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A class of n-bundles on $\mathrm{Gr}(k, n)$

By *Giorgio Ottaviani* at Firenze*)

Introduction

In this paper we construct a class $\Gamma_{k,n}$ of holomorphic vector bundles of rank n on each $\mathrm{Gr}(k, n)$. $\mathrm{Gr}(k, n)$ is the Grassmannian of linear subspaces $\mathbb{P}^k$ in $\mathbb{P}^n$ on the field $\mathbb{C}$. These bundles seem to be interesting because they are indecomposable, not uniform, and their rank is quite small with respect to the dimension $(n-k)(k+1)$ of the base space.

On the variety $\mathrm{Gr}(1, 3)$ these bundles have been studied by R. Hernandez and I. Sols [12].

We denote respectively by S and Q the universal bundle and the quotient bundle on $\mathrm{Gr}(k, n)$. Then every bundle $E \in \Gamma_{k,n}$ arises from an exact sequence

$$0 \to \mathcal{O} \to S^* \oplus Q \to E \to 0,$$

thus E has the same Chern classes as $S^* \oplus Q$. $\mathrm{Aut}(\mathrm{Gr}(k, n))$ acts transitively on $\Gamma_{k,n}$.

In order to study the behaviour of these bundles we introduce the number $\delta(k, n) = (n-1) - |(n+1) - 2(k+1)|$ which measures "how far $\mathrm{Gr}(k, n)$ is from being a projective space". We have $0 \leqq \delta(k, n) \leqq n-1$ and $\delta(k, n) = 0$ if and only if $\mathrm{Gr}(k, n) \cong \mathbb{P}^n$.

If $E \in \Gamma_{k,n}$ we obtain that:

(i) E is always indecomposable,

(ii) E is simple if and only if $\delta(k, n) > 0$ (except for the case $n = 1$),

(iii) E is Mumford-Takemoto semistable if and only if $\delta(k, n) \geqq n-2$,

(iv) E is Gieseker-Maruyama semistable if and only if $\delta(k, n) = n-1$,

(v) E is stable if and only if $\delta(k, n) = n-1$.

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Next we describe the splitting on lines and the subvariety of jumping lines for every $E \in \Gamma_{k,n}$. This subvariety determines the isomorphism class of the bundle.

Now let (x_i, y_j) be homogeneous coordinates in $\check{\mathbb{P}}^n \times \mathbb{P}^n$ and let

$$M = (\check{\mathbb{P}}^n \times \mathbb{P}^n) \setminus \left(\sum_{i=0}^{n} x_i y_i = 0 \right).$$

We show that if $\delta(k, n) > 0$, M is a fine moduli space for the isomorphism classes of bundles $E \in \Gamma_{k,n}$ and if $2(k+1) = n+1$, M is a Zariski open set of the coarse moduli space of all stable vector bundles of rank n with the same Chern classes as $S^* \oplus Q$.

Finally we obtain that the generic section of each $E \in \Gamma_{k,n}$ vanishes on a smooth subvariety with two connected components, each of them is a Schubert cycle isomorphic to $\mathrm{Gr}(k-1, n-2)$. The degeneracy locus of two generic sections of each $E \in \Gamma_{k,n}$ is a smooth subvariety $X_{k,n}$ isomorphic to $\mathbb{P}^1 \times \mathrm{Gr}(k-1, n-2)$. The normal bundle of $X_{k,n}$ is not ample. $X_{k,n}$ is a "scroll" over $\mathbb{P}^1$, i.e. its fibres (projecting over $\mathbb{P}^1$) are Schubert cycles. On the variety $\mathrm{Gr}(1, 3)$ the fact that the normal bundle is not ample has been checked by N. Goldstein [5], [6].

This paper is divided into four sections.

In the first one we fix basic notations and we give some preliminary results, in particular the stability of S and Q.

In the second section we construct the class $\Gamma_{k,n}$ of the above mentioned bundles and we study their stability.

In the third one we describe the splitting on lines and from this description we construct the moduli space of the class.

In the fourth one we describe the degeneracy locus of one or two generic sections.

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1. Notations and preliminaries

Let $\mathbb{P}^n$ be a fixed projective space of dimension n on $\mathbb{C}$. If A, B are linear subspaces of $\mathbb{P}^n$ we denote by $\langle A, B \rangle$ the linear subspace spanned by A and B.

We denote by $\mathrm{Gr}(k, n)$ the Grassmannian of linear subspaces $\mathbb{P}^k$ in $\mathbb{P}^n$ [15]. We denote by $F(r, s, n)$ the flag manifold which parametrizes the pairs $(\mathbb{P}^r, \mathbb{P}^s)$ of linear subspaces in $\mathbb{P}^n$ such that $\mathbb{P}^r \subset \mathbb{P}^s \subset \mathbb{P}^n$. This variety is the incidence variety in the product $\mathrm{Gr}(r, n) \times \mathrm{Gr}(s, n)$ and its dimension is $(s+1)(n-s) + (r+1)(s-r)$.

If we fix two subspaces $\mathbb{P}_0^{k-1}$, $\mathbb{P}_0^{k+1} \subset \mathbb{P}^n$ with $\mathbb{P}_0^{k-1} \subset \mathbb{P}_0^{k+1}$, the subvariety $\{\mathbb{P}^k | \mathbb{P}_0^{k-1} \subset \mathbb{P}^k \subset \mathbb{P}_0^{k+1}\} \subset \mathrm{Gr}(k, n)$ is called a line on the Grassmannian. These subvarieties are mapped by the Plücker embedding of $\mathrm{Gr}(k, n)$ into straight lines in $\mathbb{P}^{\binom{n+1}{k+1}-1}$. Then $F(k-1, k+1, n)$ is the variety of lines in $\mathrm{Gr}(k, n)$ [10].

On $\mathrm{Gr}(k, n)$ there is a canonical exact sequence

$$0 \to S \to \mathcal{O}^{n+1} \to Q \to 0, \tag{1.1}$$

where S has rank $k+1$ and is called the universal (tautological) bundle, Q has rank $n-k$ and is called the quotient bundle. If E is a vector bundle we denote by E^* its dual. We refer to [18] for basic facts about vector bundles. It is well known ([4], [22]) that S^* and Q are generated by their global sections and that

$$\mathrm{h}^i(S^*) = \mathrm{h}^i(Q) = \begin{cases} n+1 & \text{for } i=0, \\ 0 & \text{otherwise,} \end{cases} \quad \mathrm{h}^i(S) = \mathrm{h}^i(Q^*) = 0 \quad \text{for each } i,$$

where, for every coherent sheaf F, $\mathrm{h}^i(F) = \dim_{\mathbb{C}} \mathrm{H}^i(\mathrm{Gr}(k, n), F)$.

We have the following geometrical description of sections of S^* and Q:

1) every linear functional on $\mathbb{C}^{n+1}$ defines by restriction a linear functional on each subspace $\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}$ and then a holomorphic section of S^*.

2) every vector of $\mathbb{C}^{n+1}$ defines by quotient projection an element of $\mathbb{C}^{n+1}/\mathbb{C}^{k+1}$ for each subspace $\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}$ and then a holomorphic section of Q.

It is well known that in this way we obtain all holomorphic sections of S^* and Q. In particular, by looking at the zero loci of these sections, we get that the top Chern classes of these bundles are represented by some particular Schubert cycles, which we denote by ([9], [19])

$$c_{k+1}(S^*) = \{\mathbb{P}^k \mid \mathbb{P}^k \subset \mathbb{P}^{n-1} \text{ fixed}\} := Z_1, \tag{1.2a}$$

$$c_{n-k}(Q) = \{\mathbb{P}^k \mid p \in \mathbb{P}^k,\ p \text{ fixed point in } \mathbb{P}^n\} := Z_2. \tag{1.2b}$$

We recall also that the Chern class $c_i(E)$ of a globally generated vector bundle E of rank n is represented by the degeneracy locus of $n-i+1$ generic sections of E ([9]).

The cycles which represent the Chern classes of Q were classically called special cycles. It is well known that such cycles generate the cohomology ring

$$\mathrm{H}^*(\mathrm{Gr}(k, n), \mathbb{Z}).$$

This ring is generated as a $\mathbb{Z}$-module by the Schubert cycles $(a_0, \ldots, a_k)$ (for notations see [15]).

In [14], cap. XIV, p. 364, the degree of the Schubert cycles, mapped in $\mathbb{P}^{\binom{n+1}{k+1}-1}$ by the Plücker embedding, is computed. The degree can be intrinsically defined as the intersection of the cycle with a power of the generator h of

$$\mathrm{H}^{2(n-k)(k+1)-2}(\mathrm{Gr}(k, n), \mathbb{Z}) \cong \mathbb{Z}$$

corresponding to the bundle $\mathcal{O}(1)$, with exponent equal to the dimension of the cycle.

