $x_1, \dots, x_k$ is also tangent along a positive dimensional variety. We do not know if a similar phenomenon takes place also for k-twd.

Relations between k-weak defectivity and k-twd are probably stronger than expected, at least as far as one is interested in k-identifiability. We do not further develop this analysis.

Notice that when we deal with inductive steps in the proofs, we will need an even more complicated notion of weak defectivity. Compare with Definition 3.1.

For our purposes, Proposition 2.4 establishes that k-not tangentially weakly defectivity implies k-identifiability, when $N \geq k(n+1)$.

Remark 2.7. Let us notice that, by Proposition 2.2, if $N+1 < k(\dim X + 1)$, then k-identifiability is excluded. When $k = (N+1)/(n+1)$ the criterion of Proposition 2.4 does not apply because the span of the tangent spaces is expected to fill the ambient space. For example, our criterion could not be applied to study the 2-identifiability of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Now we are already able to prove the first cases of Theorem 1.2.

Proof of Theorem 1.2 *in case* $a \leq 7$. For $a = 2$ the theorem is well known. For $a \geq 3$, the proof is a straightforward application of Proposition 2.4. A random choice of $k(a)$ points satisfies condition (iii) of Proposition 2.4. Then X is k-identifiable. The Macaulay2 files which we used are available as ancillary files in the arXiv submission of this paper. The user may open the file *weakhiddenlaunchspan.m*2 and in the comments at the beginning he/she can find the instructions to run it. The other file is called automatically by this one. For the Macaulay2 package, we refer to [GS]. ∎

Remark 2.8. More powerful computers and/or better suited algorithms will eventually allow us to check the condition (iii) for larger values of a, and we encourage experts in numerical algebraic geometry in going further. We stopped at $a = 7$, because for $a = 8$ our algorithm on a common PC consumed too much time and memory. In the next section we show how the computation for larger values of a can be reduced to other computations for smaller values of a.

3. The inductive statement. We remind from the glossary in section 2 that if $x = u \otimes v \otimes w$ is a point of $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ embedded in $\mathbb{P}(A \otimes B \otimes C)$, then the projective tangent space $\mathbb{T}_x X$ is the projectification of the linear tangent space $T_x X = A \otimes v \otimes w + u \otimes B \otimes w + u \otimes v \otimes C$. We call these three summands, respectively, $[T_x X]_i$ for $i = 1, 2, 3$, so that $T_x X = [T_x X]_1 + [T_x X]_2 + [T_x X]_3$.

The idea is to fix two linear subspaces A', A'' of A, such that $A = A' \oplus A''$, then split the set of k points in two subsets and specialize them to the two spaces $\mathbb{P}(A') \times \mathbb{P}(B) \times \mathbb{P}(C)$ and $\mathbb{P}(A'') \times \mathbb{P}(B) \times \mathbb{P}(C)$. Then, the implication (iii) $\Longrightarrow$ (i) of Proposition 2.4 suggests that one could play induction.

Unfortunately, the situation is a little bit more complicated, since one cannot translate condition (ii) of Proposition 2.4 into the analogous condition on lower-dimensional spaces.

Instead, following the idea of [AOP] (Theorem 3.4) (suggested also from the splitting method of [BCS]), we need a more elaborated condition.

Definition 3.1. *A triple product* $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ *is called* (k, p, q, r)*-not weakly defective if*

- *for* k *general points* $x_1, \dots, x_k \in X$,
- *for* p *general points* $u_i \in X$,
- *for* q *general points* $v_i \in X$,
- *for* r *general points* $w_i \in X$,

then the span of $T_{x_i} X$, $[T_{u_i} X]_1$,$[T_{v_i} X]_2$,$[T_{w_i} X]_3$ *contains* $T_x X$ *if and only if* $x = x_i$, *for some* $i = 1, \dots, k$. *Otherwise* X *is called* (k, p, q, r)*-weakly defective.*

Clearly, $(k, 0, 0, 0)$ *weak defectivity coincides with* k*-twd.*

Remark 3.2. In order to compare the previous definition with the one in [AOP], we remind that the statement $T(a, b, c; k; p, q, r)$ given in the Definition 3.2 of [AOP] means that

- for k general points $x_1, \dots, x_k \in X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$,
- for p general points $u_i \in X$,
- for q general points $v_i \in X$,
- for r general points $w_i \in X$,

then the span of $T_{x_i} X$, $[T_{u_i} X]_1$,$[T_{v_i} X]_2$,$[T_{w_i} X]_3$ has the expected dimension

$$\min\left(abc, k(a + b + c - 2) + pa + qb + rc\right).$$

Remark 3.3. We will often use the computer algorithm, available in our arXiv submission, to prove that some triple Segre product is (k,p,q,r)-not weakly defective.

For instance, the algorithm shows that $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$ is $(1,2,1,1)$-not and $(2,1,1,1)$-not weakly defective. This is rather interesting, because the 4th secant variety of $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$ has dimension smaller than expected, namely $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$ it is 4-defective.

Example 3.4. Consider A, B, C, all of dimension 2 with basis $\{u_1,u_2\}$, $\{v_1,v_2\}$, $\{w_1,w_2\}$.

Call $x_1 = u_1v_1w_1$, $x_2 = u_2v_2w_1$. Then $T_{x_1} + [T_{x_2}]_3 = T_{x_2} + [T_{x_1}]_3$. This shows that $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ is $(1,0,0,1)$ weakly defective. Nevertheless, $T_{x_1} + [T_{x_2}]_3$ has the expected (affine dimension) 6 and it does not fill the ambient space.

Remark 3.5. (a) With the previous notation, by semicontinuity it is clear that when X is (k,p,q,r)-not weakly defective, then it is also (k',p',q',r')-not weakly defective, whenever $(k',p',q',r') \le (k,p,q,r)$, in the strict ordering.

(b) By semicontinuity, X is (k,p,q,r)-not weakly defective whenever one gets that for *particular* sets of points $\{x_i\}$, $\{u_i\}$, $\{v_i\}$, and $\{w_i\}$ as above, then the span of $T_{x_i}X$, $[T_{u_i}X]_1$,$[T_{v_i}X]_2$,$[T_{w_i}X]_3$ contains T_xX if only if $x = x_i$ for some $i = 1,\ldots k$.

