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Some Extensions of Horrocks Criterion to Vector Bundles on Grassmannians and Quadrics (*).

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Summary. – *In this paper we prove that a vector bundle E on a grassmannian (resp. on a quadric) splits as a direct sum of line bundles if and only if certain cohomology groups involving E and the quotient bundle (resp. the spinor bundle) are zero. When rank $E = 2$ a better criterion is obtained considering only finitely many suitably chosen cohomology groups.*

A well known criterion of Horrocks ([13], [14], [17]) says that a vector bundle E on the complex projective space $\boldsymbol{P}^n$ splits (i.e. is isomorphic to a direct sum of line bundles) if and only if the cohomology groups $H^i(\boldsymbol{P}^n, E(t))$ are zero for $0 < i < n = \dim \boldsymbol{P}^n$ and for all $t \in Z$, where $E(t)$ denotes $E \bigotimes_{\mathcal{O}_{\boldsymbol{P}^n}} \mathcal{O}_{\boldsymbol{P}^n}(t)$.

Let Gr (k, n) be the Grassmannian of linear k-planes in $\boldsymbol{P}^n$ and let Q_n be the smooth quadric hypersurface in $\boldsymbol{P}^{n+1}$.

In this paper we obtain some extensions of Horrocks criterion and some related result on Gr (k, n) and Q_n.

Gr (k, n) and Q_n $(n \geqslant 3)$ are the simplest rational homogeneous manifolds of rank one [23] besides $\boldsymbol{P}^n$.

Most of the results contained in this paper have been announced in [19].

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The paper is divided as follows.

In section 1 we fix basic notations and in particular we recall the Bott theorem for homogeneous vector bundles on Grassmannians.

In section 2 our main result is theorem 2.1. In particular we have the following splitting criterion:

Let E be a vector bundle on Gr (k, n). Then E splits if and only if

$$H^i(\mathrm{Gr}\,(k, n), \overset{i_1}{\bigwedge} Q^* \otimes \ldots \otimes \overset{i_s}{\bigwedge} Q^* \otimes E(t)) = 0 \qquad \forall i_1, \ldots, i_s$$

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such that $0 \leq i_1, \ldots, i_s \leq n-k$, $s \leq k$; $\forall t \in Z$; $\forall i$ such that $0 < i < (k+1)(n-k) = \dim \mathrm{Gr}(k, n)$ where Q = quotient bundle on $\mathrm{Gr}(k, n)$, Q^* = dual of Q.

When $k = 0$ or $k = n-1$ then $\mathrm{Gr}(k, n) \simeq P^n$ and we get exactly the Horrocks criterion. Obviously in the statement above we can replace Q^* by Q (it is sufficient to apply Serre duality and observe that E splits if and only if E^* splits).

Then we specialize to the case: rank $E = 2$. In this case, by a simple argument involving the Koszul complex of a line in the Grassmannian, we are able to prove that the bundle E is uniform when finitely many suitably chosen cohomology groups are zero (theorem 2.9). On the projective plane this result was proved in [18]. Uniform 2-bundles on Grassmannians have been classified by VAN DE VEN [24] and GUYOT [11]. So our result implies a strong improvement of the splitting criterion quoted above. When the Grassmannian is a projective space, we get another proof of a result of Chiantini and Valabrega [7].

In section 3 we use some results from [20]. In [20] we have defined some vector bundles on the quadric Q_n which are the natural generalization of the universal bundle and the dual of the quotient bundle on $Q_4 \simeq \mathrm{Gr}(1, 3)$. We have called them spinor bundles.

Spinor bundles appear in the main result of this section which is theorem 3.3.

In particular we have the following splitting criterion:

Let E be a vector bundle on Q_n $(n \geq 3)$, let S be a spinor bundle on Q_n: Then E splits if and only if

$$H^i(Q_n, E(t)) = 0 \qquad \text{for } 2 \leq i \leq n-1$$
$$H^i(Q_n, S \otimes E(t)) = 0 \qquad \text{for } 1 \leq i \leq n-2\,, \qquad \text{for all } t \in Z\,.$$

When rank $E = 2$, the analog of theorem 2.9 for quadrics is theorem 3.8.

1. – Notations and preliminaries.

For basic facts about vector bundles we refer to [17]. When $X = \mathrm{Gr}(k, n)$ or $X = Q_n$ $(n \geq 3)$ we have $\mathrm{Pic}(\mathrm{Gr}(k, n)) = \mathrm{Pic}(Q_n) = Z$. So it is natural to keep the notation $E(t) = E \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(t)$ for $t \in Z$ when E is a vector bundle on a Grassmannian or on a quadric.

The first Chern class of E can be considered as an integer.

We use the definition of stability of Mumford-Takemoto.

We denote by E^* the dual of the vector bundle E.

If Z is a subvariety of X we denote $E \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_Z$ by $E|_Z$. $\mathcal{J}_Z$ is the ideal sheaf of Z.

If F is a sheaf on X, we denote by $h^i(F)$ the dimension of the complex vector space $H^i(X, F)$. We shall need the following lemma:

LEMMA 1.1. – (i) Let
$$0 \to A_n \to \ldots \to A_1 \to B \to 0$$
be an exact sequence of sheaves on a variety X, let r be an integer ≥ 0.

If $H^{r+i-1}(X, A_i) = 0$ for $i = 1, \ldots, n$ then $H^r(X, B) = 0$.

(ii) Let

$$0 \to A_n \xrightarrow{a_n} \ldots \to A_1 \xrightarrow{a_1} B \to 0$$
$$0 \to A_n \xrightarrow{a_n} \ldots \to A_1 \xrightarrow{a_1} B' \to 0$$

be two exact sequences of sheaves on a variety X.

If

$$H^i(X, A_i) = 0 \qquad \text{for } i = 1, \ldots, n-2$$

and

$$H^{n-1}(X, A_n) = 0 \qquad \text{or } H^{n-1}(X, A_{n-1}) = 0$$

then

$$H^0(B) = H^0(B') .$$

PROOF. – We get (i) cutting the sequence into short exact sequences, or by a spectral argument.

Curting the first sequence of (ii) into short exact sequences, we get:

(1) $$0 \to \operatorname{Ker} a_1 \to A_1 \xrightarrow{a_1} B \to 0$$

(2) $$0 \to \operatorname{Ker} a_2 \to A_2 \to \operatorname{Ker} a_1 \to 0$$

and so on until: $0 \to A_n \xrightarrow{a_n} A_{n-1} \to \operatorname{Coker}(a_n) \to 0$. Then

$$\begin{aligned} h^0(B) &= (\text{from } (1)) \\ &= h^0(A_1) - h^0(\operatorname{Ker} a_1) + h^1(\operatorname{Ker} a_1) = (\text{from } (2)) \\ &= h^0(A_1) - h_0(A_2) + h^1(A_2) + h^0(\operatorname{Ker} a_2) - h^1(\operatorname{Ker} a_2) + h^2(\operatorname{Ker} a_2) . \end{aligned}$$

Thus, after n steps, we get $h_0(B)$ as a sum involving only some cohomology groups of the sheaves A_i (in fact $\operatorname{Ker} a_n = A_n$).

This gives the thesis.

In the case (ii) of lemma 1.1 we can prove in the same way a little more:

LEMMA 1.2. – Let

(3*i*) $$0 \to A_n \xrightarrow{a^n} \ldots \to A_1 \xrightarrow{a_1} B \to 0$$

(3*ii*) $$0 \to A_n \xrightarrow{a'_n} \ldots \to A_1 \xrightarrow{a'_1} B' \to 0$$

be two exact sequences of sheaves on a variety X.

Then:

$$|h^0(B) - h^0(B')| \leq \sum_{i=1}^{n-1} h^i(A_i)$$

$$|h^0(B) - h^0(B')| \leq \sum_{i=1}^{n-2} h^i(A_i) + h^{n-1}(A_n) .$$

PROOF. – Set $\chi^i(F) = \sum_{j=0}^{i} (-1)^j h^j(F)$ for a sheaf F.

Then, cutting (3i) and (3ii) into short exact sequences as in lemma 1.1 we have:

$$h^0(B) - h^0(B') \leq \chi^1(\mathrm{Ker}\, a_1') - \chi^1(\mathrm{Ker}\, a_1) + h^1(A_1)$$

and:

$$\chi^i(\mathrm{Ker}\, a_i') - \chi^i(\mathrm{Ker}\, a_i) \leq \chi^{i+1}(\mathrm{Ker}\, a_{i+1}) - \chi^{i+1}(\mathrm{Ker}\, a_{i+1}') + h^{i+1}(A_{i+1})$$

for $i = 1, \dots, n-1$.

The same inequalities are true interchanging a_i and a_i'.

As $\mathrm{Ker}\, a_{n-1} = \mathrm{Ker}\, a_{n-1}' = A_n$, it follows that

$$|h^0(B) - h^0(B')| \leq \sum_{i=1}^{n-1} h^i(A_i) .$$

In the same way we can prove the other inequality.