Let ([15]) $q=\sum_{i=0}^{k} a_i-\frac{k(k+1)}{2}$ be the dimension of the Schubert cycle $(a_0, \dots, a_k)$ and $d=(n-k)(k+1)$ the dimension of $\mathrm{Gr}(k, n)$. The degree of the Schubert cycle $(a_0, \dots, a_k)$ is given by

$$\text{(1. 3)} \qquad \deg(a_0, \dots, a_k)=\frac{q!}{a_0! \cdots a_k!} \prod_{\lambda>\mu} (a_\lambda-a_\mu).$$

In particular, the degree of the variety $\mathrm{Gr}(k, n)$, corresponding to the intersection multiplicity h^d, is given by

$$\text{(1. 4)} \qquad t=\frac{d!\,k!\,(k-1)! \cdots 2!}{(n-k)!\,(n-k+1)! \cdots n!}.$$

In (complex) codimension two, the cohomology module is generated by $c_2(S^*)$ and $c_2(Q)$, which correspond respectively to the cycles

$$(n-k-1, n-k, n-k+2, \dots, n-1, n) \quad \text{and} \quad (n-k-2, n-k+1, n-k+2, \dots, n-1, n).$$

Their degrees are then given by

$$\text{(1. 5a)} \qquad c_2(S^*) \cdot \mathrm{h}^{d-2}=\frac{t}{2(d-1)} k(n-k+1),$$

$$\text{(1. 5b)} \qquad c_2(Q) \cdot \mathrm{h}^{d-2}=\frac{t}{2(d-1)}(k+2)(n-k-1).$$

We shall need also the following formulas, obtained by the Hirzebruch-Riemann-Roch theorem ([11], [13]), where χ is the Euler-Poincaré characteristic and $F(m)$ denotes $F \otimes \mathcal{O}(m)$ $(m \in \mathbb{Z})$ for every coherent sheaf F:

$$\text{(1. 6)} \qquad \chi(S^*(m))=\frac{m^d}{d!} t(k+1)+\frac{m^{d-1}}{(d-1)!} \cdot \frac{t[2+(k+1)(n+1)]}{2}+\cdots+(n+1),$$

$$\text{(1. 7)} \qquad \chi(Q(m))=\frac{m^d}{d!} t(n-k)+\frac{m^{d-1}}{(d-1)!} \cdot \frac{t[2+(n-k)(n+1)]}{2}+\cdots+(n+1),$$

$$\text{(1. 8)} \qquad \chi(\mathcal{O}(m))=\frac{m^d}{d!} t+\frac{m^{d-1}}{(d-1)!} \cdot \frac{t(n+1)}{2}+\cdots+1.$$

Let us now introduce the number

$$\text{(1. 9)} \qquad \delta(k, n):=(n-1)-|(n+1)-2(k+1)|$$

which measures "how far $\mathrm{Gr}(k, n)$ is from being a projective space". We observe that $\delta(k, n)=\delta(n-k-1, n)$ and thus δ is invariant for the duality $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k-1, n)$. We have that

$$0 \leqq \delta(k, n) \leqq n-1 \quad \text{and} \quad \delta(k, n)=0 \quad \text{if and only if} \quad \mathrm{Gr}(k, n) \cong \mathbb{P}^n.$$

$\delta(k, n) = n-1$ (for $n>1$) if and only if there exist automorphisms of $\mathrm{Gr}(k, n)$ which are not induced by linear projective transformations of $\mathbb{P}^n$ but which come from polarities [3]. The automorphism group of these Grassmannians, which we shall call Grassmannians with polarities, consists of two connected components, each of them is isomorphic to $\mathrm{PGL}(n+1)$. For example on $\mathrm{Gr}(1, 3)$ there exist automorphisms that exchange the cycle (03) with the cycle (12) (for notations see [15]).

We recall also the following definitions for a vector bundle E on $\mathrm{Gr}(k, n)$.

E is called decomposable if $E = F \oplus G$ with F, G vector bundles of positive rank; indecomposable if it is not decomposable.

E is called simple if $\mathrm{End}\, E = \mathrm{H}^0(E \otimes E^*) \cong \mathbb{C}$.

E is called Mumford-Takemoto (semi)stable, shortly MT-(semi)stable, if for every coherent subsheaf F of E with E/F torsion-free and with $0 < \mathrm{rank}\, F < \mathrm{rank}\, E$ we have

$$\frac{c_1(F)}{\mathrm{rank}\, F} < (\leqq) \frac{c_1(E)}{\mathrm{rank}\, E},$$

where we use the canonical isomorphism $\mathrm{H}^2(\mathrm{Gr}(k, n), \mathbb{Z}) \cong \mathbb{Z}$, namely we identify $c_1(\mathcal{O}(1)) = 1 \in \mathbb{Z}$.

E is called Gieseker-Maruyama (semi)stable, shortly GM-(semi)stable, if for every coherent subsheaf F of E with E/F torsion-free and with $0 < \mathrm{rank}\, F < \mathrm{rank}\, E$, we have

$$\frac{\chi(F(m))}{\mathrm{rank}\, F} < (\leqq) \frac{\chi(E(m))}{\mathrm{rank}\, E} \quad \text{for every sufficiently large } m.$$

We have the following implications:

MT-stable ⇒ GM-stable ⇒ simple ⇒ indecomposable,

MT-stable ⇒ GM-stable ⇒ GM-semistable ⇒ MT-semistable.

In fact the proofs given in [18] for projective spaces extend trivially to Grassmannians.

E is called uniform if $E|_r \cong E|_{r'}$ for every r, r' lines in $\mathrm{Gr}(k, n)$. For example S and Q are uniform and we have

$$S|_r \cong \mathcal{O}_r^k \oplus \mathcal{O}_r(-1), \quad Q|_r \cong \mathcal{O}_r^{n-k-1} \oplus \mathcal{O}_r(1) \quad \text{for every line } r.$$

$S^* \otimes Q \cong T_G$ is the tangent bundle and we have

$$T_{G|r} \cong \mathcal{O}_r^{k(n-k-1)} \oplus \mathcal{O}_r(1)^{n-1} \oplus \mathcal{O}_r(2) \quad \text{for every line } r.$$

The uniform bundles have been studied in [10].

We shall use also the following theorem of A. Papantonopoulou ([19]), to which we refer for the necessary definitions.

Theorem 1. 1 (Papantonopoulou). *If* $C \subset \mathrm{Gr}(k, n)$ *is a smooth curve, then*

$$Q|_C \text{ is not ample} \Leftrightarrow C \text{ lies in some cycle } Z_1 \text{ (see (1. 2))},$$

$$S^*|_C \text{ is not ample} \Leftrightarrow C \text{ lies in some cycle } Z_2 \text{ (see (1. 2))}.$$

In the proof of the Lemma 1. 3 we shall need the vanishing theorem of Bott on homogeneous vector bundles. We sketch only the weak form of this theorem that we shall use and we refer to [2], [8], [21], [22] for a complete treatment of this topic.

We consider the Grassmannian $\mathrm{Gr}(k, n)$ as the complex homogeneous manifold $\mathrm{SL}(n+1, \mathbb{C})/P$ where

$$P = \left\{ \begin{pmatrix} h_1 & 0 \\ h_3 & h_4 \end{pmatrix} \in \mathrm{SL}(n+1) \mid h_4 \in \mathrm{GL}(k+1) \right\}.$$

$\mathfrak{sl}(n+1) = \{A \in M(n+1) \mid \mathrm{tr}\, A = 0\}$ is the semisimple Lie algebra of $\mathrm{SL}(n+1)$ and

$$\mathfrak{h} = \{A \in \mathfrak{sl}(n+1) \mid A \text{ is diagonal}\}$$

is a Cartan subalgebra of $\mathfrak{sl}(n+1)$. Let $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ be a base of the root system Φ of $\mathfrak{sl}(n+1)$ with respect to $\mathfrak{h}$ and let $\lambda_1, \ldots, \lambda_n \in \mathfrak{h}^*$ be the fundamental weights of $\mathfrak{sl}(n+1)$, characterized by the property $(\lambda_i, \alpha_j) = \delta_{ij}$, where $\frac{1}{2(n+1)}(\ ,\ )$ is the Killing form in $\mathfrak{h}^*$ and δ_{ij} is the Kronecker symbol. Let $e_{ij} \in \mathfrak{gl}$ be the matrix with 1 at the place (i, j) and all other entries equal to zero, $\{e'_{ij}\}$ the dual basis of $\{e_{ij}\}$. Then the

$$x_i = e_{i,i} - e_{i+1,i+1} \quad (i = 1, \ldots, n)$$

are a basis for $\mathfrak{h}$, the λ_i are the dual basis, and $\alpha_i = e'_{i,i} - e'_{i+1,i+1}$.

We set $\delta = \sum_{i=1}^{n} \lambda_i = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. A weight $\lambda = \sum_{i=1}^{n} n_i \lambda_i$ $(n_i \in \mathbb{Z})$ is called singular if $(\lambda, \alpha) = 0$ for at least one $\alpha \in \Phi$, otherwise is called regular. A homogeneous vector bundle E_ϱ of rank r on $\mathrm{Gr}(k, n) \cong \mathrm{SL}(n+1)/P$ is by definition a bundle arising from a representation $\varrho\colon P \longrightarrow \mathrm{GL}(r)$.

Theorem 1. 2 (Bott). *Let* E_ϱ *be a homogeneous vector bundle on*

$$\mathrm{Gr}(k, n) \cong \mathrm{SL}(n+1)/P$$

which is defined by an irreducible representation ϱ, *and let* λ *be the highest weight of* $\mathrm{D}\varrho\colon \mathfrak{p} \longrightarrow \mathfrak{gl}(r)$. *If* $\lambda + \delta$ *is singular then*

$$\mathrm{H}^i(\mathrm{Gr}(k, n), E_\varrho) = 0 \ \forall i.$$

Lemma 1. 3. *On the variety* $\mathrm{Gr}(k, n)$, *the following are true*

$$\mathrm{h}^i(S \otimes Q) = \begin{cases} \binom{n+1}{2} & \text{for } i=0,\ k=n-1 \\ 0 & \text{otherwise,} \end{cases}$$

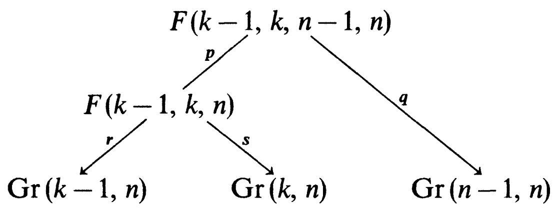
$$\mathrm{h}^i(S^* \otimes Q^*) = \begin{cases} \binom{n+1}{2} & \text{for } i=0,\ k=0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\mathrm{h}^0(S \otimes Q) = \mathrm{h}^0(S^* \otimes Q^*) = 0$ *if and only if the Grassmannian* $\mathrm{Gr}(k, n)$ *is not a projective space.*

Proof. Because of the duality $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k-1, n)$ it suffices to verify the first formula. If $k=n-1$ then $S \otimes Q \cong \Omega^1_{\mathbb{P}^n}(2)$, if $k=0$ then $S \otimes Q \cong T\mathbb{P}^n(-2)$ and in these cases the result is well known [18].

Thus we may suppose $0<k<n-1$. For the convenience of the reader we give first an argument which does not need the Bott vanishing theorem and works for $i=0$. This will suffice until Section 3.