(c) By Proposition 2.4, one soon gets that $(k,0,0,0)$-not weakly defective implies k-identifiable.

We will often apply the following reduction step.

LEMMA 3.6. *Assume that* $\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1}$ *is* (k,p,q,r)*-not weakly defective. Then* $\mathbb{P}^{a'}\times\mathbb{P}^{b'}\times\mathbb{P}^{c'}$ *is* (k,p,q,r)*-not weakly defective for any triple* $(a',b',c') > (a-1,b-1,c-1)$ *(in the strict ordering).*

Proof. We need just prove the statement for $(a',b',c') = (a,b-1,c-1)$. Write $X = \mathbb{P}^a\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1} = \mathbb{P}(A')\times\mathbb{P}(B)\times\mathbb{P}(C)$ so that $\dim(A') = a+1$.

Assume that X is (k,p,q,r)-weakly defective; then, for k general points $x_1,\ldots,x_k \in X$, p general points $u_i \in X$, q general points $v_i \in X$, r general points $w_i \in X$, the span Λ of the tangent spaces to X at the x_i's and the spaces $[T_{u_i}(X)]_1$, $[T_{v_i}(X)]_2$, $[T_{w_i}(X)]_3$ is also tangent in another point y.

Take a general point $P = (u,v,w) \in \mathbb{P}^a\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1}$ and consider the projection π of X from $L = u\otimes B\otimes C$. The image of the projection is $Y = \mathbb{P}(A)\times\mathbb{P}(B)\times\mathbb{P}(C)$, where $A \subset A'$ has codimension 1. Furthermore, by the generality of P, L does not meet Λ, as well as any line spanned by y, x_i. It follows that the span of the tangent spaces to Y at the general points $\pi(x_1),\ldots,\pi(x_k)$ and containing the spaces $[T_{\pi(u_i)}(Y)]_1$, $[T_{\pi(v_i)}(Y)]_2$, $[T_{\pi(w_i)}(Y)]_3$ is also tangent in another point $\pi(y)$. Thus Y is (k,p,q,r)-weakly defective. By induction, we get a contradiction. ◻

Now we are ready to state and prove our inductive criterion.

Let $X' = \mathbb{P}(A')\times\mathbb{P}(B)\times\mathbb{P}(C)$, $X'' = \mathbb{P}(A'')\times\mathbb{P}(B)\times\mathbb{P}(C)$. Note that $A\otimes B\otimes C = (A'\otimes B\otimes C)\oplus(A''\otimes B\otimes C)$. Denote by π' and π'' the two projections.

PROPOSITION 3.7 (inductive step). *Assume that* X' *is* $(k_1, p+k_2, q_1, r_1)$*-not weakly defective and* X'' *is* $(k_2, p+k_1, q_2, r_2)$*-not weakly defective. Then* X *is* $(k_1+k_2, p, q_1+q_2, r_1+r_2)$*-not weakly defective.*

Proof. We specialize k_1+k_2 points on X in order that k_1 of them belong to X_1 and k_2 of them belong to X_2. Let $x_1,\ldots,x_{k_1} \in X'$ and $y_1,\ldots,y_{k_2} \in X''$.

Let $z_i \in X$ for $i = 1,\ldots,p$.

We specialize q_1+q_2 points on X in order that the first q_1 of them (say $v'_1,\ldots,v'_{q_1}$) belong to X' and the last q_2 of them (say $v''_1,\ldots,v''_{q_2}$) belong to X''. Call $Q_1 = [T_{v'_1}(X')]_2 + \cdots + [T_{v'_{q_1}}(X')]_2$, $Q_2 = [T_{v''_1}(X'')]_2 + \cdots + [T_{v''_{q_2}}(X'')]_2$.

We specialize r_1+r_2 points on X in order that the first r_1 of them (say $w'_1,\ldots,w'_{r_1}$)

belong to X' and the last r_2 of them (say $w''_1, \dots, w''_{r_2}$) belong to X''. Call $R_1 = [T_{w'_1}(X')]_3 + \cdots + [T_{w'_{r_1}}(X')]_3$, $Q_2 = [T_{w''_1}(X'')]_3 + \cdots + [T_{w''_{r_2}}(X'')]_3$.

We want to prove that $T = T_{x_1}X + \cdots + T_{x_{k_1}}X + T_{y_1}X + \cdots + T_{y_{k_2}}X + [T_{z_1}X]_1 + \cdots + [T_{z_p}X]_1 + Q_1 + Q_2 + R_1 + R_2$ is tangent to X only at $x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}$.

Let $T_xX \subset T$, with $x = u \otimes v \otimes w$. Then $\pi_1(T_xX) \subset \pi_1(T)$. Let $u = u' + u''$. At least one among u' and u'' is nonzero, so let's assume $u' \neq 0$. Then we get $\pi_1(T_xX) = A' \otimes v \otimes w + u' \otimes B \otimes w + u' \otimes v \otimes C$ while $\pi_1(T) = T_{x_1}X' + \cdots + T_{x_{k_1}}X' + [T_{y_1}X']_1 + \cdots + [T_{y_{k_2}}X']_1 + [T_{z_1}X']_1 + \cdots + [T_{z_p}X']_1 + Q_1 + R_1$. By the assumption that X' is $(k_1, p + k_2, q_1, r_1)$-not weakly defective it follows that $u' \otimes v \otimes w$ is one among x_i.

If also $u'' \neq 0$, then the same argument shows that $u'' \otimes v \otimes w$ is one among y_i, which is a contradiction. Then $u'' = 0$, that is $x = u' \otimes v \otimes w$ is one among x_i. It follows that X is $(k_1 + k_2, p, q_1 + q_2, r_1 + r_2)$-not weakly defective, as we wanted. ∎

The inductive procedure stops eventually when we find some condition on weak defectivity, which does not hold. This does not necessarily mean that our starting example was not k-identifiable, but merely that we specialized the points too much, in order to expect a meaningful answer.