On the Grassmannian $\mathrm{Gr}\,(k, n)$ we have the canonical exact sequence

$$0 \to S \to \mathcal{O}_{\mathrm{Gr}}^{\oplus n+1} \to Q \to 0 . \tag{4}$$

The universal bundle S has rank $k+1$, the quotient bundle Q has rank $n-k$. We have $c_1(S) = -1$, $c_1(Q) = +1$. Considering the isomorphism $\mathrm{Gr}\,(k, n) \simeq \mathrm{Gr}\,(n-k-1, n)$, the canonical exact sequence on $\mathrm{Gr}\,(n-k-1, n)$ is the dual sequence of (4).

We consider $\mathrm{Gr}\,(k, n)$ as the complex homogeneous manifold $SL(n+1)/P$ where

$$P = \left\{ \begin{bmatrix} h_1 & 0 \\ h_3 & h_4 \end{bmatrix} \in SL(n+1) \colon h_4 \in GL(k+1) \right\} \text{ (see [26])} .$$

$\mathfrak{sl}(n+1) = \{A \in M(n+1) \colon \mathrm{tr}\, A = 0\}$ is the simple Lie algebra of $SL(n+1)$ and $\mathfrak{h} = \{A \in \mathfrak{sl}(n+1) \colon A \text{ is diagonal}\}$ is a Cartan subalgebra of $\mathfrak{sl}(n+1)$.

Let $e_{ij} \in \mathfrak{gl}(n+1)$ be the matrix with the (i, j) entry equal to 1 and all other entries equal to zero, $\{e_{ij}'\}$ the dual basis of $\{e_{ij}\}$. Then: $x_i = e_{i,i} - e_{i+1,i+1}$ for $i = 1, \dots, n$ give a basis for $\mathfrak{h}$. We call $\lambda_1, \dots, \lambda_n \in \mathfrak{h}^*$ the dual basis of $x_1, \dots, x_n$

and set

$$\alpha_i = e'_{i,i} - e'_{i+1,i+1} \in \mathfrak{h}^* .$$

It is well known that $(\lambda_i, \alpha_j) = \delta_{ij}$ where $1/(2(n+1))(\ ,\ )$ is the Killing form in $\mathfrak{h}^*$ and δ_{ij} is the Kronecker symbol.

$\alpha_1, \ldots, \alpha_n$ gives a basis of the root system Φ of $\mathfrak{sl}(n+1)$ with respect to $\mathfrak{h}$.

It is well known that $\Phi = \Phi^+ \cup \Phi^-$ where

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_j : 1 \leqslant i \leqslant j \leqslant n\}$$

is the set of positive roots and $\Phi^- = -\Phi^+$.

A weight $\lambda = \sum_{i=1}^{n} n_i \lambda_i$ $(n_i \in Z)$ is called singular if $(\lambda, \alpha) = 0$ for at least one $\alpha \in \Phi$, and regular with index p if it is not singular and there exists exactly p roots $\alpha \in \Phi^+$ such that $(\lambda, \alpha) < 0$. We set: $\delta = \sum_{i=1}^{n} \lambda_i = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

A homogeneous vector bundle E_ϱ of rank r on $\mathrm{Gr}(k, n) \simeq SL(n+1)/P$ is by definition a bundle arising from a representation $\varrho : P \to GL(r)$. In particular a homogeneous bundle satisfies the condition: $f^* E_\varrho \simeq E_\varrho$ $\forall f \in \mathrm{Aut}(\mathrm{Gr}(k, n))^0$, where $\mathrm{Aut}(\mathrm{Gr}(k, n))^0$ is the connected component of the group of all automorphisms of $\mathrm{Gr}(k, n)$.

We recall the fundamental theorem of Bott ([5], th. IV′, [26])

THEOREM (Bott). - Let E_ϱ be a homogeneous vector bundle on $\mathrm{Gr}(k, n) \simeq \simeq SL(n+1)/P$, defined by an irreducible representation ϱ, and let λ be the highest weight of $D\varrho : \mathfrak{p} \to \mathfrak{gl}(r)$.

(i) If $\lambda + \delta$ is singular then $H^i(\mathrm{Gr}(k, n), E_\varrho) = 0$ $\forall i$.

(ii) If $\lambda + \delta$ is regular with index p then $H^i(\mathrm{Gr}(k, n), E_\varrho) = 0$ for all $i \neq p$

and the dimension of $H^p(\mathrm{Gr}(k, n), E_\varrho)$ is the dimension of the representation of $\mathfrak{sl}(n+1)$ with highest weight $s(\lambda + \delta) - \delta$. Here, $s(\lambda + \delta)$ denotes the uniquely determined element of the Weyl chamber of $\mathfrak{sl}(n+1)$ which is congruent to $\lambda + \delta$ under the action of the Weyl group of reflections r_i with respect to the hyperplane orthogonal to α_i. ┘

We have

$$r_j(\lambda_i) = \begin{cases} \lambda_i & i \neq j \\ \lambda_{j-1} - \lambda_j + \lambda_{j+1} & i = j \end{cases}$$

where we set $\lambda_0 = \lambda_{n+1} = 0$.

The bundle $\bigwedge^i Q$ (i-th exterior power of Q) belongs to the irreducible representation with highest weight λ_i.

LEMMA 1.3. – Let $0 < i < \dim \mathrm{Gr}(k, n) = (k+1)(n-k)$.
If $s \leqslant k$

$$H^i(\mathrm{Gr}(k, n), \overset{i_1}{\wedge} Q \otimes \ldots \otimes \overset{i_s}{\wedge} Q(t)) = 0 \quad \forall t \in Z, \quad \text{for } 0 \leqslant i_1, \ldots, i_s \leqslant n-k$$

$$H^i(\mathrm{Gr}(k, n), \overset{i_1}{\wedge} Q \otimes \ldots \otimes \overset{i_{k+1}}{\wedge} Q(t)) = \begin{cases} C & \text{if } i_1 = \ldots = i_{k+1} = j; \\ & t = -n + j - 1; \\ & i = (n-k-j)(k+1) \text{ for } 0 < j < n-k; \\ 0 & \text{otherwise} \end{cases}$$

PROOF. – The bundle $\overset{i_1}{\wedge} Q \otimes \ldots \otimes \overset{i_s}{\wedge} Q$ belongs to a representation not irreducible but fully reducible. In fact Q is given by the representation

$$P \to GL(n-k)$$
$$\begin{bmatrix} h_1 & 0 \\ h_3 & h_4 \end{bmatrix} \mapsto h_1$$

which is a surjective projection. So we limit ourselves to studying the representations belonging to $\overset{i_1}{\wedge} Q \otimes \ldots \otimes \overset{i_s}{\wedge} Q$ as $GL(n-k)$-representations (i.e. homomorphisms $GL(n-k) \to \mathrm{Aut}(V)$, V a vector space).

The bundle Q belongs to the standard representation φ of $GL(n-k)$ and $\overset{i_1}{\wedge} Q \otimes \ldots \otimes \overset{i_s}{\wedge} Q$ belongs to $\overset{i_1}{\wedge} \varphi \otimes \ldots \otimes \overset{i_s}{\wedge} \varphi$. By Littlewood-Richardson rule we can decompose these representations into a direct sum with each summand isomorphic to $Q^{n_1, \ldots, n_r}$ for some $n_1 \geqslant \ldots \geqslant n_r$. We have found in [4] (pag. 879) a clear explanation of how to handle Littlewood-Richardson rule.

We consider $Q^{n_1, \ldots, n_r}$ as a bundle on $\mathrm{Gr}(k, n)$. It corresponds to a Young diagram with the i-th row given by n_i elements.

In particular $Q^{\overbrace{1,1,\ldots,1}^{i \text{ times}}} = \overset{i}{\wedge} Q$, $Q^p = S^p Q$ (p-th symmetric power of Q). As $\det Q = \overset{n-k}{\wedge} Q = \mathcal{O}(1)$, we have:

$$Q^{n_1, \ldots, n_r}(t) = Q^{n_1+t, \ldots, n_r+t, \overbrace{t, \ldots, t}^{n-k-r \text{ times}}}.$$

It is convenient to set $n_i = 0$ for $i > r$.

If $Q^{n_1, \ldots, n_r}$ is a direct summand of $\overset{i_1}{\wedge} Q \otimes \ldots \otimes \overset{i_s}{\wedge} Q$ then $n_i \leqslant s$ $\forall i$. This follows by Littlewood-Richardson rule. In fact $\overset{i_1}{\wedge} Q$ corresponds to a Young diagram with i_1 rows each of them with only one element. $\overset{i_1}{\wedge} Q \otimes \overset{i_2}{\wedge} Q$ decomposes in some summands, e·ch of them corresponds to a Young diagram consisting of rows with at most two elements.

Thus, $\overset{i_1}{\bigwedge} Q \otimes \ldots \otimes \overset{i_s}{\bigwedge} Q$ decomposes into summands, each corresponding to a Young diagram consisting of rows with at most s elements.

It is well known that the highest weight of the irreducible representation $Q^{n_1,\ldots,n_r}$ is $\lambda = \lambda_1(n_1 - n_2) + \lambda_2(n_2 - n_3) + \ldots + \lambda_r n_r$. A reference for this fact is [12] theorem A7, where h_i is, in our notation, equal to $\lambda_i - \lambda_{i-1}$.