We consider the following projection maps (with obvious notations):



We set $\tilde{\mathrm{H}}_{k-1} = r^*\mathcal{O}(1)$, $\tilde{\mathrm{H}}_k = s^*\mathcal{O}(1)$, which form a basis for $\mathrm{Pic}\, F(k-1, k, n)$ [4] and $\mathrm{H}_{k-1} = p^*\tilde{\mathrm{H}}_{k-1}$, $\mathrm{H}_k = p^*\tilde{\mathrm{H}}_k$, $\mathrm{H}_{n-1} = q^*\mathcal{O}(1)$ which form a basis for

$$\mathrm{Pic}\, F(k-1, k, n-1, n).$$

T. Fujita in [4], p. 413, with notations slightly different from ours, shows that

$$s_*(\tilde{\mathrm{H}}_{k-1} - \tilde{\mathrm{H}}_k) = S,\ R^i s_*(\tilde{\mathrm{H}}_{k-1} - \tilde{\mathrm{H}}_k) = 0 \quad \text{for} \quad i>0,$$
$$p_*\mathrm{H}_{n-1} = s^*Q,\ R^i p_*\mathrm{H}_{n-1} = 0 \quad \text{for} \quad i>0.$$

Then we get

$$\begin{aligned} \mathrm{H}^0(\mathrm{Gr}(k, n), S \otimes Q) &\cong \mathrm{H}^0(F(k-1, k, n), (\tilde{\mathrm{H}}_{k-1} - \tilde{\mathrm{H}}_k) \otimes s^*Q) \\ &\cong \mathrm{H}^0(F(k-1, k, n-1, n), \mathrm{H}_{k-1} - \mathrm{H}_k + \mathrm{H}_{n-1}). \end{aligned}$$

The last group is zero by [4], Theorem 4. 10, and this gives the result for $i=0$.

In general we observe that $S \otimes Q$ is induced on $\mathrm{Gr}(k, n) \cong \mathrm{SL}(n+1)/P$ by the representation $\varrho \colon P \to \mathrm{GL}((n-k)(k+1))$ defined by $\varrho\left(\begin{pmatrix} h_1 & 0 \\ h_2 & h_4 \end{pmatrix}\right) = h_1 \otimes h_4$ ([22]). It is easy to check that this representation is irreducible and that the highest weight of $\mathrm{D}\varrho$ is $\lambda = \lambda_1 - \lambda_{n-k} + \lambda_{n-k+1}$ ([22]). Thus $(\lambda + \delta, \alpha_{n-k}) = 0$, $\lambda + \delta$ is singular and by Theorem 1. 2 all the cohomology groups of $S \otimes Q$ vanish.

Now we want to prove that S and Q are stable. We need first a result about rational normal curves. If C' is a rational normal curve in $\mathbb{P}^n$, we denote by C the curve in $\mathrm{Gr}(k, n)$ obtained by taking the k-obsculating planes to C'. We observe that by Theorem 1. 1, $S^*|_C$ and $Q|_C$ are both ample. The set of all curves in $\mathrm{Gr}(k, n)$ obtained in this way is in bijection with the set of all rational normal curves in $\mathbb{P}^n$, which is in a natural way a complex variety isomorphic to the homogeneous space $\mathrm{PGL}(n+1)/\mathrm{PGL}(2)$ (all rational normal curves are projectively equivalent [9]).

Lemma 1. 4. *If S is a subvariety in* $\mathrm{Gr}(k, n)$ *of codimension* $\geqq 2$, *there exists a curve C of k-obsculating planes to some rational normal curve in $\mathbb{P}^n$ such that* $C \cap S = \emptyset$.

Proof. Let $W = \mathrm{PGL}(n+1)/\mathrm{PGL}(2)$ be the variety of curves of k-obsculating planes to some rational curve in $\mathbb{P}^n$. In the product $\mathrm{Gr}(k, n) \times W$ we consider the subvariety T of all pairs (p, C) such that $p \in C$. We have the natural projection maps

$$\begin{array}{ccc} & T & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathrm{Gr}(k, n) & & W \end{array}$$

which are both surjective. We need only to prove that $q(p^{-1}(S))$ is a proper subvariety of W. Now $\dim T = \dim W + 1$ and so every fibre of p has dimension $\dim W + 1 - \dim \mathrm{Gr}(k, n)$. Thus

$$\dim p^{-1}(S) = \dim W + 1 - \dim \mathrm{Gr}(k, n) + \dim S < \dim W$$

and then $\dim q(p^{-1}(S)) < \dim W$ as we claimed.

Theorem 1. 5. *S and Q are* MT-*stable, in particular simple.*

Proof. We will show that S^* is MT-stable. Let E be a coherent subsheaf of S^* with S^*/E torsion-free and with $0 < \operatorname{rank} E < \operatorname{rank} S^*$. We have the following exact sequence of sheaves

$$0 \to E \to S^* \to S^*/E \to 0.$$

As $c_1(S^*) = 1$, by definition it suffices to prove that $c_1(S^*/E) \geqq 1$. Assume that $c_1(S^*/E) \leqq 0$. As S^*/E is free outside of a closed set of codimension $\geqq 2$ [18], by Lemma 1. 4 there exist a curve C of k-obsculating planes to some rational normal curve C' in $\mathbb{P}^n$ on which $(S^*/E)|_C$ is a quotient bundle of $S^*|_C$. Then by the Theorem 1. 1, $(S^*/E)|_C$ would be ample on the rational C, and this contradicts the assumption $c_1(S^*/E) \leqq 0$. Then S^* is stable and S is stable too. The analogous result for Q follows by the duality

$$\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k-1, n).$$

Lemma 1. 6. *On the variety* Gr(k, n) *the following are true*:

$$\mathrm{h}^i(S \otimes S^*) = \mathrm{h}^i(Q \otimes Q^*) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Because of the duality Gr$(k, n) \cong$ Gr$(n-k-1, n)$ it suffices to verify the formula for $S \otimes S^*$. By [8], p. 134 we have

$$\mathrm{h}^i(S^* \otimes Q) = \mathrm{h}^i(T\mathrm{Gr}(k, n)) = \begin{cases} \dim \mathfrak{sl}(n+1) = (n+1)^2 - 1 & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, after tensoring the sequence (1. 1) by S^* we obtain

$$0 \to S \otimes S^* \to (S^*)^{n+1} \to S^* \otimes Q \to 0.$$

By Theorem 1. 5, $\mathrm{h}^0(S \otimes S^*) = 1$, and then the cohomology exact sequence gives the thesis.

2. The construction of the class of bundles $\Gamma_{k,n}$ and the study of the stability

Let $q^*: (\mathbb{C}^{n+1})^* \setminus \{0\} \to \check{\mathbb{P}}^n$, $q: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the natural quotient projections. $S^* \oplus Q$ is an $(n+1)$-bundle on Gr(k, n) generated by its global sections. A section of S^* is given by a nonzero linear functional $F: \mathbb{C}^{n+1} \to \mathbb{C}$ and vanishes on a Z_1-cycle corresponding to the hyperplane $q^*(F) = \mathbb{P}^{n-1} \subset \mathbb{P}^n$. A section of Q is given by a vector $v \in \mathbb{C}^{n+1} \setminus \{0\}$ and vanishes on a Z_2-cycle corresponding to the point $q(v) = p \in \mathbb{P}^n$.

If $p \notin \mathbb{P}^{n-1}$, we have obviously $Z_1 \cap Z_2 = \emptyset$, and then the pair of sections just considered define a nowhere vanishing section of $S^* \oplus Q$, that is, a trivial line subbundle of $S^* \oplus Q$. Thus, if we fix a nowhere vanishing section of $S^* \oplus Q$, we get the following exact sequence of vector bundles

(2. 1) $$0 \to \mathcal{O} \to S^* \oplus Q \to E \to 0$$

where E is defined by the sequence and has rank n.

We denote by $\Gamma_{k,n}$ the set of bundles E defined in this way. Every $E \in \Gamma_{k,n}$ is generated by its global sections. Because of the geometric description given above, Aut(Gr(k, n)) acts transitively on $\Gamma_{k,n}$.

If $n = 1$, $E \cong \mathcal{O}_{\mathbb{P}^1}(2)$, so from now on we can assume $n \geqq 2$.

Lemma 2. 1. *Let $E \in \Gamma_{k,n}$. The following are true*:

(i) $c_r(E) = c_r(S^* \oplus Q) = \sum_{i=0}^{r} c_i(S^*)\, c_{r-i}(Q)$ *for every* $r = 0, \ldots, n$,

(ii) $c_r(E) = 2 \sum_{i=0}^{[\frac{r}{2}]} c_{2i}(S^*)\, c_{r-2i}(Q)$.

Proof. (i) is obvious from (2. 1). From the canonical sequence (1. 1) we obtain, for every $r \geqq 1$:

$$\sum_{i=0}^{r} (-1)^i c_i(S^*)\, c_{r-i}(Q) = 0.$$

This identity together with (i) gives (ii).

Remark 2. 2. In particular, if we write $c_r(E) = \sum a_i \xi_i$ where $a_i \in \mathbb{Z}$ and ξ_i are Schubert cycles, the a_i's can have only the values 0 or 2. The Schubert calculus determines which coefficients among the a_i's are zero, but this description will not be needed in the sequel.

Example 2. 3. On the variety $\mathrm{Gr}(1, 5)$, following the notations of [15] we get for every $E \in \Gamma_{1,5}$: $c_1(E) = 2(35)$, $c_2(E) = 2(25) + 2(34)$, $c_3(E) = 2(15) + 2(24)$, $c_4(E) = 2(05) + 2(14) + 0(23)$, $c_5(E) = 2(04) + 0(13)$.