Proof of Theorem 1.2 *in cases* $a = 8, 9, 10$. In the case $a = 8$ we start with 22 points and we want to apply iteratively Proposition 3.7. Splitting one 8-dimensional vector space of the product in a direct sum of two 4-dimensional spaces, one sees that the $(22, 0, 0, 0)$-not weak defectivity of $\mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7$ follows if one knows that $\mathbb{P}^3 \times \mathbb{P}^7 \times \mathbb{P}^7$ is $(11, 11, 0, 0)$-not weakly defective. Repeating the procedure with the second factor, everything reduces to prove that $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^7$ is $(5, 7, 6, 0)$-not weakly defective and $(6, 4, 5, 0)$-not weakly defective. The first statement reduces to show that $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ is $(3, 3, 3, 2)$-not weakly defective and $(2, 4, 3, 3)$-not weakly defective. These statements have finally a reasonable size and can be checked with a random choice of points with our Macaulay2 algorithm. The last statement reduces to show that $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ is $(3, 2, 3, 3)$-not weakly defective and $(3, 2, 2, 3)$-not weakly defective, which follows from the above check and by the Remark 3.5 (a).

In the case $a = 9$ we start with 27 points and we split the nine-dimensional space in *three* three-dimensional summands. The inductive step is better explained by the following table:

a	b	c	k	p	q	r
9	9	9	27	0	0	0
3	9	9	9	18	0	0
3	3	9	3	6	6	0
3	3	3	1	2	2	2

The last statement can be checked again with Macaulay2.

The $a = 10$ case starts as follows:

a	b	c	k	p	q	r
10	10	10	32	0	0	0
5	10	10	16	16	0	0
5	5	10	8	8	8	0
5	5	5	4	4	4	4

The second statement reduces to show that $\mathbb{P}^1 \times \mathbb{P}^4 \times \mathbb{P}^4$ is $(1, 7, 2, 2)$-not weakly defective and $\mathbb{P}^2 \times \mathbb{P}^4 \times \mathbb{P}^4$ is $(3, 5, 2, 2)$-not weakly defective. Both these statements can be checked with Macaulay2. This concludes the proof. ∎

4. Proof of Theorem 1.1. In order to use the inductive step, we need a starting point.

LEMMA 4.1. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *is* $(1,0,0,0)$*-not and* $(0,1,1,1)$*-not weakly defective.*

Proof. The first fact is true for any smooth variety; see Example 2.5. For the second one, we consider $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$, where A, B, C all have dimension 2 and we choose basis spanning each space, $A = \langle a_0, a_1\rangle$, $B = \langle b_0, b_1\rangle$, $C = \langle c_0, c_1\rangle$. Then, without loss of generality, we may consider the span $T = A \otimes b_0 \otimes c_0 + a_0 \otimes B \otimes c_1 + a_1 \otimes b_1 \otimes C$. In the monomial basis of $A \otimes B \otimes C$ this span contains all the monomials with the only exception of $a_0 \otimes b_1 \otimes c_0$ and $a_1 \otimes b_0 \otimes c_1$. Then, a vector $v = \sum x_{ijk} a_i \otimes b_j \otimes c_k$, belongs to $X \cap \mathbb{P}(T)$ if all the 2×2-minors of the two following flattening matrices vanish:

$$\begin{bmatrix} x_{000} & x_{001} & x_{100} & 0 \\ 0 & x_{011} & x_{110} & x_{111} \end{bmatrix}, \qquad \begin{bmatrix} x_{000} & 0 & x_{100} & x_{110} \\ x_{001} & x_{011} & 0 & x_{111} \end{bmatrix}.$$

We recall that the flattening matrices of v are the matrices associated to the contraction maps induced by v like $B^\vee \otimes C^\vee \to A$, and the other maps induced by permutation of the three spaces. A straightforward check on the minors shows that $X \cap \mathbb{P}(T)$ consists of the following six lines in the five-dimensional space $\mathbb{P}(T) = \{x_{010} = x_{101} = 0\}$:

$r_0 = V(x_{001}, x_{000}, x_{100}, x_{110})$ (the variety given by the equations $x_{001} = x_{000} = x_{100} = x_{110} = 0$),

$r_1 = V(x_{000}, x_{100}, x_{110}, x_{111})$,

$r_2 = V(x_{100}, x_{110}, x_{111}, x_{011})$,

$r_3 = V(x_{110}, x_{111}, x_{011}, x_{001})$,

$r_4 = V(x_{111}, x_{011}, x_{001}, x_{000})$,

$r_5 = V(x_{011}, x_{001}, x_{000}, x_{100})$,

which have the property that, for $i \neq j$,

$$r_i \cap r_j = \begin{cases} \text{one point} & \text{if } i = j+1, j-1 \bmod 6, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed it is straightforward to verify that at the points of the six lines both the flattening matrices drop rank. Since $\deg X \cap \mathbb{P}(T) = \deg X = 6$, by degree reasons $X \cap \mathbb{P}(T)$ cannot contain anything else. It follows that $\mathbb{P}(T)$ is not tangent anywhere, because the tangent space at a point meets X in three concurrent lines, and there are no three concurrent lines in $X \cap \mathbb{P}(T)$. This proves that X is $(0,1,1,1)$-not weakly defective. ∎

Remark 4.2. We will use affine spaces whose dimension is a power of 2, as well as sets of points or subspaces whose number is expressed in terms of powers of 2, essentially because they allow the following recursive application of Lemma 3.7.

Assume we want to prove that $\mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma-1} = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ is $(2x, 2^u, 2^v, 2^w)$-not weakly defective. Then, by splitting the first linear space A in a direct sum of two subspaces of dimension $2^{\alpha-1}$ and balancing the splitting of the number of points and linear spaces, by Proposition 3.7 it is sufficient to prove that $\mathbb{P}^{2^{\alpha-1}-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma-1}$ is $(x, 2^u, 2^{v-1}, 2^{w-1})$-not weakly defective.

We will use this trick so often, in the arguments below.

The final statement will be that if we order the dimensions so that $1 \le \alpha \le \beta \le \gamma$, then $X = \mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma-1}$ is $(k,0,0,0)$-not weakly defective for $k \le 2^{\alpha+\beta-2}$.

Before showing this fact, we need a series of lemmas.

Proposition 4.3. *Assume that* $X = \mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma-1}$ *is* $(k,0,0,0)$*-not weakly defective. Then also* $X' = \mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\gamma}$ *is* $(k,0,0,0)$*-not weakly defective.*

Proof. The proof follows immediately by Lemma 3.6. ∎

So, in order to prove Theorem 1.1, we can reduce ourselves to the case $\beta = \gamma$, $k = 2^{\alpha+\beta-2}$.