Observe that $r \leqslant n - k = \operatorname{rank} Q$. We recall that the line bundle $\mathcal{O}(t)$ $(t \in Z)$ belongs to the representation with highest weight $t\lambda_{n-k}$.

Let first $s \leqslant k$. Then $n_i \leqslant k$, in particular $n_i - n_{i+1} \leqslant k$. Then we claim that $\lambda + t\lambda_{n-k} + \delta$ is a singular weight for $-n - n_1 \leqslant t \leqslant -1 - ni$, is regular of index 0 for $t \geqslant -n_r$, is regular of index $(k+1)(n-k)$ for $t \leqslant -n - n_1 - 1$.

For, let first $-n - n_1 \leqslant t \leqslant -n + k - n_1$. Then,

$$(\lambda + t\lambda_{n-k} + \delta, \alpha_1 + \ldots + \alpha_{-t-n_1}) = (\lambda, \alpha_1 + \ldots + \alpha_{-t-n_1}) +$$
$$+ (t\lambda_{n-k}, \alpha_1 + \ldots + \alpha_{-t-n_1}) + (\delta, \alpha_1 + \ldots + \alpha_{-t-n_1}) = n_1 + t + (-t - n_1) = 0\,,$$

so that $\lambda + t\lambda_{n-k} + \delta$ is singular.

Let now $-n + k - n_1 + 1 \leqslant t \leqslant -1 - n_r$. Consider the following decreasing sequence of integers:

$$\begin{aligned}
a_1 &:= (\lambda + t\lambda_{n-k} + \delta, \alpha_1 + \ldots + \alpha_{n-k}) = n_1 + t + n - k \\
a_2 &:= (\lambda + t\lambda_{n-k} + \delta, \alpha_2 + \ldots + \alpha_{n-k}) = n_2 + t + n - k - 1 \\
&\vdots \\
a_{n-k} &:= (\lambda + t\lambda_{n-k} + \delta, \alpha_{n-k}) = n_{n-k} + t + 1
\end{aligned}$$

We have

$$0 \leqslant a_{i-1} - a_i = n_{i-1} - n_i + 1 \leqslant s + 1 \leqslant k + 1$$

By hypothesis: $a_1 \geqslant 1$, $a_{n-k} \leqslant 0$. Let a_{j+1} be the first element of the sequence which is nonpositive. Then $a_j \geqslant 1$, so that:

$$-k \leqslant a_{j+1} \leqslant 0\,.$$

Thus

$$(\lambda + t\lambda_{n-k} + \delta, \alpha_{j+1} + \ldots + \alpha_{n-k-a_{j+1}}) = (\lambda + t\lambda_{n-k} + \delta, \alpha_{j+1} + \ldots + \alpha_{n-k}) +$$
$$+ \sum_{f=1}^{-a_{j+1}} (\lambda + t\lambda_{n-k} + \delta, \alpha_{n-k+f}) = a_{j+1} + \sum_{f=1}^{-a_{j+1}} 1 = a_{j+1} - a_{j+1} = 0\,,$$

so that $\lambda + t\lambda_{n-k} + \delta$ is singular.

If $t \geqslant -n_r$, then $(\lambda + t\lambda_{n-k} + \delta, \alpha) > 0$ for each $\alpha \in \Phi^+$, so that $\lambda + t\lambda_{n-k} + \delta$ is regular of index 0 (for all s). If $t \leqslant -n - n_1 - 1$, then $(\lambda + t\lambda_{n-k} + \delta, \alpha) < 0$ exactly for $\alpha = \alpha_i + \ldots + \alpha_j$ with $1 \leqslant i \leqslant n - k \leqslant j \leqslant n$, and positive otherwise. Then $\lambda + t\lambda_{n-k} + \delta$ is regular of index $(k+1)(n-k)$ (for all s).

Thus, if $s \leqslant k$, we get the result from Bott theorem.

If $s = k + 1$, we point out that when $-n - n_1 \leqslant t \leqslant -1 - n_r$ the proof above shows that $\lambda + t\lambda_{n-k} + \delta$ is singular for $a_{j+1} \neq -k - 1$.

When $a_{j+1} = -k - 1$, then $a_j = 1$ and

$$n_j - n_{j+1} = k + 1 \,,$$

so that the corresponding Young diagram has a row with exactly $k + 1$ elements more than the above one.

After twisting by some line bundle, the corresponding bundle $Q^{n'_1, \ldots, n'_r}$ satisfies the condition:

$$\begin{cases} 0 \leqslant n'_i \leqslant k + 1 & \forall i \\ n'_j - n'_{j+1} = k + i \end{cases}$$

so that

$$n'_i = \begin{cases} k + 1 & 1 \leqslant i \leqslant j \\ 0 & j + 1 \leqslant i \end{cases}$$

$Q^{n'_1, \ldots, n'_r}$ is then a direct summand of $\overbrace{\bigwedge^j Q \otimes \ldots \otimes \bigwedge^j Q}^{k+1 \text{ times}}$.

Consider the corresponding weight:

$$\lambda = (k + 1)\lambda_j + t\lambda_{n-k} \quad \text{with } t \in Z \,.$$

When t changes, $\lambda + \delta$ is regular of index different from 0, $(k + 1)(n - k)$ only when $t = -n + j - 1$, and in this case $((k + 1)\lambda_j + (-n + j - 1)\lambda_{n-k} + \delta, \alpha) < 0$ exactly when $\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_p$ with $j + 1 \leqslant i \leqslant n - k \leqslant p \leqslant n$, so that the index of $(k + 1)\lambda_j + (-n + j - 1)\lambda_{n-k} + \delta$ is $(n - k - j)(k + 1)$.

By applying Bott theorem again, it remains only to show that $(k + 1)\lambda_j + (-n + j - 1)\lambda_{n-k} + \delta$ is congruent to δ under the action of the Weyl group.

This is explained by the following example:

Let $n = 6$, $k = 2$, $j = 1$ so that:

$$(k + 1)\lambda_6 + (-n + j - 1)\lambda_{n-k} + \delta = 4\lambda_1 + \lambda_2 + \lambda_3 - 5\lambda_4 + \lambda_5 + \lambda_6 \,.$$

We apply to this weight a sequence of reflections (elements of the Weyl group) obtaining:

$$\text{step 1)} \begin{cases} (\text{apply } r_4)\colon & 4\lambda_1 + \lambda_2 - 4\lambda_3 + 5\lambda_4 - 4\lambda_5 + \lambda_6 \\ (\text{apply } r_5)\colon & 4\lambda_1 + \lambda_2 - 4\lambda_3 + \lambda_4 + 4\lambda_5 - 3\lambda_6 \\ (\text{apply } r_6)\colon & 4\lambda_1 + \lambda_2 - 4\lambda_3 + \lambda_4 + \lambda_5 + 3\lambda_6 \end{cases}$$

$$\text{step 2)}\begin{cases}(\text{apply } r_3)\colon & 4\lambda_1-3\lambda_2+4\lambda_3-3\lambda_4+\lambda_5+3\lambda_6\\(\text{apply } r_4)\colon & 4\lambda_1-3\lambda_2+\lambda_3+3\lambda_4-2\lambda_5+3\lambda_6\\(\text{apply } r_5)\colon & 4\lambda_1-3\lambda_2+\lambda_3+\lambda_4+2\lambda_5+\lambda_6\end{cases}$$

$$\text{step 3)}\begin{cases}(\text{apply } r_2)\colon & \lambda_1+3\lambda_2-2\lambda_3+\lambda_4+2\lambda_5+\lambda_6\\(\text{apply } r_3)\colon & \lambda_1+\lambda_2+2\lambda_3-\lambda_4+2\lambda_5+\lambda_6\\(\text{apply } r_4)\colon & \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6=\delta\,.\end{cases}$$

In general, if $j = n - k$ the claim is obvious.

If $j < n - k$ we apply to $(k+1)\lambda_j + (-n+j-1)\lambda_{n-k} + \delta$ the following sequence of reflections:

step 1) $\quad r_n\circ\ldots\circ r_{n-k+1}\circ r_{n-k}$ (this is sufficient if $j = n-k-1$)

step 2) $\quad r_{n-1}\circ\ldots\circ r_{n-k}\circ r_{n-k-1}$ (this is sufficient if $j = n-k-2$)

$\vdots$

step $n-k-j$) $\quad r_{j+k-1}\circ\ldots\circ r_{j+2}\circ r_{j+1}$.

In the end we obtain δ, as the reader can convince himself.

This completes the proof of lemma 1.3.

As a corollary of lemma 1.3 we get the following well known statement (look at the duality $\mathrm{Gr}(k, n) \simeq \mathrm{Gr}(n-k-1, n)$):

PROP. 1.4. – Let $0 < i < \dim \mathrm{Gr}(k, n)$

(i) We have $H^i(\mathrm{Gr}(k, n), \mathcal{O}(t)) = 0 \;\; \forall t \in Z$

(ii) $$H^i(\mathrm{Gr}(k, n), Q(t)) = \begin{cases} C & k = 0,\ t = -n,\ i = n-1\\ 0 & \text{otherwise}\end{cases}$$

$$H^i(\mathrm{Gr}(k, n), S^*(t)) = \begin{cases} C & k = n-1,\ t = -n,\ i = n-1\\ 0 & \text{otherwise}\,.\end{cases}$$

2. – Splitting criteria on Grassmannians.