Lemma 2. 4. *Let* $E \in \Gamma_{k,n}$. *Then*:

$$\mathrm{h}^0(E(m)) = 0 \quad \textit{for} \quad m \leqq -2, \quad \mathrm{h}^0(E(-1)) = \begin{cases} 1 & \textit{for} \quad k = 0,\ n-1, \\ 0 & \textit{for} \quad 0 < k < n-1, \end{cases}$$

for $m \geqq 0$:

$$\mathrm{h}^0(E(m)) = \chi(E(m)) = \frac{m^d}{d!} \cdot tn + \frac{m^{d-1}}{(d-1)!} \cdot \frac{t(n^2+n+4)}{2} + \cdots + (2n+1),$$

$$\mathrm{h}^i(E^*) = \begin{cases} 1 & \textit{for} \quad i = 1 \\ 0 & \textit{for} \quad i \neq 1. \end{cases}$$

If $0 < k < n-1$, *we have* $\mathrm{h}^i(E(m)) = 0$ $\forall m$, $\forall i$: $0 < i < d-1$.

Proof. It follows from (1. 6), (1. 7), (1. 8) and the long cohomology sequences associated to (2. 1) and to its dual sequence.

Lemma 2. 5. *Let* $E \in \Gamma_{k,n}$. *We have*

$$\dim_{\mathbb{C}} \mathrm{Hom}(E, S^*) = \mathrm{h}^0(E^* \otimes S^*) = \begin{cases} = 0 & \textit{for} \quad k \neq 0 \\ \geqq \binom{n}{2} & \textit{for} \quad k = 0, \end{cases}$$

$$\dim_{\mathbb{C}} \mathrm{Hom}(E, Q) = \mathrm{h}^0(E^* \otimes Q) = \begin{cases} = 0 & \textit{for} \quad k \neq n-1 \\ \geqq \binom{n}{2} & \textit{for} \quad k = n-1. \end{cases}$$

Proof. After tensoring by S^* the sequence dual to (2. 1) we get

(2. 2) $$0 \to E^* \otimes S^* \to (S \otimes S^*) \oplus (Q^* \otimes S^*) \to S^* \to 0.$$

We suppose first $k \neq 0$. Then, as by Lemma 1. 3, $\mathrm{H}^0(Q^* \otimes S^*) = 0$ and by Theorem 1. 5 S is simple, in cohomology we obtain

$$0 \to \mathrm{H}^0(E^* \otimes S^*) \to \underset{\substack{\| \\ \mathbb{C}}}{\mathrm{H}^0(S \otimes S^*)} \xrightarrow{\varphi} \mathrm{H}^0(S^*).$$

Let F be the functional on $\mathbb{C}^{n+1}$ which defines the section of S^* that corresponds to the bundle E. The map φ acts in the following way: if $a \in \mathrm{End}(S)$, we have $\varphi(a) = a \circ F = a' \cdot F$ where $a' \in \mathbb{C}$. Then φ is injective, and this shows that $\mathrm{H}^0(E^* \otimes S^*) = 0$. If $k = 0$ we have the sequence (see Lemma 1. 3)

$$0 \to \mathrm{H}^0(E^* \otimes S^*) \to \mathbb{C} \oplus \mathbb{C}^{\binom{n+1}{2}} \to \mathbb{C}^{n+1}$$

and thus $\mathrm{h}^0(E^* \otimes S^*) \geqq \binom{n+1}{2} + 1 - n - 1 = \binom{n}{2}$. The proof in the case of $E^* \otimes Q$ is similar.

We want to prove now the following

Theorem 2. 6. *Let* $E \in \Gamma_{k,n}$. *Then*

(i) *E is always indecomposable,*

(ii) *E is simple if and only if* $\delta(k, n) > 0$,

(iii) *E is* MT-*semistable if and only if* $\delta(k, n) \geqq n - 2$,

(iv) *E is* GM-*semistable if and only if* $\delta(k, n) = n - 1$,

(v) *E is* MT-*stable (and then* GM-*stable) if and only if* $\delta(k, n) = n - 1$.

Theorem 2. 6 is proved by the following four Lemmas 2. 7, 2. 8, 2. 9 and 2. 10.

Lemma 2. 7. *Let* $E \in \Gamma_{k,n}$. *If the Grassmannian* Gr(k, n) *is not a projective space then E is simple. If the Grassmannian* Gr(k, n) *is a projective space then E is not simple but it is indecomposable.*

Proof. After tensoring the sequence (2. 1) by E^* we obtain

$$(2.3) \qquad 0 \to E^* \to (S^* \otimes E^*) \oplus (Q \otimes E^*) \to E \otimes E^* \to 0.$$

Let first $\delta(k, n) > 0$. Then the cohomology exact sequence, by Lemmas 2. 4 and 2. 5, gives

$$0 \to \mathrm{H}^0(E \otimes E^*) \to \mathbb{C}$$

and then E is simple.

On the contrary, if $\delta(k, n) = 0$, in the same way we obtain

$$0 \to \mathbb{C}^{\binom{n}{2}} \to \mathrm{H}^0(E \otimes E^*).$$

Then if $n \geqq 3$, E cannot be simple. Also if $n=2$, E is not simple. This will follow by Lemma 2. 9 recalling that a 2-bundle on $\mathbb{P}^2$ is simple if and only if it is stable [18]. However, in the case $n=2$ the class $\Gamma_{k,n}$ is well known: these bundles arise as the restrictions to hyperplanes of null-correlation bundles on $\mathbb{P}^3$ ([18], p. 181).

To finish the proof we need only to show that on projective spaces every $E \in \Gamma_{k,n}$ is indecomposable. This fact is merely topologic. In fact from the Euler sequence (which is (1. 1)), it follows that the Chern polynomial [18] of $T\mathbb{P}^n(-1)$ is

$$p(T\mathbb{P}^n(-1)) = \frac{1}{1-t} = 1 + t + \cdots + t^n,$$

so that

$$p(E) = \frac{p(\mathcal{O}(1)) \cdot p(T\mathbb{P}^n(-1))}{p(\mathcal{O})} = \frac{1+t}{1-t} = 1 + 2t + \cdots + 2t^n.$$

This last polynomial is irreducible in $\mathbb{Z}[t]$. It is easy to verify this claim by projecting elements in $\mathbb{Z}/2\mathbb{Z}[t]$ (in this ring 1 is the only unit).

Lemma 2. 8. *Let $E \in \Gamma_{k,n}$. If $\delta(k, n) = n-1$ then E is* MT-*stable. If $\delta(k, n) \geqq n-2$ then E is* MT-*semistable.*

Proof. If $\delta(k, n) \geqq n-2$ we have to prove that each $E \in \Gamma_{k,n}$ is MT-semistable. For this let F be a coherent subsheaf of E with E/F torsion-free and with $0 < \operatorname{rank} F < \operatorname{rank} E$. It suffices to show that

$$\frac{2}{n} = \frac{c_1(E)}{\operatorname{rank} E} \leqq \frac{c_1(E/F)}{\operatorname{rank}(E/F)}.$$

There is a diagram:

$$\begin{array}{ccccccccc} & & & & S^* \oplus Q & & & & \\ & & & & \downarrow & \searrow^{\beta} & & & \\ 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & E/F & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

from which, if we set $P := \operatorname{Ker} \beta$, we get the exact sequence of sheaves

$$(2.4) \qquad 0 \to P \to S^* \oplus Q \to E/F \to 0.$$

Then we consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & P & \overset{\gamma}{\searrow} & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & S^* & \longrightarrow & S^* \oplus Q & \longrightarrow & Q & \longrightarrow & 0
\end{array}$$

We can identify $\operatorname{Ker} \gamma$ with a subsheaf of S^*, and we get the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} \gamma & \longrightarrow & P & \longrightarrow & \operatorname{Im} \gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^* & \longrightarrow & S^* \oplus Q & \longrightarrow & Q & \longrightarrow & 0
\end{array}$$

which, by the snake lemma, can be completed into the following (commutative) diagram of sheaves with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} \gamma & \longrightarrow & P & \longrightarrow & \operatorname{Im} \gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^* & \longrightarrow & S^* \oplus Q & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^*/\operatorname{Ker} \gamma & \longrightarrow & E/F & \longrightarrow & Q/\operatorname{Im} \gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2. 5}$$

Because of the hypothesis $0 < \operatorname{rank}(E/F)$, either

$$\operatorname{rank} \operatorname{Ker} \gamma < \operatorname{rank} S^* \quad \text{or} \quad \operatorname{rank} \operatorname{Im} \gamma < \operatorname{rank} Q.$$

In any case, by Theorem 1. 5 we get $c_1(\operatorname{Ker}\gamma)+c_1(\operatorname{Im}\gamma)\leqq 1$ so that

$$c_1(P)\leqq 1 \quad \text{by the first row of (2. 5)}$$

$$c_1(E/F)\geqq 1 \quad \text{by (2. 4).} \tag{2. 6}$$

There are the following three cases:

Case 1. If $\operatorname{rank}\operatorname{Ker}\gamma<\operatorname{rank}S^*$ and $\operatorname{rank}\operatorname{Im}\gamma<\operatorname{rank}Q$ then from Theorem 1. 5 it follows that $c_1(\operatorname{Ker}\gamma)\leqq 0$, $c_1(\operatorname{Im}\gamma)\leqq 0$ so that

$$c_1(E/F)\geqq 2\geqq 2\cdot\frac{\operatorname{rank}(E/F)}{n}$$

as claimed.

Case 2. If $\operatorname{rank}\operatorname{Ker}\gamma=\operatorname{rank}S^*$ we suppose first $\operatorname{rank}\operatorname{Im}\gamma\geqq 1$. In this case, by the first row of (2. 5)

$$\operatorname{rank}P\geqq\operatorname{rank}\operatorname{Ker}\gamma+1=\operatorname{rank}S^*+1$$

and by (2. 4)

$$\operatorname{rank}(E/F)\leqq(n+1)-(\operatorname{rank}S^*+1)=n-\operatorname{rank}S^*.$$

By the hypothesis $\delta(k,n)\geqq n-2$ we have $\operatorname{rank}S^*\geqq\frac{n}{2}$ and then $\operatorname{rank}(E/F)\leqq\frac{n}{2}$. Thus (2. 6) says just that $\frac{2}{n}\leqq\frac{c_1(E/F)}{\operatorname{rank}(E/F)}$ as we claimed.