Lemma 4.4. *Take* $X = \mathbb{P}^{2^{a_1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$*, with* $a_1, a_2, a_3 \geq 1$*. Pick non-negative integers* u_1, u_2, u_3 *such that* $u_i \leq a_j + a_k - 2$*, whenever* $\{i,j,k\} = \{1,2,3\}$*. Then* X *is* $(0, 2^{u_1}, 2^{u_2}, 2^{u_3})$*-not weakly defective.*

Proof. We make induction on the sum $a_1 + a_2 + a_3$.

If $a_1 = a_2 = a_3 = 1$, then the numerical conditions imply that $u_1 = u_2 = u_3 = 0$ and the conclusion follows from the fact that $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is $(0,1,1,1)$-not weakly defective, which holds by Lemma 4.1.

Assume (without loss of generality) $a_1 > 1$ and split the first projective space in a sum of two spaces of dimension 2^{a_1-1}. Then there are three possibilities:

(1) Assume $u_2 = u_3 = 0$. Then, by using Lemma 3.7, the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$ is $(0, 2^{u_1}, 1, 1)$-not weakly defective and it is $(0, 2^{u_1}, 0, 0)$-not weakly defective. The first consition implies the second condition. Since $a_1 > 1$, the six numbers $a_1 - 1, a_2, a_3, u_1, 0, 0$ fulfill the numerical inequalities of the statement. Hence the claim follows by induction in this case.

(2) Assume $u_3 > u_2 = 0$. Then the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$ is $(0, 2^{u_1}, 1, 2^{u_3-1})$-not weakly defective and it is $(0, 2^{u_1}, 0, 2^{u_3-1})$-not weakly defective. The second condition is contained in the first. One checks that the six numbers $a_1-1, a_2, a_3, u_1, 0, u_3-1$ fulfill the numerical inequalities of the statement. Hence the claim follows by induction.

(3) Assume $u_2, u_3 > 0$. Then the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$ is $(0, 2^{u_1}, 2^{u_2-1}, 2^{u_3-1})$-not weakly defective. One checks that the six numbers $a_1-1, a_2, a_3, u_1, u_2-1, u_3-1$ fulfill the numerical inequalities of the statement. Hence the claim follows by induction. ∎

Lemma 4.5. *Take* $X = \mathbb{P}^{2^{a_1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$*, with* $a_1, a_2, a_3 \geq 1$*. Pick non-negative integers* u_1, u_2, u_3 *such that* $u_i \leq a_j + a_k - 2$*, whenever* $\{i,j,k\} = \{1,2,3\}$*. Then* X *is* $(1, 2^{u_1}-1, 2^{u_2}-1, 2^{u_3}-1)$*-not weakly defective.*

Proof. We make induction on the sum $a_1 + a_2 + a_3$.

If $a_1 = a_2 = a_3 = 1$, then the numerical conditions imply that $u_1 = u_2 = u_3 = 0$ and the conclusion follows from the fact that $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is $(1,0,0,0)$-not weakly defective (Lemma 4.1).

Assume $a_1 > 1$ and split the first projective space in a sum of two spaces of dimension 2^{a_1-1}. Then there are three possibilities:

(1) Assume $u_2 = u_3 = 0$. Then, by using Lemma 3.7, the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$ is $(0, 2^{u_1}, 1, 1)$-not weakly defective and it is $(1, 2^{u_1}-1, 0, 0)$-not weakly defective. The first condition follows by the previous Lemma 4.4. For the second condition, notice that since $a_1 > 1$, the six numbers $a_1-1, a_2, a_3, u_1, 0, 0$ fulfill the numerical inequalities of the statement (and $0 = 2^0 - 1$). Hence the claim follows by induction in this case.

(2) Assume $u_3 > u_2 = 0$. Then the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$ is $(0, 2^{u_1}, 1, 2^{u_3-1})$-not weakly defective and it is $(1, 2^{u_1}-1, 0, 2^{u_3-1}-1)$-not weakly defective. The first condition follows by Lemma 4.4. The second condition follows by induction, since one checks that the six numbers $a_1-1, a_2, a_3, u_1, 0, u_3-1$ fulfill the numerical inequalities of the statement.

(3) $u_2, u_3 > 0$. Then the claim reduces to prove that $\mathbb{P}^{2^{a_1-1}-1} \times \mathbb{P}^{2^{a_2}-1} \times \mathbb{P}^{2^{a_3}-1}$ is $(0, 2^{u_1}, 2^{u_2-1}, 2^{u_3-1})$-not weakly defective and it is $(1, 2^{u_1}-1, 2^{u_2-1}-1, 2^{u_3-1}-1)$-not weakly defective. One checks that the numerical conditions in the statement are still fulfilled, by the six numbers $a_1-1, a_2, a_3, u_1, u_2-1, u_3-1$. Hence the claim follows by induction. ∎

Now we are ready to prove the following theorem.

THEOREM 4.6. *$X = \mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\beta-1}$ is $(k,0,0,0)$-not weakly defective, for $k \leq 2^{\alpha+\beta-2}$.*

Proof. Write $\alpha+\beta-2 = 2p+e$, where e is the remainder.

Now we start our reduction.

(A_1) One can split the vector space in the middle as a sum of two spaces of dimension $2^{\beta-1}$. By using Proposition 3.7, it turns out that X is $(2^{\alpha+\beta-2},0,0,0)$-not weakly defective when $\mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^{\beta-1}-1} \times \mathbb{P}^{2^\beta-1}$ is $(2^{\alpha+\beta-3}, 0, 2^{\alpha+\beta-3}, 0)$-not weakly defective.