Consider now the problem of finding some cohomological conditions for a vector bundle E on $\mathrm{Gr}(k, n)$ that are equivalent to the splitting of E.

By prop. 1.4 we get that the condition $H^i(\mathrm{Gr}(k, n), E(t)) = 0$ for all $t \in Z$, for $0 < i < \dim \mathrm{Gr}(k, n)$ is always necessary but is sufficient only when the Grassmannian $\mathrm{Gr}(k, n)$ is isomorphic to a projective space (i.e. $k = 0, n-1$). So it is natural to look for more vanishing conditions.

The answer is given by the following theorem.

THEOREM 2.1. – Let E be a vector bundle on $\mathrm{Gr}(k, n)$.
The following conditions are equivalent:

a) E splits

b) $H^i(\mathrm{Gr}(k, n), \overset{i_1}{\bigwedge} Q^* \otimes \ldots \otimes \overset{i_s}{\bigwedge} Q^* \otimes E(t)) = 0 \ \forall i_1, \ldots, i_s$ such that $0 \leqslant i_1, \ldots, i_s \leqslant n - k$, $s \leqslant k$, $\forall t \in Z$, $\forall i$: $0 < i < (k+1)(n-k) = \dim \mathrm{Gr}(k, n)$

c) $H^i(\mathrm{Gr}(k, n), \overset{i_1}{\bigwedge} Q^* \otimes \ldots \otimes \overset{i_s}{\bigwedge} Q^* \otimes E(t)) = 0$, $\forall t \in Z$, $\forall i_1, \ldots, i_s, i$ s.t.:

$$\begin{cases} \sum_{n=1}^{s} i_n \leqslant i < \sum_{n=1}^{s} i_n + \dim \mathrm{Gr}(k-s, n-s) \\ 0 < i, \quad 0 \leqslant i_1, \ldots, i_s \leqslant n-k \end{cases}$$

where we set, $\overset{0}{\bigwedge} Q^* = \mathcal{O}_{\mathrm{Gr}}$, $\dim \mathrm{Gr}(p, q) = 0$ if $p < 0$.

PROOF. – a) $\Rightarrow$ b) It follows from lemma 1.3 and Serre duality as cohomology commutes with direct sums.

b) $\Rightarrow$ c) is trivial, because if $s > k$ condition c) is empty.

c) $\Rightarrow$ a) The proof is by induction on k and follows the pattern of the proof of Horrocks criterion given in [3].

For $k = 0$ the implication is exactly the Horrocks criterion on $\boldsymbol{P}^n$. Consider now a generic section s of S (Q is globally generated): it has zero locus $Z \simeq \mathrm{Gr}(k-1, n-1)$. Observe that $Q|_Z \simeq Q_Z$. The first step in our proof is to show that $E|_Z$ splits. In order to use the induction hypothesis, we claim that

$$H^i(Z, \overset{j_1}{\bigwedge} Q^* \otimes \ldots \otimes \overset{j_s}{\bigwedge} Q^* \otimes E(t)|_Z) = 0\,, \qquad \forall t \in Z\,, \quad \forall j_1 \ldots j_s, i \text{ s.t.:}$$

$$\begin{cases} \sum j_n \leqslant i < \sum j_n + \dim \mathrm{Gr}(k-1-s, n-1-s) \\ i > 0\,. \end{cases}$$

For, we consider the Koszul complex of s, after tensoring it by $E(t)$:

$$(5) \qquad 0 \to \overset{n-k}{\bigwedge} Q^* \otimes E(t) \to \overset{n-k-1}{\bigwedge} Q^* \otimes E(t) \to \ldots \to$$
$$\to \overset{2}{\bigwedge} Q^* \otimes E(t) \to Q^* \otimes E(t) \to E(t)|_Z \to 0\,.$$

This sequence is exact.

We tensor (5) by $\overset{j_1}{\bigwedge} Q^* \otimes \ldots \otimes \overset{j_s}{\bigwedge} Q^*$.

Our hypothesis together with lemma 1.1, (i) proves our claim. So we can construct a splitting bundle F on Gr (k, n) and a isomorphism $\alpha_0\colon F|_Z \to E|_Z$, $\alpha_0 \in H^0(Z, (F^*\otimes E)|_Z)$.

Our second step is to show that α_0 can be extended to an isomorphism $\alpha \in H^0(\mathrm{Gr}\,(k, n), F^*\otimes E)$. The obstruction to this extension lies in $H^1(\mathrm{Gr}\,(k, n), \mathfrak{J}_Z\otimes F^*\otimes E)$.

The Koszul complex of s gives an exact sequence:

$$(6) \qquad 0 \to \bigwedge^{n-k} Q^* \to \ldots \to \bigwedge^{2} Q^* \to Q^* \to \mathfrak{J}_Z \to 0\,.$$

We tensor (6) by $F^*\otimes E$.

Our hypothesis together with lemma 1.1, (i) gives:

$$H^1(\mathrm{Gr}\,(k, n), \mathfrak{J}_Z\otimes F^*\otimes E) = 0\,.$$

Then there exists a morphism $\alpha\colon F \to E$, and then a morphism: $\det\alpha\colon \det F \to \det E$. We obtain

$$\det\alpha \in H^0(\mathrm{Gr}\,(k, n), (\det F)^*\otimes \det E) = H^0\big(\mathrm{Gr}\,(k, n), \mathcal{O}(c_1(E) - c_1(F))\big) = H^0(\mathrm{Gr}\,(k, n), \mathcal{O}_{\mathrm{Gr}(k,n)}) \simeq \boldsymbol{C}\,.$$

Then $\det\alpha$ is a constant; as it is nonzero on Z, it is nonzero everywhere on Gr (k, n). Thus α must be an isomorphism. q.e.d.

REMARK 2.2. – Theorem 2.1 is useful if $k+1 \leqslant n-k$. Otherwise we can perform the duality Gr $(n-k-1, n) \simeq$ Gr (k, n) and use the dual of theorem 2.1 with S at the place of Q^*.

REMARK 2.3. – The computation in lemma 1.3 for $s = k+1$ shows that the bound $s \leqslant k$ in (*b*) of theorem 2.1 is sharp.

EXAMPLE 2.4. – Let E be a vector bundle on Gr (1, 4). Theorem 2.1 says that E splits if and only if:

$$H^i(\mathrm{Gr}\,(1, 4), E(t)) = 0 \qquad \text{for } 1 \leqslant i \leqslant 5, \text{ for all } t \in Z$$

$$H^i(\mathrm{Gr}\,(1, 4), Q^*\otimes E(t)) = 0 \qquad \text{for } 1 \leqslant i \leqslant 3, \text{ for all } t \in Z$$

$$H^i\big(\mathrm{Gr}\,(1, 4), \bigwedge^2 Q^*\otimes E(t)\big) = 0 \qquad \text{for } 2 \leqslant i \leqslant 4, \text{ for all } t \in Z\,.$$

On Gr (1, 3) $\simeq Q_4$ a better criterion will be found in section 3 (theorem 3.3).

EVANS and GRIFFITH have proved in [8], th. 2.4, that if E is a vector bundle on $\boldsymbol{P}^n$

and $H^i(\boldsymbol{P}^n, E(t)) = 0$ for all $t \in Z$, for all i such that $0 < i < \operatorname{rank} E$, then E splits.

This improves Horrocks criterion when rank E is small.

Using the result of Evans and Griffith, by a proof similar to that of theorem 2.1, we obtain the following:

THEOREM 2.5. – Let E be a vector bundle on Gr (k, n). The following conditions are equivalent:

a) E splits

b) $H^i(\operatorname{Gr}(k, n), \overset{i_1}{\bigwedge} Q^* \otimes \dots \otimes \overset{i_s}{\bigwedge} Q^* \otimes E(t)) = 0$ for all $t \in Z$, $\forall i_1, \dots, i_s, i$ such that:

$$\begin{cases} \sum_{n=1}^{s} i_n \leqslant i < \sum_{n=1}^{s} i_n + \min\{\operatorname{rank} E, \dim \operatorname{Gr}(k-s, n-s)\} \\ 0 < i, \ 0 \leqslant i_1, \dots, i_s \leqslant n-k \,. \end{cases}$$

We recall now that Horrocks gave the following characterization of the bundle $\overset{j}{\bigwedge} Q^*$ on $\boldsymbol{P}^n$ (recall that $Q \simeq T\boldsymbol{P}^n(-1) = (\Omega^1(1))^*$ on $\boldsymbol{P}^n$):

$E \simeq (\overset{j}{\bigwedge} Q^*)^{\oplus r}$ if and only if $(n \geqslant 2)$:

E does not contain any line subbundle as direct summand, and

$$H^i(\boldsymbol{P}^n, E(t)) = \begin{cases} \boldsymbol{C}^r & \text{if } i = j, \ t = -j \\ 0 & \text{otherwise}. \end{cases}$$

We obtain the following result (for $k \geqslant 2$) exactly in the same way we obtained theorem 2.1:

THEOREM 2.6. – Let j such that $1 \leqslant j \leqslant n-k-1$, and let $k \geqslant 2$. Let E be a vector bundle on Gr (k, n). The following conditions are equivalent:

a) $E \simeq (\overset{j}{\bigwedge} Q^*)^{\oplus r}$;

b) E does not contain any line subbundle as direct summand and:

$$H^i(\operatorname{Gr}(k, n), \overset{i_1}{\bigwedge} Q^* \otimes \dots \otimes \overset{i_s}{\bigwedge} Q^* \otimes E(t)) = \begin{cases} 0 & \begin{array}{l} \forall i_1, \dots, i_s \text{ such that } s \leqslant k, \\ 0 \leqslant i_1, \dots, i_s \leqslant n-k, \text{ for all } t \in Z, \\ \forall i\colon 0 < i < \dim \operatorname{Gr}(k, n) \\ \text{with the only exception } s = k, \\ i_1 = \dots = i_s = j, \ i = j(k+1), \\ t = -j \end{array} \\ \boldsymbol{C}^r & \begin{array}{l} \text{if } i_1 = \dots = i_s = j, \ s = k \\ i = j(k+1), \ t = -j; \end{array} \end{cases}$$

c) E does not contain any line subbundle as direct summand and:

$$H^i\big(\mathrm{Gr}\,(k,n), \bigwedge^{i_1} Q^*\otimes\ldots\otimes\bigwedge^{i_s} Q^*\otimes E(t)\big)=0 \qquad \forall i_1,\ldots,i_s, i \text{ such that:}$$

$$\begin{cases} \sum_{n=1}^{s} i_n \leqslant i < \sum_{n=1}^{s} i_n + \dim \mathrm{Gr}\,(k-s, n-s) \\ 0<i,\ 0\leqslant i_1,\ldots,i_s\leqslant n-k, \\ \text{with the only exception } s=k,\ i_1=\ldots=i_s=j,\ i=j(k+1),\ t=-j \end{cases}$$

$$H^{j(k+1)}\big(\mathrm{Gr}\,(k,n), \overbrace{\bigwedge^{j} Q^*\otimes\ldots\otimes\bigwedge^{j} Q^*}^{k \text{ times}}\otimes E(-j)\big)=\boldsymbol{C}^r$$

$$H^i\big(\mathrm{Gr}\,(k,n), \bigwedge^{i} Q^*\otimes \bigwedge^{n-k-i} Q^*\otimes E\big)=0 \quad \text{for } 1\leqslant i\leqslant n-k\,.$$

REMARK 2.7. – As $Q \simeq \bigwedge^{n-k-1} Q^*(1)$, theorem 2.6 gives also a cohomological characterization of the quotient bundle.

From now on, we specialize to the case rank $E=2$. We point out that in this case: $E^*\simeq E(-c_1(E))$, regarding $c_1(E)$ as an integer.

It is well known that if l is a line on Gr (k,n) (i.e. a Schubert cycle of dimension 1, consisting of all $\boldsymbol{P}^k$ such that $\boldsymbol{P}_0^{k-1}\subset\boldsymbol{P}^k\subset\boldsymbol{P}_0^{k+1}$ with $\boldsymbol{P}_0^{k-1}, \boldsymbol{P}_0^{k+1}$ fixed subspaces of $\boldsymbol{P}^n$) then $E|_l\simeq \mathcal{O}_l(p)\oplus\mathcal{O}_l(q)$ with $p+q=c_1(E)$.

When p, q do not depend on the line l, the bundle E is called uniform. VAN DE VEN [24] and GUYOT [11] have shown that uniform 2-bundles always split on Gr (k,n) $(n\geqslant 3)$, except in the case $k=1$ when also the 2-bundle $S(t)$ is uniform, and $k=n-2$ when also the 2-bundle $Q(t)$ is uniform.

Let us consider the bundle

$$F=Q^{\oplus k}\oplus\mathcal{O}(1)^{\oplus n-k-1}$$

F is a globally generated vector bundle of rank $(k+1)(n-k)-1$. Note that $\det F=\mathcal{O}(n-1)$.

A generic section of F vanishes on a line l, and the following Koszul complex is exact (we set $d=(k+1)(n-k)$):

$$0\to\bigwedge^{d-1}F^*\to\ldots\to\bigwedge^{2}F^*\to F^*\to\mathcal{O}_{\mathrm{Gr}}\to\mathcal{O}_l\to 0\,. \tag{7}$$

We recall also that twisting by $\mathcal{O}(t)$, we can suppose that $c_1(E)=0$ or $c_1(E)=-1$. In fact E is uniform if and only if $E(t)$ is uniform. A 2-bundle with $c_1(E)=0$ or -1 is called normalized.

Observe that an iterated application of the canonical decomposition

$$\bigwedge^{n}(A\oplus B)=\bigoplus_{i=0}^{n}\big(\bigwedge^{i}A\otimes\bigwedge^{n-i}B\big)\,, \quad \text{where } A, B \text{ are vector spaces}\,,$$

shows that $\bigwedge^{i} F^*$ is the direct sum of some bundles isomorphic to

$$\bigwedge^{r_1} Q^* \otimes \ldots \otimes \bigwedge^{r_k} Q^*\Big(\sum_{j=1}^{k} r_j - i\Big) \quad \text{with } 0 \leq \sum_{j=1}^{k} r_j \leq i\,.$$

LEMMA 2.8. – On Gr $(1, n)$, we have, for all i, j such that $0 < i < 2n-2$, $1 \leq j \leq n-2$, for all $t \in Z$:

$$H^i\Big(\mathrm{Gr}\,(1, n), \bigwedge^{j} Q^* \otimes S(t)\Big) = 0$$

with the only exceptions:

$$H^1(\mathrm{Gr}\,(1, n), Q^* \otimes S) \simeq H^{2n-3}\Big(\mathrm{Gr}\,(1, n), \bigwedge^{n-2} Q^* \otimes S(1-n)\Big) = \boldsymbol{C}\,.$$

PROOF. – It is convenient to use Serre duality first. Then the lemma is a standard application of Bott theorem. In fact S^* belongs to the irreducible representation with highest weight λ_n.

We get the following

THEOREM 2.9. – Let E be a normalized 2-bundle on Gr (k, n) $(n \geq 3)$.

(i) If $c_1(E) = 0$, E splits if and only if either one of the following holds:

a) $H^i\big(\mathrm{Gr}\,(k, n), \bigwedge^{i-1} F^* \otimes E(-1)\big) = 0$ for $i = 1, \ldots, d-2$;

a') $H^i\big(\mathrm{Gr}\,(k, n), \bigwedge^{i} F^* \otimes E(-1)\big) = 0$ for $i = 2, \ldots, d-1$.

(ii) If $c_1(E) = -1$, E splits if and only if either one of the following holds:

b) $H^i\big(\mathrm{Gr}\,(k, n), \bigwedge^{i-1} F^* \otimes E(-1)\big) = 0$ for $i = 1, \ldots, d-1$;

b') $H^i\big(\mathrm{Gr}\,(k, n), \bigwedge^{i} F^* \otimes E\big) = 0$ for $i = 1, \ldots, d-1$.

(iii) If $c_1(E) = -1$, E splits or $k = 1$ and $E \simeq S$ if and only if either one of the following holds:

*b*1) $H^i\big(\mathrm{Gr}\,(k, n), \bigwedge^{i-1} F^* \otimes E(-1)\big) = 0$ for $i = 1, \ldots, d-2$;
$H^{d-1}(\mathrm{Gr}\,(k, n), E(-n)) = 0$

*b*1') $H^i\big(\mathrm{Gr}\,(k, n), \bigwedge^{i} F^* \otimes E\big) = 0$ for $i = 2, \ldots, d-1$;
$H^1(\mathrm{Gr}\,(k, n), E) = 0$

(iv) If $c_1(E) = -1$, E is uniform if and only if either one of the following holds:

c) $H^i\big(\mathrm{Gr}\,(k, n), \bigwedge^{i-1} F^* \otimes E\big) = 0$ for $i = 1, \ldots, d-1$

c') $H^i(\mathrm{Gr}\,(k,n), \overset{i}{\wedge} F^* \otimes E(-1)) = 0$ for $i = 1, \ldots, d-1$

d) $H^i(\mathrm{Gr}\,(k,n), \overset{i-1}{\wedge} F^* \otimes E) = 0$ for $i = 1, \ldots, d-2$;

$H^{d-1}(\mathrm{Gr}\,(k,n), E(-n+1)) = 0$

d') $H^i(\mathrm{Gr}\,(k,n), \overset{i}{\wedge} F^* \otimes E(-1)) = 0$ for $i = 2, \ldots, d-1$;

$H^1(\mathrm{Gr}\,(k,n), E(-1)) = 0$

PROOF. – First observe that a) and a'), b) and b') and so on, are equivalent by Serre duality and by the isomorphism $\overset{i}{\wedge} F^* \simeq \overset{d-i-1}{\wedge} F \otimes \det F^*$ (we recall that $K_{\mathrm{Gr}(k,n)} \simeq \mathcal{O}(-n-1)$ is the canonical bundle).

If E splits, all conditions hold by theorem 2.1. If $k = 1$ and $E \simeq S$, condition $b1$) holds by lemma 2.8. If E is uniform, conditions c) and d) hold by theorem 2.1 and lemma 2.8.