On the contrary if $\operatorname{rank}\operatorname{Im}\gamma=0$, as $\operatorname{Im}\gamma$ is torsion-free (because is a subsheaf of Q) it must be zero, and then $Q\cong Q/\operatorname{Im}\gamma$. By diagram (2. 5), $S^*/\operatorname{Ker}\gamma$ is a rank zero subsheaf of E/F which is torsion-free, and then it must be zero, too. Then $E/F\cong Q/\operatorname{Im}\gamma\cong Q$ and we get a surjective morphism from E to Q, and this contradicts Lemma 2. 5.

Case 3. If $\operatorname{rank}\operatorname{Im}\gamma=\operatorname{rank}Q$, we suppose first $\operatorname{rank}\operatorname{Ker}\gamma\geqq 1$. In this case we have by the first row of (2. 5) $\operatorname{rank}P\geqq\operatorname{rank}Q+1$ and thus by (2. 4)

$$\operatorname{rank}E/F\leqq(n+1)-(\operatorname{rank}Q+1)=n-\operatorname{rank}Q.$$

By the hypothesis $\delta(k,n)\geqq n-2$ we have $\operatorname{rank}Q\geqq\frac{n}{2}$, so that $\operatorname{rank}E/F\leqq\frac{n}{2}$ and as in case 2 we can conclude that (2. 6) gives the thesis.

On the contrary if $\operatorname{rank}\operatorname{Ker}\gamma = 0$, then as before $\operatorname{Ker}\gamma = 0$, $P \cong \operatorname{Im}\gamma$ and the diagram (2. 5) becomes the following

$$
\text{(2. 7)}\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & P & \xrightarrow{\cong} & \operatorname{Im}\gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^* & \longrightarrow & S^* \oplus Q & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^*/\operatorname{Ker}\gamma & \longrightarrow & E/F & \longrightarrow & Q/\operatorname{Im}\gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We want to show that the rank zero sheaf $Q/\operatorname{Im}\gamma$ is torsion-free. Consider the diagram (2. 7). We have a morphism $t_1 \colon Q \to S^* \oplus Q$ which splits the middle row, so by the assumption $P \cong \operatorname{Im}\gamma$ we can construct a morphism $t_2 \colon Q/\operatorname{Im}\gamma \to E/F$ which splits the last row, in such a way that the right-down square of (2. 7) becomes

$$
\begin{array}{ccc}
S^* \oplus Q & \overset{t_1}{\rightleftarrows} & Q \\
\downarrow & & \downarrow \\
E/F & \overset{t_2}{\rightleftarrows} & Q/\operatorname{Im}\gamma
\end{array}
$$

We obtain

$$E/F \cong (S^*/\operatorname{Ker}\gamma) \oplus (Q/\operatorname{Im}\gamma).$$

At last $Q/\operatorname{Im}\gamma$ is torsion-free and then it is zero. We get $E/F \cong S^*/\operatorname{Ker}\gamma \cong S^*$ and we have a surjective morphism from E to S^*, which once again contradicts Lemma 2. 5. The proof of MT-semistability is so completed.

If $\delta(k, n) = n - 1$, we observe that n must be odd. In this case then

$$\text{h.c.f.}(\operatorname{rank} E, c_1(E)) = \text{h.c.f.}(n, 2) = 1$$

and with this assumption it is a trivial fact that stability is equivalent to semistability [18]. This concludes the proof of Lemma 2. 8.

Lemma 2. 9. *Let $E \in \Gamma_{k,n}$. If $\delta(k, n) < n - 1$ then E is not* MT-*stable. If $\delta(k, n) < n - 2$ then E is not* MT-*semistable.*

Proof. From the sequence (2. 1) we get that S^* and Q inject as subsheaves of E (but not as subbundles!). If $\delta(k, n) < n-1$, in the case $2(k+1) \leqq n$ we obtain

$$\frac{c_1(E)}{\operatorname{rank} E} = \frac{2}{n} \leqq \frac{1}{k+1} = \frac{c_1(S^*)}{\operatorname{rank} S^*}$$

and this inequality contradicts stability. In the case $2(k+1) \geqq n+2$ we can consider Q in place of S^* and that contradicts stability again.

If $\delta(k, n) < n-2$ in a similar way we can suppose $2(k+1) \leqq n-1$ so that $2(k+1) < n$ and we obtain

$$\frac{c_1(E)}{\operatorname{rank} E} = \frac{2}{n} < \frac{1}{k+1} = \frac{c_1(S^*)}{\operatorname{rank} S^*}.$$

This inequality contradicts semistability.

Lemma 2. 10. *If $\delta(k, n) = n-2$ then E is not* GM*-semistable.*

Proof. By the duality $\operatorname{Gr}(k, n) \cong \operatorname{Gr}(n-k-1, n)$ it suffices to consider Grassmannians $\operatorname{Gr}(k, 2k+2)$. Then we claim that the subsheaf S^* of E satisfies

$$\frac{\chi(S^*(m))}{\operatorname{rank} S^*} > \frac{\chi(E(m))}{\operatorname{rank} E} \quad \text{for } m \text{ sufficiently large.}$$

We emphasize that in this case the MT-semistability is not contradicted.

We have to show that $2\chi(S^*(m)) > \chi(E(m))$ for $m \gg 0$. From the sequence (2. 1) and from the canonical sequence (1. 1) we get the identities

$$\chi(S(m)) + \chi(Q(m)) = (2k+3)\,\chi(\mathcal{O}(m)),$$
$$\chi(E(m)) = \chi(S^*(m)) + \chi(Q(m)) - \chi(\mathcal{O}(m)) \qquad \forall m \in \mathbb{Z}$$

and then it suffices to prove that

$$\text{(2. 8)} \qquad \chi(S(m)) + \chi(S^*(m)) > (2k+2)\,\chi(\mathcal{O}(m)) \quad \text{for} \quad m \gg 0.$$

Set $d = (k+1)(k+2) = \dim_{\mathbb{C}} \operatorname{Gr}(k, 2k+2)$ and $t = \deg \operatorname{Gr}(k, 2k+2)$ as in (1. 4). Both the members of (2. 8) are polynomials in the variable m of degree d. We denote by $[X]_p$ the component of codimension p of the element X in the Chow ring $A(\operatorname{Gr}(k, 2k+2)) \otimes \mathbb{Q}$, and by ξ the generator in codimension zero.

By the Hirzebruch-Riemann-Roch theorem ([11], [13]) the two members of (2. 8) are respectively equal to

$$[\operatorname{ch}(\mathcal{O}(m)) \cdot (\operatorname{ch} S + \operatorname{ch} S^*) \cdot \operatorname{td} \operatorname{Gr}(k, 2k+2)]_d = \frac{t(2k+2)}{d!} m^d + \frac{t(2k+2)(2k+3)}{(d-1)!2} m^{d-1} + \cdots \text{(lower deg. terms)}$$

$$(2k+2)[\operatorname{ch} \mathcal{O}(m) \cdot \operatorname{td} \operatorname{Gr}(k, 2k+2)]_d = \frac{t(2k+2)}{d!} m^d + \frac{t(2k+2)(2k+3)}{(d-1)!2} m^{d-1} + \cdots \text{(lower deg. terms)}$$

where we set ch = Chern character, td = Todd class.

Then the main point is the sign of the term

$$\mathcal{O}(1)^{d-2} \cdot [(\operatorname{ch} S + \operatorname{ch} S^* - (2k+2)\xi) \cdot (\operatorname{td} \operatorname{Gr}(k, 2k+2))]_2.$$

As $[\operatorname{ch} S + \operatorname{ch} S^* - (2k+2)\xi]_1 = [\operatorname{ch} S + \operatorname{ch} S^* - (2k+2)\xi]_0 = 0$ it suffices to prove (if we set $\mathrm{H} = \mathcal{O}(1)$) that

$$\text{(2. 9)} \qquad \mathrm{H}^{d-2} \cdot (c_1(S^*)^2 - 2c_2(S^*)) > 0.$$

From the canonical sequence (1. 1) we obtain the identity

$$c_2(S^*) + c_2(Q) - c_1(S^*)^2 = 0$$

and then it suffices to show that

$$c_2(Q) \cdot \mathrm{H}^{d-2} > c_2(S^*) \cdot \mathrm{H}^{d-2}.$$

Substituting $n = 2k+2$ in (1. 5), the last inequality becomes

$$\frac{t}{2(d-1)}(k+2)(k+1) > \frac{t}{2(d-1)} k(k+3).$$

This is trivially true, and so we have proved (2. 9) and then (2. 8).

This concludes also the proof of Theorem 2. 6.

3. The splitting on lines and the moduli space of the class $\Gamma_{k,n}$

Now let $(x_0, \ldots, x_n, y_0, \ldots, y_n)$ be homogeneous coordinates in $\check{\mathbb{P}}^n \times \mathbb{P}^n$ and let $M := (\check{\mathbb{P}}^n \times \mathbb{P}^n) \setminus \left(\sum_{i=0}^{n} x_i y_i = 0 \right)$. With the identification $(\mathbb{P}^{n-1}, p) \in \check{\mathbb{P}}^n \times \mathbb{P}^n$, we exclude exactly the pairs $(\mathbb{P}^{n-1}, p)$ for which $p \in \mathbb{P}^{n-1}$.

By definition, each $E \in \Gamma_{k,n}$ arises from a pair of sections of S^* and Q and by the description given in Section 2 we have a natural map

$$\pi: \Gamma_{k,n} \to M. \tag{3.1}$$

Now we consider the natural projection maps

$$\begin{array}{ccc} & W := M \times \mathrm{Gr}(k, n) & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ M & & \mathrm{Gr}(k, n) \end{array}$$

We define a bundle U on W by the sequence

$$0 \to \mathcal{O}_W \xrightarrow{f} p^*(S^* \oplus Q) \to U \to 0 \tag{3.2}$$

where f is constant on the fibres of q and is defined obviously in such a way that

$$U|_{\{\pi(E)\} \times \mathrm{Gr}(k,n)} \cong E.$$

Let now $\tilde{\Gamma}_{k,n}$ be the set of isomorphism classes of the bundles belonging to $\Gamma_{k,n}$.

Lemma 3.1. *If $\delta(k, n) > 0$ then π factors through the following diagram*

$$\begin{array}{ccc} \Gamma_{k,n} & \xrightarrow{\pi} & M \\ \downarrow & \nearrow{\scriptstyle \pi'} & \\ \tilde{\Gamma}_{k,n} & & \end{array}$$

and π' is a bijective map.