(A_2) Splitting now the third vector space as a sum of two spaces of dimension $2^{\beta-1}$, and using Proposition 3.7, this reduces to prove that

$$\mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^{\beta-1}-1} \times \mathbb{P}^{2^{\beta-1}-1} \text{ is } (2^{\alpha+\beta-4}, 0, 2^{\alpha+\beta-4}, 2^{\alpha+\beta-4})\text{-not weakly defective.}$$

(A_3) Now repeat the procedure, splitting the space in the middle: Everything reduces to prove that

$$\mathbb{P}^{2^\alpha-1}\times\mathbb{P}^{2^{\beta-2}-1}\times\mathbb{P}^{2^{\beta-1}-1} \text{ is } (2^{\alpha+\beta-5}, 0, 2^{\alpha+\beta-4}+2^{\alpha+\beta-5}, 2^{\alpha+\beta-5})\text{-not weakly defective.}$$

Now split again the third vector space, and repeat the steps. At the end of the $(\alpha+\beta-2)$th step, after the computation, we find out that we need just to prove that

$$\mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^{\beta-p-e}-1} \times \mathbb{P}^{2^{\beta-p}-1} \text{ is } \left(1, 0, \sum_{i=0}^{p+e-1} 2^i, \sum_{i=0}^{p-1} 2^i\right)\text{-not weakly defective.}$$

Notice that all of these steps can be performed because $\beta-p \geq \beta-p-e \geq 1$. Indeed we have $\alpha \leq \beta$, thus $2\beta-2 \geq 2p+e$; hence $2\beta \geq 2p+e+2 > 2p+2e$.

Now, $\sum_{i=0}^{p+e-1} 2^i = 2^{p+e}-1$ while $\sum_{i=0}^{p-1} 2^i = 2^p-1$. Moreover

$$p+e \leq \alpha + (\beta-p) - 2 \qquad \text{since } 2p+e = \alpha+\beta-2,$$

$$p \leq \alpha + (\beta-p-e) - 2 \qquad \text{since } 2p = \alpha+\beta-e-2.$$

Thus we may apply Lemma 4.5 and see that $\mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^{\beta-p-e}-1} \times \mathbb{P}^{2^{\beta-p}-1}$ is $(1, 0, 2^{p+e}-1, 2^p-1)$-not weakly defective. The result is settled. ∎

When $\alpha = \beta$, i.e., when the product is balanced, we find that X is k-identifiable for $k \leq 2^{2\alpha-2}$.

Proof of Theorem 1.1. Fix α, β maximal such that $2^\alpha \leq a$ and $2^\beta \leq b$. Then, by Theorem 4.6, $\mathbb{P}^{2^\alpha-1} \times \mathbb{P}^{2^\beta-1} \times \mathbb{P}^{2^\beta-1}$ is $(k,0,0,0)$-not weakly defective, for $k \leq 2^{\alpha+\beta-2} = 2^\alpha 2^\beta/4$. Thus also $\mathbb{P}(A)\otimes\mathbb{P}(B)\otimes\mathbb{P}(C) = \mathbb{P}^{a-1}\times\mathbb{P}^{b-1}\times\mathbb{P}^{c-1}$ is $(k,0,0,0)$-not weakly defective, for $k \leq 2^\alpha 2^\beta/4$. The conclusion follows. ∎

Comparing our result with the maximal k for which the identifiability of $\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ makes sense, i.e.,

$$k_{max} = \left\lfloor \frac{abc}{a+b+c-2} \right\rfloor .$$

(see Proposition 2.2) and considering that $ab/3 \leq k_{max} \leq ab$, we see that the bound in Theorem 1.1 is, at least log-asymptotically, sharp for cubic tensors, as explained in the introduction.

In any event, it improves Kruskal's bound for identifiability.

Remark 4.7. In principle, there are no obstructions in repeating the argument of Theorem 1.1, when we substitute powers of 2 with powers of any other integer $p > 1$. The final statement is

$$X = \mathbb{P}^{p^{\alpha}-1} \times \mathbb{P}^{p^{\beta}-1} \times \mathbb{P}^{p^{\beta}-1} \text{ is } (k,0,0,0)\text{-not weakly defective, for } k \leq p^{\alpha+\beta-2}.$$

The proof is achieved very similarly, by splitting, step by step, a vector space of dimension p^n into p spaces of dimension p^{n-1} (see, e.g., the case $a = 9$ in the proof of Theorem 1.2).

We can use this statement, instead of Theorem 4.6, in the proof of Theorem 1.1, obtaining another bound which implies k-identifiability.

In most cases, however, the new bound is weaker than the one of Theorem 1.1. On the other hand, in some specific case, typically when powers of 3 are involved, it can be stronger.

To give an example, let us consider $X = \mathbb{P}^{26} \times \mathbb{P}^{26} \times \mathbb{P}^{26}$. Using Theorem 1.1, we obtain k-identifiability for $k \leq 2^{4+4-2} = 64$. Using powers of $p = 3$, instead, we get k-identifiability for $k \leq 3^{3+3-2} = 81$. It is an improvement, but still a long way from $k_{max} = 249$.

5. Some examples in low dimension and the proof of Theorem 1.4. In this section, we study the k-identifiability of Segre products $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$, when the dimensions a, b, c are small. We also provide a proof for Theorem 1.4.

Proof of Theorem 1.3. Consider $X = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$. This product is 5-identifiable, by Kruskal's criterion. On the other hand, accordingly with Proposition 2.2, one may ask about the 6-identifiability of X.

We are able to prove that X is *not* 6-identifiable, and the general point in $S_6(X)$ sits in exactly two 6-secant 5-planes. From the tensorial point of view, this means that a general $4 \times 4 \times 4$ tensor of rank 6 can be written as a sum of six decomposable tensors in exactly two ways (up to scalar multiplication and permutations).

The reason relies in the fact that through six general points $x_1, \dots, x_6$ of $X = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$, one can draw an elliptic normal curve Γ of degree 12, which spans a projective space $L = \mathbb{P}^{11}$, containing the linear span of $x_1, \dots, x_6$. So, a general point $u \in S_6(X)$ lies in a linear space L spanned by an elliptic normal curve $\Gamma \subset X$. By [CC2], Proposition 5.2, it is known that Γ has 6-secant order 2, i.e., there are exactly two 5-planes, 6-secant to Γ, inside L. By Proposition 2.4 of [CC2], if we prove that Γ coincides with the contact locus of a general 6-tangent hyperplane, then X must have 6-secant order equal to 2. This last fact can be checked by our Macaulay2 algorithm. Unfortunately, the existence of an elliptic normal curve, of degree 12, passing through 6 randomly chosen points of X, gives only a probabilistic argument for the existence of such a curve passing through six general points of X. To overcome this problem, we offer the following theoretical argument.