Let now $c_1(E) = 0$. If a) holds, we want to show that E is uniform. We tensor (7) by $E(-1)$. Then from our hypothesis and from lemma 1.1 (ii) we get that if l, l' are any two lines in $\mathrm{Gr}\,(k,n)$:

$$H^0(l, E(-1)|_l) \simeq H^0(l', E(-1)|_{l'}) .$$

This means exactly that E is uniform.

Since the bundles $S(t), Q(t)$ have odd first Chern class, then they are not isomorphic to E. So E must split, as claimed.

Let now $c_1(E) = -1$. The proof is similar, but in order to show that E is uniform, it is sufficient to verify that:

$$H^0(l, E|_l) = H^0(l', E|_{l'}) \tag{8}$$

or:

$$H^0(l, E(-1)|_l) = H^0(l', E(-1)|_{l'}) \tag{9}$$

for l', l' any two lines in $\mathrm{Gr}\,(k,n)$. From b) or $b1$) we get (9). From c) or d) we get (8). In case b) the possibilities $E \simeq S$ for $k = 1$ or $E \simeq Q^*$ for $k = n-1$ are excluded by lemma 2.8 and lemma 1.3. q.e.d.

By the well known Hartshorne-Serre correspondence between vector bundles of rank 2 and 2-codimensional subcanonical smooth subvarieties (see [25] theorem 2.1 and 2.2) we can state Theorem 2.9 in the following equivalent form (for simplicity we state only the cases a), b) and $b1$)).

THEOREM 2.10. – Let $X \subset \mathrm{Gr}\,(k,n)$ be a smooth subvariety of codimension 2. Suppose that $K_X \simeq \mathcal{O}_{\mathrm{Gr}(k,n)}(a)|_X$ for some $a \in Z$ (i.e. X is a-subcanonical).

(i) If $a+n+1$ is even, then X is a complete intersection if and only if one of the following holds:

$$a)\ H^i\left(\mathrm{Gr}(k,n), \bigwedge^{i-1} F^* \otimes \mathcal{J}_X\left(\frac{a+n-1}{2}\right)\right) = 0 \qquad \text{for } i = 1, \dots, d-2$$

$$a')\ H^i\left(\mathrm{Gr}(k,n), \bigwedge^{i} F^* \otimes \mathcal{J}_X\left(\frac{a+n-1}{2}\right)\right) = 0 \qquad \text{for } i = 2, \dots, d-2;$$

$$H^1\left(\mathrm{Gr}(k,n), \mathcal{J}_X\left(\frac{a+n-1}{2}\right)\right) = 0$$

(ii) If $a+n+1$ is odd, then X is a complete intersection if and only if one of the following holds:

$$b)\ H^i\left(\mathrm{Gr}(k,n), \bigwedge^{i-1} F^* \otimes \mathcal{J}_X\left(\frac{a+n-2}{2}\right)\right) = 0 \qquad \text{for } i = 1, \dots, d-2;$$

$$H^1\left((\mathrm{Gr}, k, n), F^* \otimes \mathcal{J}_X\left(\frac{a+n}{2}\right)\right) = 0$$

$$b')\ H^i\left(\mathrm{Gr}(k,n), \bigwedge^{i} F^* \otimes \mathcal{J}_X\left(\frac{a+n}{2}\right)\right) = 0 \qquad \text{for } i = 1, \dots, d-2;$$

$$H^1\left(\mathrm{Gr}(k,n), \mathcal{J}_X\left(\frac{a+n-2}{2}\right)\right) = 0$$

(iii) If $a+n+1$ is odd then X is a complete intersection or $k=1$ and X is the zero locus of a section of $S(t)$ if and only if one of the following holds:

$$b1)\ H^i\left(\mathrm{Gr}(k,n), \bigwedge^{i-1} F^* \otimes \mathcal{J}_X\left(\frac{a+n-2}{2}\right)\right) = 0 \qquad \text{for } i = 1, \dots, d-2;$$

$$H^1\left(\mathrm{Gr}(k,n), \mathcal{J}_X\left(\frac{a+n}{2}\right)\right) = 0$$

$$b2)\ H^i\left(\mathrm{Gr}(k,n), \bigwedge^{i} F^* \otimes \mathcal{J}_X\left(\frac{a+n}{2}\right)\right) = 0 \qquad \text{for } i = 2, \dots, d-2;$$

$$H^1\left(\mathrm{Gr}(k,n), \mathcal{J}_X\left(\frac{a+n}{2}\right)\right) = H^1\left(\mathrm{Gr}(k,n), \mathcal{J}_X\left(\frac{a+n-2}{2}\right)\right) = 0$$

PROOF. – The normal bundle of X in Gr (k, n) extends to a 2-bundle E on Gr (k, n), with $c_1(E) = a+n+1$, $E|_X \simeq N_{X|\mathrm{Gr}(k,n)}$.

We have an exact sequence

$$(10) \qquad 0 \to \mathcal{O}_{\mathrm{Gr}} \to E \to \mathcal{J}_X(a+n-1) \to 0\,.$$

We normalize E after twisting by $\mathcal{O}(-(a+n+1)/2)$ when $a+n+1$ is even, and by $\mathcal{O}(-(a+n+2)/2)$ when $a+n+1$ is odd. Then, we can tensor (10) by suitable wedge powers of F^*, and then we apply theorem 2.9, lemma 1.3 and Serre duality.

REMARK 2.11. - When $k = 0$ or $k = n-1$, the Grassmannian $\mathrm{Gr}(k, n)$ is isomorphic to the projective space $\boldsymbol{P}^n$. In this case $F^* = \mathcal{O}(-1)^{\oplus n-1}$, and in theorems 2.9 and 2.10 we can read $\mathcal{O}_{P^n}(-i)$ in place of $\bigwedge^i F^*$.

Condition $a)$ is exactly Cor. 1.8 (i) of [7] (our proof is different).

On $\boldsymbol{P}^n$ conditions $b)$ and $b1)$ of theorem 2.9 are equivalent (observe that in this case we can ask that b or $b1)$ be fulfilled only for $i = 1, \dots, [n/2]$ by Serre duality), and are exactly Cor. 1.8 (ii) of [7].

Condition $c)$ is weaker than Cor. 1.8 (iii) of [7].

Condition $d)$ is apparently new for $n \geq 4$.

We want to point out the following

THEOREM 2.12 (Sommese). - Let $X \subset \mathrm{Gr}(k, n)$ be a smooth subvariety of codimension 2.

If $n \geq 6$ then $\mathrm{Pic}(X)$ is generated by the hyperplane section. In particular X is subcanonical.

PROOF. - In [21] ((3.5) and (3.6.3)) is proved that, if $x_0 \in X$:

$$\pi_j(\mathrm{Gr}(k, n), X, x_0) = 0 \qquad \text{for } j \leq n + 1 - 2 \operatorname{codim} X$$

Then, by the relative Hurewicz theorem ([22] ch. 7 sect. 5.4)

$$H_j(\mathrm{Gr}(k, n), X, Z) = 0 \qquad \text{for } j \leq n + 1 - 2 \operatorname{codim} X .$$

By (10), cor. 23.14, it follows that $H^j(\mathrm{Gr}(k, n), X, Z) = 0$ for $j \leq n + 1 - 2 \operatorname{codim} X$.

So in our hypothesis $H^j(\mathrm{Gr}(k, n), X, Z) = 0$ for $j \leq n-3$. As $n \geq 6$, we get in particular $H^j(\mathrm{Gr}(k, n), X, Z) = 0$ for $j \leq 3$. From the exact cohomology sequence of the pair $(\mathrm{Gr}(k, n), X)$ it follows that $H^1(X, Z) = 0$ and that $H^2(X, Z) = Z$ is generated by the hyperplane section. Observe that by Hodge decomposition $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Now from the cohomology sequence associated to the exponential sequence

$$0 \to Z \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

we get the result.

Let E be a 2-bundle and let $l \subset \mathrm{Gr}(k, n)$ be a line. If $E|_l \simeq \mathcal{O}_l(a) \oplus \mathcal{O}_l(b)$ we define, as usual:

$$d_l(E) := \begin{cases} \frac{1}{2}|b-a| & \text{if } c_1(E) \text{ is even} \\ \frac{1}{2}(|b-a|+1) & \text{if } c_1(E) \text{ is odd} \end{cases}$$

and $d(E) := d_l(E)$ for generic l.

THEOREM 2.13. – Let E be a 2-bundle on Gr (k, n). Let l, l' be any two lines on Gr (k, n). Let $F = Q^{\oplus k} \oplus \mathcal{O}(1)^{\oplus n-k-1}$. Then the following inequalities hold:

$$|d_l(E) - d_{l'}(E) \leq \sum_{i=1}^{d-1} h^i(\bigwedge^{i-1} F^* \otimes E(k))$$

$$|d_l(E) - d_{l'}(E)| \leq \sum_{i=1}^{d-2} h^i(\bigwedge^{i-1} F^* \otimes E(k)) + h^{d-1}(\bigwedge^{d-1} F^* \otimes E(k))$$

$$\text{for } \left|k + \frac{c_1(E)}{2} + 1\right| \leq d(E) .$$

PROOF. – It is easy to check that, if

$$\left|k + \frac{c_1(E)}{2} + 1\right| \leq d(E) , \tag{11}$$

then

$$h^0(E(k)|_l) = d_l(E) + k + \left[\frac{c_1(E)}{2}\right] + 1 .$$

Thus, for k in the range of (11) we have

$$|d_l(E) - d_{l'}(E)| = |h_0(E(k)|_l) - h_0(E(k)|_{l'})| .$$

Now it is sufficient to look at the Koszul complexes of l (which is (7)) and l' and apply lemma 1.2.