Proof. Let $E, E' \in \Gamma_{k,n}$ and $\phi: E \to E'$ be a bundle isomorphism. On Gr(k, n) we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & S^* \oplus Q & \longrightarrow & E & \longrightarrow & 0 \\ & & & & & \searrow{\scriptstyle \phi'} & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & S^* \oplus Q & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

where $\phi' \in \mathrm{H}^0((S \oplus Q^*) \otimes E') = \mathrm{Hom}(S^* \oplus Q, E')$.

We ask now if ϕ' can be extended to $\phi'': S^* \oplus Q \to S^* \oplus Q$ in such a way that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & S^* \oplus Q & \longrightarrow & E & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \phi''} & \searrow{\scriptstyle \phi'} & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & S^* \oplus Q & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

commutes.

After tensoring the sequence (2. 1) by $S^* \oplus Q$ we obtain the exact sequence

$$0 \to S \oplus Q^* \to \underset{\mathscr{E}nd\,(S^* \oplus Q)}{(S \oplus Q^*) \otimes (S^* \oplus Q)} \to \underset{\mathscr{H}om\,(S^* \oplus Q,\, E')}{(S \oplus Q^*) \otimes E'} \to 0,$$

that is,

$$0 \to S \oplus Q^* \to (S \otimes S^*) \oplus (S \otimes Q) \oplus (Q^* \otimes S^*) \oplus (Q^* \otimes Q) \to (S \otimes E') \oplus (Q^* \otimes E') \to 0.$$

The well known vanishing of $H^0(S \oplus Q^*)$ and $H^1(S \oplus Q^*)$ guarantees that there exists a unique extension ϕ''. Lemma 1. 3 and Theorem 1. 5 guarantee that ϕ'' is defined by two different homothetics on S^* and on Q. Then we have a natural injective map $\pi': \tilde{\Gamma}_{k,n} \to M$ and this map is surjective by construction.

We know from Section 1 that the flag manifold $F(k-1, k+1, n)$ is in a natural way the variety of lines in $\mathrm{Gr}(k, n)$. If we fix two subspaces $\mathbb{P}_0^{k-1} \subset \mathbb{P}_0^{k+1} \subset \mathbb{P}^n$ then the cycle $r = \{\mathbb{P}^k \subset \mathbb{P}^n \mid \mathbb{P}_0^{k-1} \subset \mathbb{P}^k \subset \mathbb{P}_0^{k+1}\} \subset \mathrm{Gr}(k, n)$ is a line.

Let $E \in \Gamma_{k,n}$. In order to describe the splitting of E on lines, we identify $\pi(E) = (\tilde{\mathbb{P}}^{n-1}, \tilde{p}) \in M$ (see (3. 1)) and we set

$$F'(E) := \{r \in F(k-1, k+1, n) \mid \tilde{p} \in \mathbb{P}_0^{k+1}, \mathbb{P}_0^{k-1} \subset \tilde{\mathbb{P}}^{n-1}\} \subset F(k-1, k+1, n).$$

Theorem 3. 2. $E|_r \cong \mathcal{O}_r^{n-2} \oplus \mathcal{O}_r(1)^2$ *if and only if* $r \notin F'(E)$,

$E|_r \cong \mathcal{O}_r^{n-1} \oplus \mathcal{O}_r(2)$ *if and only if* $r \in F'(E)$.

So $F'(E)$ *is the subvariety of jumping lines of* E: *it is a smooth subvariety of codimension* $n-1$ *in* $F(k-1, k+1, n)$.

Proof. If r is any line in $\mathrm{Gr}(k, n)$, by restricting to r the sequence (2. 1) we get the sequence

$$0 \to \mathcal{O}_r \xrightarrow{f} \underbrace{\mathcal{O}_r^k \oplus \mathcal{O}_r(1)}_{S^*|_r} \oplus \underbrace{\mathcal{O}_r^{n-k-1} \oplus \mathcal{O}_r(1)}_{Q|_r} \to E|_r \to 0.$$

After tensoring this sequence by $\mathcal{O}_r(-1)$ and looking at cohomology we obtain that the only possible splittings for $E|_r$ are those claimed in the theorem. We consider now the two maps defined by f:

$$0 \longrightarrow \mathcal{O}_r \xrightarrow{f_1} S^*|_r \cong \mathcal{O}_r^k \oplus \mathcal{O}_r(1),$$

$$0 \longrightarrow \mathcal{O}_r \xrightarrow{f_2} Q|_r \cong \mathcal{O}_r^{n-k-1} \oplus \mathcal{O}_r(-1).$$

f_1, f_2 are defined by the restrictions to r of the sections of S^* and Q which define E.

It is easy to see that $r \in F'(E)$ if and only if f_1, f_2 give sections (of $S^*|_r$ and $Q|_r$ respectively) both with a unique simple zero. In this case f_1, f_2 projected respectively onto the trivial summands $\mathcal{O}^k$ and $\mathcal{O}^{n-k-1}$ must be zero and so $E|_r$ must contain $\mathcal{O}^{n-1}$ as direct summand.

On the contrary if $r \notin F'(E)$ we identify

$$f_1 = (a_1, \dots, a_k, s_1), \quad f_2 = (b_1, \dots, b_k, s_2)$$

with $a_i, b_j \in \mathbb{C}$; $s_1, s_2 \in \mathrm{H}^0(r, \mathcal{O}_r(1))$.

We may assume that $a_1 \neq 0$. If $s_1 \equiv 0$, then $\mathcal{O}_r(1)$ is a subbundle of $E|_r$ and we have proved the thesis. If $s_1 \not\equiv 0$ let us consider

$$\tilde{f}_1 = (0, \dots, 0, s_1) \in \mathrm{H}^0(r, S^*|_r), \quad \tilde{f}_2(0, \dots, 0, 0) \in \mathrm{H}^0(r, Q|_r).$$

This pair $(\tilde{f}_1, \tilde{f}_2)$ projects onto an element of $\mathrm{H}^0(r, E|_r)$ which vanishes exactly where $(\tilde{f}_1, \tilde{f}_2)$ is proportional to (f_1, f_2) and then exactly on the unique zero of s_1. So we have constructed a section of $E|_r$ with just one simple zero. So we must exclude the case $E|_r \cong \mathcal{O}_r^{n-1} \oplus \mathcal{O}_r(2)$ and we have $E|_r \cong \mathcal{O}_r^{n-2} \oplus \mathcal{O}_r(1)^2$ as we claimed.

Remark 3. 3. In the case of the projective spaces the jumping lines are all the lines passing through the point $\tilde{p}$ defined by $\pi(E)$.

Now let X, S be complex spaces and let $p: S \times X \to S$ be the natural projection map. If E is a bundle on $S \times X$, we set $\mathsf{E}_s := \mathsf{E}|_{p^{-1}(s)}$.

For lack of a suitable reference we give a proof of the well known

Lemma 3. 4. *Let E, E' be vector bundles on $S \times X$ and S be reduced. If $\mathsf{E}_s, \mathsf{E}'_s$ are simple bundles $\forall s \in S$, then the following facts are equivalent:*

(i) $\mathsf{E}_s \cong \mathsf{E}'_s \quad \forall s \in S$.

(ii) *There exists a line bundle L on S such that $\mathsf{E}' \cong p^*(L) \otimes \mathsf{E}$.*

Proof. (ii) $\Rightarrow$ (i) is trivial. Thus we assume (i). $\mathsf{E}^* \otimes \mathsf{E}'$ is a vector bundle on $S \times X$, flat on S because p is flat. By hypothesis we have $\mathrm{H}^0(\{s\} \times X, (\mathsf{E}^* \otimes \mathsf{E}')_s) \cong \mathbb{C}$ and by the base-change theorem [18] we obtain that $p_*(\mathsf{E}^* \otimes \mathsf{E}')$ is a line bundle on S and its fibre at the point s is the vector space $\mathrm{H}^0(\{s\} \times X, (\mathsf{E}^* \otimes \mathsf{E}')_s)$ (at this point it is necessary to assume S reduced). We set $L = p_*(\mathsf{E}^* \otimes \mathsf{E}')$. The natural map

$$p^* L \to \mathsf{E}^* \otimes \mathsf{E}' \cong \mathcal{H}om(\mathsf{E}, \mathsf{E}')$$

induces a map

$$h: p^* L \otimes \mathsf{E} \to \mathcal{H}om(\mathsf{E}, \mathsf{E}') \otimes \mathsf{E}.$$

It is now easy to check that the composition of h with the contraction map

$$\mathcal{H}om(\mathsf{E}, \mathsf{E}') \otimes \mathsf{E} \to \mathsf{E}'$$

is a bundle isomorphism. Thus we obtain $\mathsf{E}' \cong p^*(L) \otimes \mathsf{E}$ and (ii) holds.

Now let S be a normal complex space. If $\delta(k, n) > 0$ let us define

$$\Theta_{\Gamma_{k,n}}(S) = \{\mathsf{E} | \mathsf{E} \text{ vector bundle on } S \times \mathrm{Gr}(k, n) \text{ such that } \mathsf{E}_s \in \Gamma_{k,n} \forall s \in S\}/\sim$$

where $\mathsf{E} \sim \mathsf{E}'$ if E, E' satisfy any of the two conditions (i), (ii) of Lemma 3. 4 (E_s is simple by Lemma 2. 7).

$\Theta_{\Gamma_{k,n}}$ is a functor from the category of normal complex spaces to the category of sets.

Theorem 3.5. *If $\delta(k,n)>0$ then the functor $\Theta_{\Gamma_{k,n}}$ is represented by the pair $\left((\mathbb{P}^n\times\mathbb{P}^n)\setminus\left(\sum_{i=0}^{n} x_i y_i=0\right), U\right)$, that is $(\mathbb{P}^n\times\mathbb{P}^n)\setminus\left(\sum_{i=0}^{n} x_i y_i=0\right)$ is a fine moduli space for $\Gamma_{k,n}$.*

Proof. Let S be a normal complex space and let $\mathsf{E}\in\Theta_{\Gamma_{k,n}}(S)$. Because of Lemma 3.1 we have a natural map $f\colon S\to M$ induced by π. We want to prove that this map is holomorphic, as X is normal it suffices to check it at the smooth points of S.