We remind that a normal elliptic curve in $\mathbb{P}^n$ is a smooth curve of genus one and degree $n + 1$, which is not contained in any hyperplane. This is the minimal degree for a curve of genus one not contained in any hyperplane, and every smooth elliptic curve can be obtained as a linear projection from a normal elliptic curve.

Consider the projections $z_{i1}, \dots, z_{i6}$ of $x_1, \dots, x_6$, into the ith copy of $\mathbb{P}^3$, so that $z_{i1}, \dots, z_{i6}$ are general points of $\mathbb{P}^3$. Normal elliptic curves C passing through the six points of $\mathbb{P}^3$ are given by pairs of quadrics through the points, so they are parametrized by the Grassmannian G of lines in the space $\mathbb{P}^3$ of quadrics through $z_{i1}, \dots, z_{i6}$. In order that three normal elliptic curves C, C', C'' in the three copies of $\mathbb{P}^3$ correspond to the same abstract curve, they need to differ by an element of $PGL(3)$. So, once we have C (four parameters), we can choose $\phi, \psi \in PGL(3)$ for the two remaining maps $C \to \mathbb{P}^3$ (thus a total of $4 + 15 + 15 = 34$ parameters). On the other hand, we need to impose that $\phi(C) = C'$ (resp., $\psi(C) = C''$) pass through $z_{21}, \dots, z_{26}$ (resp., $z_{31}, \dots, z_{36}$). Since each point imposes two conditions, we get a total of 24 algebraic conditions on the 34 parameters.

Moreover, if we want that after this correspondence, C, C', C'' are projections of the same curve passing through $x_1, \dots, x_6$, we also need that the projectivity $\phi : C \to C'$ (resp., $\psi : C \to C''$) composed with the automorphisms of the curves which sends z_{11} to z_{21} (resp., z_{11} to z_{31}), also sends any z_{1i} to z_{2i} (resp., z_{1i} to z_{3i}) for $i \geq 2$. This gives ten more conditions, which are algebraic on the coefficients of the two quadrics and the entries of the matrices of ϕ, ψ.

So, we have a total of 34 conditions, which are algebraic on the 34 parameters, i.e., on the projective coordinates of $G \times PGL(3) \times PGL(3)$. Thus we get at least a finite number of curves passing through $x_1, \dots, x_6$, for a general choice of the points. ∎

Remark 5.1. In the previous example, notice that if the three projections of the points $x_1, \dots, x_k$ differ by a projectivity, then the number of conditions decreases, and we find infinitely many normal elliptic curves.

It is easy to see that this implies that a point in the secant variety S_6 of any of these curves indeed belongs to infinitely many 6-secant spaces.

The case of products of projective spaces of dimension three is particularly interesting, due to its applications to statistical studies on DNA strings.

If we have many substrings of DNA strings, each formed by three positions, and we record the occurrence of the four bases in each position, we get a distribution which can be arranged in a $4 \times 4 \times 4$ tensor T. The rank k of T suggests the existence of k different types of substrings, in the probe, such that for each type, the distribution of bases is independent. So T is the sum of k tensors $T_1, \dots, T_k$, of rank 1.

An obvious question concerns the possibility of recovering the k tensors T_i, starting from T. When $k \geq 7$, this possibility is excluded, since 7 exceeds the maximum given in Proposition 2.2. For $k \leq 5$, k-identifiability (by Kruskal's criterion) tells us that, at least theoretically, the reconstruction is possible.

The amazing situation happens for $k = 6$. Although one could expect that 6-identifiability holds, Theorem 1.3 shows that there are exactly two sets of tensors of rank 1, whose sum is T. Hence, at least over the complex field, there are exactly two different sets of distributions in the six types that produce the same distribution T.

In [AOP], section 6.3, one finds the list of known Segre varieties $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C) = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$ (with $a \leq b \leq c$) such that the dimension of kth secant variety is smaller than the expected value. Recall that when the dimension of $S_k(X)$ is smaller than the expected value, i.e., when the variety X is k-defective, the k-identifiability necessarily fails.

A list of known Segre varieties X which are not k-identifiable, i.e., such that the general tensor of rank k in $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ does not have a unique decomposition, is the following (for $k < k_{max}$):

	(a,b,c)	k	*notes*
Defective Unbalanced	$c \geq (a-1)(b-1)+1$	$(a-1)(b-1) \leq k$ $k < \min(c, ab)$	[AOP, CGG]
Defective	$(3,4,4)$	5	[AOP]
Defective	$(3,b,b)$ b odd	$\frac{3b-1}{2}$	[Str]
W. defective Unbalanced	$3 \leq a$ $c \geq (a-1)(b-1)+2$	$(a-1)(b-1)+1$	$\binom{d}{(a-1)(b-1)+1}$ decompositions where $d = \binom{a+b-2}{a-1}$ (Theorem 1.4)
W. defective	$(4,4,4)$	6	2 decompositions (Theorem 1.3)
W. defective	$(3,6,6)$	8	(**)

A computer check shows that this list is complete for $c \leq 7$. In the last case marked with (**), the contact variety is a 4-fold in $\mathbb{P}^{39}$ of degree 108. This case needs an "ad hoc" analysis which goes beyond the space of the present note and will be addressed in a forthcoming paper [CMO].

In the unbalanced case, the identifiability can be proved theoretically.

Proposition 5.2. *The general tensor of rank* $\leq (a-1)(b-1)$ *in* $\mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ *has a unique decomposition as sum of* $(a-1)(b-1)$ *summands in* $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$ *for* $c \geq (a-1)(b-1)$.