REMARK 2.14. – If s is the minimum integer such that $h^0(E(s)) \neq 0$, then

$$-\left[\frac{c_1(E)}{2}\right] - s \leq d(E) \leq d_l(E)$$

for each line $l \subset$ Gr (k, n). This means that when E is « very unstable » (i.e. $s \ll 0$) then the inequalities of theorem 2.13 hold for k in a wide range. Observe that when E is not uniform, the theorem says that the right-hand sides of the inequalities are nonzero.

3. – Splitting criteria on quadrics.

We recall now from [20] the definition and some properties of spinor bundles on Q_n.

Let S_k be the spinor variety which parametrizes the family of $(k-1)$-planes in Q_{2k-1} or one of the two disjoint families of k-planes in Q_{2k}.

We have $\dim S_k = (k(k+1))/2$, Pic $(S_k) = Z$ and $h^0(S_k, \mathcal{O}(1)) = 2^k$. Spinor varieties are rational homogeneous manifolds of rank 1 [23]. When $n = 2k-1$ is odd, consider $\forall x \in Q_{2k-1}$ the variety $\{P^{k-1} \in$ Gr $(k-1, 2k) | x \in {}^{k-1} \subset Q_{2k-1}\}$. This variety is isomorphic to S_{k-1} and we denote it by $(S_{k-1})_x$. Then we have a natural embedding

$$(S_{k-1})_x \xrightarrow{i_x} S_k .$$

Considering the linear spaces spanned by these varieties, we have $\forall x \in Q_{2k-1}$ a natural inclusion $H^0((S_{k-1})_x, \mathcal{O}(1))^* \to H^0(S_k, \mathcal{O}(1))^*$ and then an embedding

$$s: Q_{2k-1} \to \mathrm{Gr}\,(2^{k-1}-1, 2^k-1) .$$

In the same way, when $n = 2k$ is even, we have two embeddings:

$$s': Q_{2k} \to \mathrm{Gr}\,(2^{k-1}-1, 2^k-1)$$
$$s'': Q_{2k} \to \mathrm{Gr}\,(2^{k-1}-1, 2^k-1)$$

If U is the universal bundle of $\mathrm{Gr}\,(2^{k-1}-1, 2^k-1)$ we call

$s^* U$ the spinor bundle on Q_{2k-1}

$s'^* U, s''^* U$ the two spinor bundles on Q_{2k} .

As $S_1 = \boldsymbol{P}^1$, $S_2 = \boldsymbol{P}^3$, it is easy to verify that on $Q_4 \simeq \mathrm{Gr}\,(1, 3)$ the two spinor bundles are the universal bundle and the dual of the quotient bundle.

We summarize the results that we need in the following theorem (see [20] theorems 1.4 and 2.3).

THEOREM 3.1. – (i) Let S', S'' be the spinor bundles on Q_{2k}, let $i: Q_{2k-1} \to Q_{2k}$ be a smooth hyperplane section. Then $i^*S' \simeq i^*S'' \simeq S$ spinor bundle on Q_{2k-1}.

(ii) Let S be the spinor bundle on Q_{2k+1}, let $i: Q_{2k} \to Q_{2k+1}$ be a smooth hyperplane section. Then $i^*S \simeq S' \oplus S''$, where S', S'' are the spinor bundles on Q_{2k}:

(iii) Let S be a spinor bundle on Q_n.

Then:

$$H^i(Q_n, S(t)) = 0 \quad \text{for } 0 < i < n , \quad \text{for all } t \in Z .$$

Consider now the problem of finding some cohomological conditions for a vector bundle E on Q_n $(n \geqslant 3)$ that are equivalent to the splitting of E.

It is well known that if E splits on Q_n then:

$$H^i(Q_n, E(t)) = 0 \quad \text{for } 0 < i < n , \quad \forall t \in Z . \tag{12}$$

As in the case of Grassmannians, by theorem 3.1 (iii) we get that condition (12) is too weak to force E to split.

So also in this case it is natural to look for more vanishing conditions.

LEMMA 3.2. – Let E be a vector bundle on Q_n $(n\geq 3)$, let S be a spinor bundle on Q_n. Then E splits if and only if

$$H^i(Q_n, E(t)) = H^i(Q_n, E\otimes S(t)) = 0 \quad \text{for } 0<t<n \quad \forall t\in Z .$$

PROOF. – If E splits, we have see that $H^i(Q_n, E(t)) = H^i(Q_n, E\otimes S(t)) = 0$ for $1\leq i\leq n-1$, for all $t\in Z$.

For the converse, we prove first the result on Q_3.

If l is a line on Q_3, then $E|_l$ splits by Grothendieck theorem, so there exists a splitting bundle F on Q_3 and a isomorphism $\alpha\colon F|_l\to E|_l$, $\alpha\in H^0(l, (F^*\otimes E)|_l)$.

We have the following exact sequence of sheaves on Q_3 (it is the Koszul complex of a section of S^*, S spinor bundle on Q_3):

$$0\to \mathcal{O}(-1)\to S\to \mathcal{J}_l\to 0 . \tag{13}$$

The obstruction to extend α to $H^0(Q_3, F^*\otimes E)$ lies in $H^1(Q_3, F^*\otimes E\otimes \mathcal{J}_l)$. We tensor (13) by $F^*\otimes E$ and we obtain the exact sequence:

$$0\to F^*\otimes E(-1)\to F^*\otimes E\otimes S\to F^*\otimes E\otimes \mathcal{J}_l\to 0 .$$

As F splits, by hypothesis:

$$H^1(Q_3, F^*\otimes E\otimes S) = 0 \qquad H^2(Q_3, F^*\otimes E(-1)) = 0$$

so that $H^1(Q_3, F^*\otimes E\otimes I_l) = 0$ and we can choose a homomorphism $\alpha'\colon F\to E$ which restricts to α on l.

As F and E have the same first Chern class,

$$\det\alpha\in H^0\big(Q_3, \mathcal{O}(c_1(E) - c_1(F))\big) = H^0(Q_3, \mathcal{O}) = \boldsymbol{C} .$$

As $\det\alpha$ is nonzero on l, it must be nonzero everywhere. Then α is an isomorphism, as we wanted.

If $n\geq 3$ the result follows by induction on n. In fact, if

$$H^i(Q_{n+1}, E(t)) = 0 \qquad \text{for } 1\leq i\leq n, \quad \forall t\in Z$$
$$H^i(Q_{n+1}, E\otimes S(t)) = 0 \qquad \text{for } 1\leq i\leq n, \quad \forall t\in Z$$

then from the exact sequences on Q_{n+1} (Q_n is a smooth hyperplane section):

$$0\to E(t-1)\to E(t)\to E(t)|_{Q_n}\to 0$$
$$0\to E(t-1)\otimes S\to E(t)\otimes S\to E(t)\otimes S|_{Q_n}\to 0 \qquad (\text{for all } t\in Z)$$

and theorem 3.1 it follows that if S_0 is a spinor bundle on Q_n then:

$$H^i(Q_n, E|_{Q_n}(t)) = 0 \qquad \text{for } 1 \leqslant i \leqslant n-1 \quad \forall t \in Z$$
$$H^i(Q_n, E \otimes S_0|_{Q_n}(t)) = 0 \quad \text{for } 1 \leqslant i \leqslant n-1 \quad \forall t \in Z\,.$$

By the induction hypothesis $E|_{Q_n}$ splits then there exists a splitting bundle B on Q_{n+1} and a isomorphism $\alpha\colon B|_{Q_n} \to E|_{Q_n}$. As in the previous cases, the vanishing of $H^1(Q_{n+1}, E(t))$ $\forall t \in \mathbf{Z}$ allows to extend α on Q_{n+1} in such a way that E splits.

COROLLARY 3.3. – Let E be a vector bundle on Q_n and let $Q_2 \subset Q_n$ be a smooth plane section. Then E splits if and only if $E|_{Q_n}$ splits.

PROOF. – Cut Q_n with hyperplanes and use theorems 3.1, 3.2 and theorem B.
We can prove now our main result:

THEOREM 3.4. – Let E be a vector bundle on Q_n $(n \geqslant 3)$, let S be a spinor bundle on Q_n. Then E splits if and only if

(i) $H^i(Q_n, E(t)) = 0$ for $2 \leqslant i \leqslant n-1$ $\forall t \in \mathbf{Z}$;

(ii) $H^i(Q_n, E \otimes S(t)) = 0$ for $1 \leqslant i \leqslant n-2$ $\forall t \in \mathbf{Z}$.