Let us consider the natural projection maps

$$\begin{array}{ccc} & S\times F(k-1,k,k+1,n) & \\ \swarrow^{p} & & \searrow^{q} \\ S\times F(k-1,k+1,n) & & S\times \mathrm{Gr}(k,n). \end{array}$$

By the semicontinuity theorem applied to the bundle $q^*\mathsf{E}$ and to the map p we obtain that the set

$$\{(s,l)\mid l \text{ is a jumping line in } \mathsf{E}_s\}\subset S\times F(k-1,k+1,n)$$

is an analytic subvariety of $S\times F(k-1,k+1,n)$. Let $s\in S$ be a smooth point, by Theorem 3.2 the subvariety of jumping lines of E_s determines uniquely the element $f(s)\in M$ and this description shows that f is holomorphic at s. By definition

$$(f^*U)_s\cong \mathsf{E}_s \quad \forall s\in S.$$

Conversely, every morphism $g\colon S\to M$ determines $g^*U\in\Theta_{\Gamma_{k,n}}(S)$ and so the functors $\Theta_{\Gamma_{k,n}}$ and $\mathrm{Hom}(-,M)$ are isomorphic.

We remember now that by M. Maruyama [16], [17] there exist a coarse moduli space $\mathscr{C}$ for GM-stable n-bundles on Grassmannians with the same Chern classes as $S^*\oplus Q$. $\mathscr{C}$ is algebraic. It is well known that $\mathscr{C}$ is smooth at the points corresponding to bundles E with $\mathrm{H}^2(E\otimes E^*)=0$ and that at a smooth point the dimension of $\mathscr{C}$ is equal to $\mathrm{h}^1(E\otimes E^*)$.

In our case we have

Lemma 3.6. *Let $E\in\Gamma_{k,n}$ and let $\delta(k,n)>0$. Then $\mathrm{h}^1(E\otimes E^*)=2n$, $\mathrm{h}^2(E\otimes E^*)=0$.*

Proof. The cohomology sequence associated to (2.2) is

$$\begin{array}{ccccccc} \mathrm{H}^0(E^*\otimes S^*) & \to & \mathrm{H}^0(S\otimes S^*) & \oplus & \mathrm{H}^0(Q^*\otimes S^*) & \to & \mathrm{H}^0(S^*) \\ \| & & \| & & \| & & \| \\ 0 & & \mathbb{C} & & 0 & & \mathbb{C}^{n+1} \end{array}$$

$$\begin{array}{ccccc} \to \mathrm{H}^1(E^*\otimes S^*) & \to & \mathrm{H}^1(S\otimes S^*) & \oplus & \mathrm{H}^1(Q^*\otimes S^*), \\ & & \| & & \| \\ & & 0 & & 0 \end{array}$$

(see Lemmas 1.3, 1.6 and 2.5), thus we have $\mathrm{h}^1(E^*\otimes S^*)=n$.

By the further part of the sequence above we obtain $h^i(E^* \otimes S^*) = 0$ for $i \geqq 2$.

In the same way we obtain

$$h^1(E^* \otimes Q) = n, \quad h^i(E^* \otimes Q) = 0 \quad \text{for} \quad i \geqq 2.$$

Then the cohomology sequence associated to (2. 3) is (using Lemmas 2. 4 and 2. 7 too)

$$\begin{array}{cccccccccc} H^0(S^* \otimes E^*) & \oplus & H^0(Q \otimes E^*) & \to & H^0(E \otimes E^*) & \to & H^1(E^*) & \to & H^1(S^* \otimes E^*) & \oplus \; H^1(Q \otimes E^*) \\ \| & & \| & & \| & & \| & & \| & \| \\ 0 & & 0 & & \mathbb{C} & & \mathbb{C} & & \mathbb{C}^n & \mathbb{C}^n \end{array}$$

$$\begin{array}{cccccccccc} \to H^1(E \otimes E^*) & \to & H^2(E^*) & \to & H^2(S^* \otimes E^*) & \oplus & H^2(Q \otimes E^*) & \to & H^2(E \otimes E^*) & \to H^3(E^*) \\ \| & & \| & & \| & & & & & \| \\ 0 & & 0 & & 0 & & & & & 0 \end{array}$$

and this gives the result.

Theorem 3. 7. *Let* $2(k+1) = n+1$. $M = (\mathbb{P}^n \times \mathbb{P}^n) \setminus \left(\sum_{i=0}^{n} x_i y_i = 0 \right)$ *is a Zariski open set of the coarse moduli space* $\mathscr{C}$ *of all* GM-*stable vector bundles of rank* n *on* Gr(k, n) *with the same Chern classes as* $S^* \oplus Q$.

Proof. First we note that by Lemma 2. 8, in the case $2(k+1) = n+1$ (Grassmannians with polarities), each bundle in $\Gamma_{k,n}$ is stable.

The bundle U on $M \times \mathrm{Gr}(k, n)$ gives (by definition of coarse moduli space) an algebraic map $g: M \to \mathscr{C}$. By Lemma 3. 1 if $p_1, p_2 \in M$ are distinct points then U_{p_1}, U_{p_2} are not isomorphic: thus g is injective. As $\dim M = 2n$, it follows by Lemma 3. 6 that g is an (analytic) open embedding. As g is flat (locally it is an isomorphism), we have that g is open in the algebraic sense.

Remark 3. 8. On the variety Gr(1, 3) Hernandez and Sols proved [12] that all stable 3-bundles with the same Chern classes as $S^* \oplus Q$ belong to $\Gamma_{k,n}$. Then $(\mathbb{P}^3 \times \mathbb{P}^3) \setminus \left(\sum_{i=0}^{n} x_i y_i = 0 \right)$ is a fine moduli space for all stable 3-bundles with the same Chern classes as $S^* \oplus Q$. We do not know if a similar result holds on all Grassmannians with polarities.

Remark 3. 9. Now we fix a hyperplane $\tilde{\mathbb{P}}^{n-1} \subset \mathbb{P}^n$. If we consider the natural inclusion

$$Z_1 = \mathrm{Gr}(k, n-1) \to \mathrm{Gr}(k, n) \quad [19]$$

we have

$$S^*|_{Z_1} = (S_{Z_1})^*, \quad Q|_{Z_1} \cong Q_{Z_1} \oplus \mathcal{O}_{Z_1}.$$

Let p be any point in $\mathbb{P}^n$ which does not lie on $\tilde{\mathbb{P}}^{n-1}$ and let $E \in \Gamma_{k,n}$ be such that $\pi(E) = (\tilde{\mathbb{P}}^{n-1}, p)$. Then by restricting the sequence (2. 1) to Z_1 we obtain $E|_{Z_1} \cong E_{Z_1} \oplus \mathcal{O}_{Z_1}$. This shows that $E|_{Z_1}$ is not ample, so neither E is.

4. The degeneracy locus of one or two generic sections

Theorem 4. 1. *The generic section of* $E \in \Gamma_{k,n}$ *vanishes on a smooth subvariety with two connected components, each of them is a Schubert cycle isomorphic to* $\mathrm{Gr}(k-1, n-2)$.

Proof. Let s be the section of $S^* \oplus Q$ which defines E. By the description given in Section 2, s is represented by a pair (F, v) where F is a nonzero linear functional on $\mathbb{C}^{n+1}$ and v is a vector in $\mathbb{C}^{n+1} \setminus \{0\}$. We have

$$\pi(E) = (q^*(F), q(v)) = (\mathbb{P}^{n-1}, p).$$

As $\mathrm{H}^1(\mathrm{Gr}(k, n), \mathcal{O}) = 0$, by the cohomology sequence associated to (2. 1) it follows that all sections of E come from sections of $S^* \oplus Q$. Let $\tilde{t}$ be a generic section of E and $\tilde{s}$ be any section of $S^* \oplus Q$ which is a lifting of $\tilde{t}$: thus $\tilde{t}$ vanishes exactly when s and $\tilde{s}$ are proportional.

Let the pair $(\tilde{F}, \tilde{v})$ represent $\tilde{s}$. Then we look for the linear subspaces $\mathbb{P}^k \subset \mathbb{P}^n$ and the pairs $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $\alpha s + \beta \tilde{s}$ vanishes on $\mathbb{P}^k$, i.e. $\mathbb{P}^k \subset q^*(\alpha F + \beta \tilde{F})$ and $q(\alpha v + \beta \tilde{v}) \in \mathbb{P}^k$. The necessary and sufficient condition for the existence of (α, β) is $(\alpha F + \beta \tilde{F})(\alpha v + \beta \tilde{v}) = 0$. From this quadratic homogeneous equation in α, β we obtain in general two distinct homogeneous solutions (α_1, β_1) and (α_2, β_2). Thus we have two distinct points on the line $\langle q(v), q(\tilde{v}) \rangle$ which lie respectively on two distinct hyperplanes of the pencil spanned by $q^*(F)$ and $q^*(\tilde{F})$.

As $\{\mathbb{P}^k \mid p_0 \in \mathbb{P}^k \subset \mathbb{P}_0^{n-1}\} \cong \mathrm{Gr}(k-1, n-2)$ we have the thesis.

Remark 4. 2. The locus found in the preceding theorem represents the Chern class $c_n(E)$. We observe that in projective spaces the zero set of a generic section consists of two points.

Now we denote by $G = \mathrm{PGL}(n+1)$ the connected component of the identity in $\mathrm{Aut}(\mathrm{Gr}(k, n))$. We recall also that $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$ and that in this group all the involutions g (i.e. $g^2 = 1$, $g \neq 1$) belong to the same conjugacy class.

Theorem 4. 3. *Two generic sections of* $E \in \Gamma_{k,n}$ *are linearly dependent on a smooth subvariety* $X_{k,n}$ *isomorphic to* $\mathbb{P}^1 \times \mathrm{Gr}(k-1, n-2)$. $X_{k,n}$ *is a scroll over* $\mathbb{P}^1$, *i.e. its fibres (projecting over* $\mathbb{P}^1$*) are Schubert cycles. There is a bijection between the orbits of* $X_{k,n}$ *under the action of* G *and the set of non involution conjugacy classes in* $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$.