Proof. Let $\phi \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ be general of rank $k \leq (a-1)(b-1)$. It induces the flattening contraction operator

$$A_\phi \colon (\mathbb{C}^c)^\vee \to \mathbb{C}^a \otimes \mathbb{C}^b$$

which has still rank k, by the assumption $c \geq (a-1)(b-1)$. Indeed, if $\phi = \sum_{i=1}^k u_i \otimes v_i \otimes w_i$ with $u_i \in \mathbb{C}^a, v_i \in \mathbb{C}^b, w_i \in \mathbb{C}^c$, where w_i can be chosen as part of a basis of C, then Im A_ϕ is the span of the representatives of $v_i \otimes w_i$ for $i = 1, \ldots, k$. It is well known that the projectification of this span, whose dimension is smaller than the codimension of the Segre variety $Y = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \subset \mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b)$, meets Y only in these k points (see, for example, the Theorem 2.6 in [CC1]). The proposition follows. ◻

Proposition 5.3. *If* $c = (a-1)(b-1)$ *or* $c = (a-1)(b-1)+1$, *then the rank of a general tensor in* $\mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ *is* $ab - a - b + 2$.

Proof. When $c \geq (a-1)(b-1)+1$, we are in the unbalanced case, according to Definition 4.2 of [AOP] in the defective setting, the range of the unbalanced case is slightly bigger than in the weakly defective setting. In this case the rank of the general tensor is $\min\{c, ab\}$ by (ii) of Theorem 4.4 of [AOP].

Assume $c = (a-1)(b-1)$. Using the same technique, we show that the secant variety $S_k(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1})$ has the expected dimension, for $k \leq (a-1)(b-1)$, and fills the ambient space, for $k = (ab - a - b + 2)$.

Indeed, with the notations of [AOP], $T(a-1, b-1, ab-a-b; (a-1)(b-1); 0, 0, 0)$ reduces to $T(a-1, b-1, 0; 1; 0, 0, ab-a-b)$, which is true and subabundant, while $T(a-1, b-1, ab-a-b; ab-a-b+2; 0, 0, 0)$ reduces (for $b \geq 3$) to $T(a-1, b-$

$1, 0; 1; 0, 0, ab - a - b + 1)$ and $T(a - 1, b - 1, 0; 2; 0, 0, ab - a - b)$, which are both superabundant and true. □

PROPOSITION 5.4. *Assume $c \geq (a-1)(b-1)+2$. Then the rank of the general tensor in $\mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$ is at least $(a-1)(b-1)+2$, and it is equal to $(a-1)(b-1)+2$ in the border case $c = (a-1)(b-1)+2$. The number of different decomposition of a general tensor of rank $(a-1)(b-1)+1$ is $\binom{d}{(a-1)(b-1)+1}$, where $d = \deg(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}) = \binom{a+b-2}{a-1}$. This number is always bigger than 1, with the only exception $a = b = 2$.*

Proof. We apply the same argument of the proof of Proposition 5.2. The unique difference is that, now, the dimension of the projectification of Im A_ϕ equals the codimension of $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$. Thus we get d points of intersection. Any choice of $(a-1)(b-1)+1$ among these d points yields a decomposition. □

Remark 5.5. The case $a = b = 3$ of Proposition 5.4 is connected to the work of ten Berge, who showed in [tB] that there are six different decompositions of a general rank 5 tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5$, chosen taking five among six possible summands. Our argument, which we gave for $c \geq 6$, can be extended to the case $c = 5$ and $k = k_{max} = 5$, and it gives a geometric explanation of this phenomenon. Indeed the six possible summands correspond to the six intersection points of $\mathbb{P}^2 \times \mathbb{P}^2$ with a general $\mathbb{P}^4$.

As a consequence of the three previous propositions, we get the proof of Theorem 1.4.

6. Products with many factors. At the cost of the growth of the notation, we can generalize the statement of our main Theorem 1.1 to products of many vector spaces.

In this section, we simply list the corresponding definitions and results. The proofs are absolutely straightforward, following the pattern of the corresponding arguments in the previous sections. Only the initial step of the induction needs an extra argument, which is displayed in Lemma 6.5.

For a given set of complex vector spaces $A_1, \ldots, A_n$, with $n \geq 3$ and $\dim A_i \geq 2$, let us give the general definition.

DEFINITION 6.1. *A Segre product $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ is called $(k, p_1, \ldots, p_n)$-not weakly defective if*

- *for k general points $x_1, \ldots, x_k \in X$,*
- *for p_i general points $w_{ij} \in X$, for $i = 1, \ldots, n$, $j = 1, \ldots, p_i$,*

the span of the spaces $T_{x_i}X$, $[T_{w_{ij}}X]_i$ contains T_xX if and only if $x = x_i$, for some $i = 1, \ldots, k$. Otherwise X is called $(k, p_1, \ldots, p_n)$-weakly defective.

Remark 6.2. (a) With the previous notation, by semicontinuity it is clear that when X is $(k, p_1, \ldots, p_n)$-not weakly defective, then it is also $(k', p'_1, \ldots, p'_n)$-not weakly defective, whenever $(k', p'_1, \ldots, p'_n) \leq (k, p_1, \ldots, p_n)$ in the strict ordering.

(b) By semicontinuity, X is $(k, p_1, \ldots, p_n)$-not weakly defective whenever one gets that for *particular* sets of points $\{x_i\}$, $\{w_{ij}\}$, as above, then the span of $T_{x_i}X$ and all $[T_{w_{ij}}X]_i$ contains T_xX if only if $x = x_i$ for some $i = 1, \ldots, k$.

(c) By Proposition 2.4, one soon gets that $(k, 0, \ldots, 0)$-not weakly defective implies k-identifiable.

LEMMA 6.3. *Consider $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ and assume that, for a choice of subspaces $A'_i \subset A_i$, the product $\mathbb{P}(A'_1) \times \cdots \times \mathbb{P}(A'_n)$ is $(k, p_1, \ldots, p_n)$-not weakly defective. Then X is $(k, p_1, \ldots, p_n)$-not weakly defective.*

The inductive criterion can be rephrased as follows, always following the lines in [AOP].

PROPOSITION 6.4 (inductive step). *Split the vector space A_i in the sum of two spaces A'_i and A''_i. Let $X' = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A'_i) \times \cdots \times \mathbb{P}(A_n)$, $X'' = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A''_i) \times \cdots \times \mathbb{P}(A_n)$,*

Assume that the product X' is $(k_1, p'_1, \dots, p_i + k_2, \dots, p'_n)$-not weakly defective and the product X'' is $(k_2, p'_1, \dots, p_i + k_1, \dots, p'_n)$-not weakly defective. Then, setting $p_j = p'_j + p''_j$ for $j \neq i$, we get that X is $(k_1 + k_2, p_1, \dots, p_i, \dots, p_n)$-not weakly defective.