PROOF. – It suffices to observe that in the proof of Theorem 3.2 the hypothesis $H^{n-1}(Q_n, E \otimes S(t)) = 0$ is not needed and the hypothesis $H^1(Q_n, E(t)) = 0$ is needed only to prove that if $E|_{Q_{n-1}}$ splits then also E splits, but this assured by Corollary 3.3.

REMARK. – Lemma 3.2 follows also from the following result, proved by KNÖRRER, BUCHWEITZ, GREUEL and SCHREIER in [6], [15] conj. B remark 2 with completely different techniques.

THEOREM 3.5. – Let E be a vector bundle on Q_n. $H^i(Q_n, E(t)) = 0$ for $0 < i < n$ $\forall t \in \mathbf{Z}$ if and only if E is isomorphic to a direct sum of line bundles and spinor bundles twisted by some $\mathcal{O}(t)$ (for $n = 2$ the line bundles must be of type $\mathcal{O}(t, t)$).
Looking at the previous theorem, we can give an elementary proof of the weaker:

THEOREM 3.6. – Let E be a vector bundle on Q_n,

(i) If $n \geqslant 3$ and $H^i(Q_n, E(t)) = 0$ for $1 \leqslant i \leqslant n-1$, $\forall t \in Z$, then E is uniform.

(ii) If $n = 2$ and $H^1(Q_2, E(t)) = 0$ for all $t \in Z$, then E is uniform separately on each family of lines on Q_2.

PROOF. – Let S be the spinor bundle on Q_3. For each line l on Q_3 there is a section of S^* which vanishes exactly on l. For any two lines l, l' tensoring by $E(t)$

the respective Koszul complexes, we get the exact sequences:

$$0 \to E(t-1) \to S \otimes E(t) \to E(t) \to E(t)|_l \to 0 ,$$
$$0 \to E(t-1) \to S \otimes E(t) \to E(t) \to E(t)|_{l'} \to 0 .$$

Considering the associated exact sequences of cohomology groups, it is an easy matter to check from our hypothesis that (see lemma 1.1 (ii)):

$$H^0(l, E(t)|_l) \simeq H^0(l', E(t)|_{l'}) \qquad \forall t \in Z .$$

This means exactly that E is uniform, as we wanted.

For $n \geqslant 3$ the result follows by induction on n using the fact that a bundle on Q_n which is uniform on every smooth hyperplane section is uniform.

For $n = 2$ the proof is similar.

We now specialize to the case: rank $E = 2$.

THEOREM 3.7. – Let E be a 2-bundle on Q_n, $n \geqslant 3$, let S be a spinor bundle.

(*a*) If $c_1(E) = 0$, E splits if and only if

$$H^i(Q_n, E(-i)) = 0 \quad \text{ for } 1 \leqslant i \leqslant \left[\frac{n}{2}\right].$$

(*b*) If $c_1(E) = -1$, E splits if and only if

$$H^i(Q_n, E(-i)) = 0 \qquad \text{for } 1 \leqslant i \leqslant n-2$$
$$H^i(Q_n, E(-i+1) \otimes S) = 0 \quad \text{for } 2 \leqslant i \leqslant n-1 .$$

(*c*) If $c_1(E) = -1$, E is uniform (and hence splits for $n \geqslant 5$) if and only if:

$$H^i(Q_n, E(-i)) = 0 \quad \text{ for } 1 \leqslant i \leqslant n-1$$

PROOF. – If E splits or is uniform, all conditions hold. In fact, by [9], all uniform 2-bundles on Q_n ($n \geqslant 3$) either split or are spinor bundles (up to tensoring by some line bundle).

Observe that by Serre duality the vanishing of $H^i(Q_n, E(-i))$ for $1 \leqslant i \leqslant [n/2]$ in case (*a*) is equivalent to the same condition for $1 \leqslant i \leqslant n-1$. In fact, if $c_1(E) = 0$ and E is a 2-bundle, then $E \simeq E^*$.

First we prove the result on Q_3: As in the proof of theorem 3.5, for each line $l \subset Q_3$ we have an exact sequence:

$$0 \to E(-2) \to E(-1) \otimes S \to E(-1) \to E(-1)|_l \to 0 .$$

Then each one of our hypothesis implies that $h^0(l, E(-1)|_l) = h^0(l', E(-1)|_{l'})$ for each lines l, l'. This means that E is uniform.

It remains to show that if $c_1(E) = -1$ and

$$H^1(Q_3, E(-1)) = H^2(Q_3, E\otimes S(-1)) = 0$$

then E splits (in fact the spinor bundle has odd first Chern class, so that there are no problems in the case $c_1(E) = 0$).

It is sufficient to note that $H^2(Q_3, S\otimes S(-1)) = \boldsymbol{C}$ (e.g. by Bott theorem) and so the case $E \simeq S$ must be excluded.

If $n \geqslant 3$ the proof is by induction on n, in the same way as in the proof of lemma 3.2, using corollary 3.3. $\lrcorner$

As in the case of Grassmannians, theorem 3.7 can be stated in the following equivalent form (for simplicity we state only the case (a), (b)):

THEOREM 3.8. – Let $X\subset Q_n$ be a smooth subvariety of codimension 2. Suppose that $K_X \simeq \mathcal{O}_{Q_n}(a)|_X$ for some $a\in \boldsymbol{Z}$ (i.e. X is a-subcanonical).

Let S be a spinor bundle on Q_n.

(i) If $n + a$ is even then X is a complete intersection if and only if

$$H^i\left(Q_n, \mathcal{J}_X\left(\frac{n+a}{2} - i\right)\right) = 0 \quad \text{for } 1 \leqslant i \leqslant [n/2]\,.$$

(ii) If $n + a$ is odd then X is a complete intersection if and only if the following hold:

$$H^i\left(Q_n, \mathcal{J}_X\left(\frac{n+a-1}{2} - i\right)\right) = 0 \qquad \text{for } 1 \leqslant i \leqslant n-2;$$

$$H^i\left(Q_n, \mathcal{J}_X\left(\frac{n+a+1}{2} - i\right)\otimes S\right) = 0 \quad \text{for } 2 \leqslant i \leqslant n-1\,.$$

PROOF. – By the Hartshorne-Serre correspondence [25], the normal bundle of X in Q_n extends to a 2-bundle E on Q_n. As $K_{Q_n} \simeq \mathcal{O}(-n)$, we have $c_1(E) = n + a$.

We get an exact sequence

$$0 \to \mathcal{O}_{Q_n} \to E \to \mathcal{J}_X(n+a) \to 0\,.$$

We normalize E after twisintg by $\mathcal{O}(-(n+a)/2)$ when $n + a$ is even and by $\mathcal{O}(-(n+a+1)/2)$ when $n + a$ is odd. Then, we apply theorem 3.7 and Serre duality.

We want to point out the following:

THEOREM 3.9 (Barth-Larsen). – Let $X\subset Q_n$ be a smooth subvariety of codimension 2. If $n\geqslant 7$ then Pic $(X) = \boldsymbol{Z}$ is generated by the hyperplane section. In particular, X is subcanonical.

PROOF. – X is a codimension 3 smooth subvariety of P^{n+1}.
Then, apply the Barth-Larsen theorem for subvarieties of P^{n+1} [16].

EXAMPLE 3.10. – Let C be a smooth subcanonical curve in P^4 which is embedded in a smooth quadric hypersurface Q_3.

If $K_C = \mathcal{O}_{P^4}(a)|_C$ with a odd, then C is a complete intersection of Q_3 and two other hypersurfaces of P^4 if and only if the restriction map

$$H_0\left(Q_3, \mathcal{O}\left(\frac{a+1}{2}\right)\right) \to H^0\left(C, \mathcal{O}\left(\frac{a+1}{2}\right)\right)$$

is surjective $\big($i.e. C is $((a+1)/2)$-normal in $Q_3\big)$.

If E is a 2-bundle on Q_n, and $l \subset Q_n$ is a line, define now $d_l(E)$ and $d(E)$ exactly as before theorem 2.13.

The proofs of the following two theorems are completely analogous to the proof of theorem 2.13 and are omitted.

THEOREM 3.11. – Let E be a 2-bundle on Q_3. Let l, l' be any two lines in Q_3 and let S be the spinor bundle on Q_3. Then the following inequalities hold:

$$|d_l(E) - d_{l'}(E)| \leqslant h^1(E(k)) + h^2(E(k) \otimes S)$$
$$|d_l(E) - d_{l'}(E)| \leqslant h^1(E(k)) + h^2(E(k-1))$$

for

$$\left|k + \frac{c_1(E)}{2} + 1\right| \leqslant d(E)\,.$$

THEOREM 3.12. – Let E be a 2-bundle on Q_4. Let l, l' be any two lines in Q_4, let S be a spinor bundle on Q_4 and let $F = S^* \otimes \mathcal{O}(1)$. Then the following inequalities hold:

$$|d_l(E) - d_{l'}(E)| \leqslant h^1(E(k)) + h^2(F^* \otimes E(k)) + h^3(\textstyle\bigwedge^2 F^* \otimes E(k))$$
$$|d_l(E) - d_{l'}(E)| \leqslant h^1(E(k)) + h^2(F^* \otimes E(k)) + h^3(E(k-1))$$

for

$$\left|k + \frac{c_1(E)}{2} + 1\right| \leqslant d(E)\,.$$

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