Proof. As in the proof of the previous theorem, now we look for the degeneracy locus of three generic sections s_1, s_2, s_3 of $S^* \oplus Q$ represented respectively by (F_1, v_1), (F_2, v_2), (F_3, v_3). We may suppose that

$$\left(\bigcap_{i=1}^{3} q^*(F_i) \right) \cap \langle q(v_1), q(v_2), q(v_3) \rangle = \emptyset.$$

Thus we look for linear subspaces $\mathbb{P}^k \subset \mathbb{P}^n$ and triples $(\alpha, \beta, \gamma) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$ such that $\alpha s_1 + \beta s_2 + \gamma s_3$ vanishes on $\mathbb{P}^k$, i.e. $\mathbb{P}^k \subset q^*(\alpha F_1 + \beta F_2 + \gamma F_3)$ and $q(\alpha v_1 + \beta v_2 + \gamma v_3) \in \mathbb{P}^k$.

Let us call $a_{ij}=F_i(v_j)$, $A=(a_{ij})$ is a 3×3 matrix. The necessary and sufficient condition for the existence of (α, β, γ) is

$$(\alpha, \beta, \gamma)\, A(\alpha, \beta, \gamma)^t = 0. \tag{4. 1}$$

The points $q(\alpha v_1+\beta v_2+\gamma v_3)$ corresponding to the solutions of this quadratic homogeneous equation in (α, β, γ) define a conic C in the plane π spanned by the $q(v_i)$. We may suppose that the conic C is smooth. We emphasize that A is in general not symmetric and so the equation of the tangent line to the point $q(\alpha_0 v_1+\beta_0 v_2+\gamma_0 v_3)$ is

$$(\alpha, \beta, \gamma)\,(A+A^t)\,(\alpha_0, \beta_0, \gamma_0)^t = 0.$$

If (α, β, γ) is a solution of (4. 1) then the hyperplane $q^*(\alpha F_1+\beta F_2+\gamma F_3)$ of the net spanned by $\{q^*(F_i)\}$ contains the point $p=q(\alpha v_1+\beta v_2+\gamma v_3)$ and contains also another point p' on the conic C. The map $p\mapsto p'$ is an automorphism σ of C.

Thus the degeneracy locus $X_{k,n}$ is obtained in this way: if

$$p' = q(\alpha' v_1+\beta' v_2+\gamma' v_3)$$

is a point on $C\subset\pi$ we consider first the hyperplane

$$(\mathbb{P}^{n-1})' = q^*(\alpha' F_1+\beta' F_2+\gamma' F_3) = \tilde\sigma(p')$$

where $\tilde\sigma\colon C\longrightarrow \check{\mathbb{P}}^n$ is defined by

$$\tilde\sigma(p') := \left\langle \langle p', \sigma(p')\rangle, \bigcap_{i=1}^{3} q^*(F_i)\right\rangle,$$

where if $p'=\sigma(p')$, the notation $\langle p', \sigma(p')\rangle$ means the tangent line to C in p'.

Thus $X_{k,n}=\{\mathbb{P}^k \mid p'\in\mathbb{P}^k\subset\tilde\sigma(p'),\ p'\in C\}$ and $X_{k,n}$ is determined by σ.

Viceversa $X_{k,n}$ determines uniquely $\tilde\sigma$ and then σ. It is easy to check that if h is a linear projective transformation of $\mathbb{P}^n$ leaving C fixed then $h(X_{k,n})$ determines $h|_C\circ\sigma\circ(h^{-1})|_C$. For this construction in the case of the variety $\mathrm{Gr}(1, 3)$ see [7].

Now we want to prove that if σ is not an involution, then $X_{k,n}$ is smooth and isomorphic to $\mathbb{P}^1\times\mathrm{Gr}(k-1, n-2)$. We may suppose $k\geqq 1$, for if $k=0$, $X_{k,n}$ is generically a smooth conic.

We describe $X_{k,n}$ using local coordinates. We can choose homogeneous coordinates in $\mathbb{P}^n$ such that $\bigcap_{i=1}^{3} q^*(F_i)$ is given by the equations $x_0=x_1=x_2=0$ and the conic C is given by

$$\begin{cases} x_0x_2-x_1^2=0\\ x_3=x_4=\cdots=x_n=0.\end{cases}$$

Then $P(t)=(1, t, t^2, 0, \dots, 0)$ gives a parametrization of C and

$$Q(t)=\left(0,\ 1,\ \frac{ct^2+(a+d)t+b}{ct+d},\ 0, \dots, 0\right)$$

lies on the same line of $P(t)$ and

$$\sigma(P(t))=\left(1,\ \frac{at+b}{ct+d},\ \left(\frac{at+b}{ct+d}\right)^2,\ 0, \dots, 0\right) \quad \text{(with } ad-bc\neq 0).$$

We fix the points

$$R_i=(\underset{\substack{\| \\ x_0}}{0}, \dots, \underset{\substack{\| \\ x_{i+3}}}{1}, \dots, 0) \quad \text{for} \quad i=0, \dots, n-3$$

and we set $R_{n-2}=Q(t)$.

Observe that $\langle R_0, \dots, R_{n-3}\rangle=\bigcap_{i=1}^{3} q^*(F_i)$.

We represent, as usual, a point of the Grassmannian $\mathrm{Gr}(p, q)$ as a $(p+1)\times(q+1)$ matrix. Then we can define the morphism $\varphi\colon C\times \mathrm{Gr}(k-1, n-2)\to \mathrm{Gr}(k, n)$ given by:

$$\left(t,\ \begin{bmatrix} x_{0,0} & \cdots & x_{0,n-2} \\ \vdots & & \vdots \\ x_{k-1,0} & \cdots & x_{k-1,n-2} \end{bmatrix}\right) \overset{\varphi}{\longmapsto} \begin{bmatrix} x_{0,0}\ R_0+\cdots+x_{0,n-2}\ R_{n-2} \\ \vdots \\ x_{k-1,0}\ R_0+\cdots+x_{k-1,n-2}\ R_{n-2} \\ P(t) \end{bmatrix}.$$

In fact this map extends in a natural way for $t=\infty$.

In matrix form we have:

$$\left(t,\ \begin{bmatrix} x_{0,0} & \cdots & x_{0,n-2} \\ \vdots & & \vdots \\ x_{k-1,0} & \cdots & x_{k-1,n-2} \end{bmatrix}\right) \overset{\varphi}{\longrightarrow} \begin{bmatrix} 0 & x_{0,n-2} & x_{0,n-2}\cdot\dfrac{ct^2+(a+d)t+b}{ct+d} & x_{0,0} & \cdots & x_{0,n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & x_{k-1,n-2} & x_{k-1,n-2}\cdot\dfrac{ct^2+(a+d)t+b}{ct+d} & x_{k-1,0} & \cdots & x_{k-1,n-3} \\ 1 & t & t^2 & 0 & \cdots & 0 \end{bmatrix}$$

By the description given it is obvious that $\mathrm{Im}(\varphi)=X_{k,n}$ and that φ is injective when σ^2 has no fixed points except those of σ, equivalently when σ is not an involution. This happens exactly when $a+d\neq 0$, then generically.

An explicit computation of φ in local coordinates shows that the jacobian matrix $\mathrm{D}\varphi$ has maximal rank, as we claimed.

Observe that, if $p \in C$ then $\varphi(\{p\} \times \mathrm{Gr}(k-1, n-2))$ is the Schubert cycle

$$\{\mathbb{P}^k \mid p \in \mathbb{P}^k \subset \tilde{\sigma}(p)\}.$$

This completes the proof.

Remark 4.4. The previous locus represents the Chern class $c_{n-1}(E)$. An analogous description for lower order Chern classes fails because the corresponding degeneracy loci contain singularities.

Remark 4.5. $X_{k,n} \subset \mathrm{Gr}(k, n)$ is not an (algebraic) complete intersection (look at its homology class).

Remark 4.6. In the case (non generic!) that

$$\left(\bigcap_{i=1}^{3} q^*(F_i)\right) \cap \langle q(v_1), q(v_2), q(v_3)\rangle$$

is a point, we obtain generically a variety $X'_{k,n}$ as degeneracy locus. $X'_{k,n}$ is a scroll over $\mathbb{P}^1$ and arises geometrically as a deformation of $X_{k,n}$. It is interesting to observe that

$$X'_{1,3} \cong \mathbb{F}_2 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \quad \text{(see [7])}.$$

$X_{1,3}$ and $X'_{1,3}$ are the only nondegenerate scroll surfaces of bidegree (2, 2) (remember that the bidegree of $c_2(E)$ is (2, 2)) in Gr(1, 3) (see [7], [20]).

We note also that in the case of Gr(1, 4) there is a topological classification of all scroll of planes found by A. Alzati [1].

Theorem 4.7. *If the Grassmannian* Gr(k, n) *is not a projective space then the normal bundle* $N_{X_{k,n}|\mathrm{Gr}(k,n)}$ *is not ample.*

Proof. Let $N = N_{X_{k,n}|\mathrm{Gr}(k,n)}$ and $F = \mathrm{Gr}(k-1, n-2)$ be any fibre of $X_{k,n}$. Let $l \subset F \subset \mathrm{Gr}(k, n)$ be a line. Then we have the exact sequence of bundles

$$0 \to \mathrm{T}X_{k,n}|_l \to \mathrm{TGr}(k, n)|_l \to N|_l \to 0.$$

Then we obtain

$$c_1(\mathrm{T}X_{k,n}|_l) = c_1(\mathrm{TGr}(k-1, n-2)|_l) = n-1,$$
$$c_1(\mathrm{TGr}(k, n)|_l) = n+1$$

so that

$$c_1(N|_l) = 2.$$

As rank $N = n-1$, it is obvious that for $n \geqq 4$, $N|_l$ is not ample: thus neither N is. The case $n = 3$ has been studied by N. Goldstein in [5], [6].

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Istituto Mat. U. Dini, Viale Morgagni 67A, I-50134 Firenze

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