Now we use again the previous criterion, when the dimension of the vector spaces are powers of 2, i.e., when $\dim(A_i) = 2^{\alpha_i}$ for all i. We agree to order the spaces, so that

$$\alpha_1 \leq \cdots \leq \alpha_n.$$

The following numerical criterion is the exact generalization of Lemmas 4.4 and 4.5.

LEMMA 6.5. *Take $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $n \geq 3$ and $\dim(A_i) = 2^{\alpha_i} \geq 2$. Pick nonnegative integers $u_1, \dots, u_n$ such that for all i*

$$u_i \leq \alpha_1 + \cdots + \hat{\alpha}_i + \cdots + \alpha_n - (n-1).$$

Then X is $(0, 2^{u_1}, \dots, 2^{u_n})$-not weakly defective and $(1, 2^{u_1}-1, 2^{u_2}-1, \dots, 2^{u_n}-1)$-not weakly defective.

Proof. The proof goes by induction. For the inductive step, one can follow the proof of Lemmas 4.4 and 4.5, rephrased for products of many vector spaces. Thus we need only check the starting points of the induction, namely that $Y_n = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is $(1, 0, \dots, 0)$-not weakly defective and $(0, 1, \dots, 1)$-not weakly defective.

The first fact follows soon, as $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is smooth, so that the general tangent hyperplane is not bitangent.

The second fact follows by induction on n. Namely it is true for $n = 3$, as observed in Lemma 4.5. For general n, write $Y_n = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $\dim(A_i) = 2$, and split A_1 in a direct sum of two 1-dimensional spaces A', A''. Using Lemma 6.3, one has thus to prove that $Y_{n-1} = \mathbb{P}^0 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is $(0, 1, 0, \dots, 0)$-not weakly defective and $(0, 0, 1, \dots, 1)$-not weakly defective. The former claim is obvious. The latter follows by induction. ∎

We get the following proposition.

PROPOSITION 6.6. *Take $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $n \geq 3$ and $\dim(A_i) = 2^{\alpha_i} \geq 2$. Order the α_i's so that $\alpha_1 \leq \cdots \leq \alpha_n$. Then X is not k-weakly defective, for $k \leq 2^{\alpha_1 + \cdots + \alpha_{n-1} - (n-1)}$.*

It follows that we have the next theorem.

THEOREM 6.7. *Take $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, with $n \geq 3$ and $\dim(A_i) = a_i \geq 2$ and, for all i, take α_i maximal, such that $a_i \geq 2^{\alpha_i}$. Then X is k-identifiable for*

$$k \leq 2^{\alpha_1 + \cdots + \alpha_{n-1} - (n-1)}.$$

Comparing our result with the maximal k for which the identifiability of $\mathbb{P}(A_1) \times$

$\cdots \times \mathbb{P}(A_n)$ makes sense, which, in the case of a product of many factors, reads as

$$k_{max} = \left\lfloor \frac{\prod_{i=1}^{n-1} a_i}{1 + \frac{\sum_{i=1}^{n-1} a_i - (n-1)}{a_n}} \right\rfloor .$$

We see again that the bound in Theorem 6.7 is log-asymptotically sharp for hypercubic tensors when $a_i = a$.

The inequality of the theorem can be written as

$$k \leq 2^{\left(\sum_{i=1}^{n-1} \lfloor \log_2 a_i - 1 \rfloor\right)}.$$

Since $2^{\alpha_i} \geq \frac{a_i+1}{2}$ we get the general tensor of rank k is k-identifiable if

$$k \leq \frac{\prod_{i=1}^{n-1}(a_i+1)}{2^{2n-2}}.$$

In [SB] Kruskal's bound was extended to the case of n factors. A sufficient condition for the k-identifiability of the general tensor of rank k is

$$2k + n - 1 \leq \sum_{i=1}^{n} \min(k, a_i).$$

To compare with our condition, in the hypercubic case where $a_i = a$, the bound in [SB] is

$$k \leq \frac{n(a-1)+1}{2},$$

while our bound is

$$k \leq 2^{(n-1)(\lfloor \log_2 a - 1 \rfloor)}.$$

For $a \geq 4$ we get also the weaker, but more handy, inequality

$$k \leq \left(\frac{a+1}{4}\right)^{n-1}.$$

When a_n is large enough, so that the format is far from the cubic one, and in the cases $k \leq a_n$, better bounds than ours are in section 5 of [Ste] and a bound for $n = 4$ is in [Lat].

Example 6.8. Instead of giving the proofs, which, we repeat, are analogue to the proofs of the statement of section 4, let us see how the reduction works in a concrete example.

Take $A_1 = \cdots = A_5 = \mathbb{C}^{16}$ and consider $X = \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_5)$. We want to prove that X is k-not weakly defective for $k = 2^{4+4+4+4-4} = 4096$.

The reduction step starts as in the following table:

A_1	A_2	A_3	A_4	A_5	k	p_1	p_2	p_3	p_4	p_5
16	16	16	16	16	4096	0	0	0	0	0
8	16	16	16	16	2048	2048	0	0	0	0
8	8	16	16	16	1024	1024	1024	0	0	0
8	8	8	16	16	512	512	512	512	0	0
8	8	8	8	16	256	256	256	256	256	0
8	8	8	8	8	128	128	128	128	128	128
4	8	8	8	8	64	192	64	64	64	64
4	4	8	8	8	32	96	96	32	32	32
4	4	4	8	8	16	48	48	48	16	16
4	4	4	4	8	8	24	24	24	24	8
4	4	4	4	4	4	12	12	12	12	12
2	4	4	4	4	2	14	6	6	6	6
2	2	4	4	4	1	7	7	3	3	3

Then use Lemma 6.5 with $u_1 = u_2 = 3$, $u_3 = u_4 = u_5 = 2$.

Remark 6.9. As in the case of triple Segre products, in principle, there are no obstructions in repeating the argument, when we substitute powers of 2 with powers of 3 (see the proof of Theorem 1.2 we gave in case $a = 9$), or any other integer $p > 1$.

For some numerical cases, the bound for identifiability that we get using powers of numbers bigger than two can be closer to the maximal value k_{max}.

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