REGULARITY OF THE MODULI SPACE
OF INSTANTON BUNDLES $MI_{P^3}(5)$

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Abstract. We prove that the moduli space of mathematical instanton bundles on $P^3$ with $c_2 = 5$ is smooth.

Introduction

Instanton bundles were defined by Atiyah, Drinfeld, Hitchin and Manin [ADHM] in order to construct all the self-dual solutions of the Yang–Mills equation over $S^4$. A mathematical instanton bundle $E$ on $P^3 := P^3(C)$ can be defined as the cohomology bundle of a monad

$$ \mathcal{O}(-1)^k \rightarrow \mathcal{O}^{2k+2} \rightarrow \mathcal{O}(1)^k $$

on $P^3$. This is equivalent to the condition that $E$ is a stable bundle of rank 2 on $P^3$ such that $c_1(E) = 0$, $c_2(E) = k$, and $H^1(E(-2)) = 0$. If $E$ is a mathematical instanton bundle, then it is easy to check by using the Hirzebruch–Riemann–Roch Theorem that $h^1(S^2E) \geq \dim(T_E MI(k)) \geq 8k - 3$ and in case of equality, $MI(k)$ is smooth at $E$. So $8k - 3$ is the expected dimension of the moduli space of mathematical instanton bundles $MI_{P^3}(k) = MI(k)$. It is not known if the moduli space $MI(k)$ is a regular variety of pure dimension $8k - 3$. It is evident in the case $k = 1$. In the cases $2 \leq k \leq 4$ it was proved in [H], [ES] and [LeP]. In [Ch] and later in [NT] it was proved that $MI(k)$ is regular at bundles $E$ with $h^0(E(1)) \neq 0$. In [R2] (see also [S]) it was proved that $MI(k)$ is regular at bundles with a jumping line of maximal order. In this article we give a general proof of the regularity of $MI(k)$ for the cases $2 \leq k \leq 5$.

Theorem 1. For $2 \leq k \leq 5$ the moduli space $MI(k)$ of mathematical instantons is a regular variety of pure dimension $8k - 3$.

Our result should be compared with [AO2] (see also [R1]), where it was proved that the closure of $MI(5)$ in the Maruyama scheme of vector bundles of rank 2 with $c_1 = 0$, $c_2 = 5$ contains singular points. Our proof requires tools both from multilinear algebra and algebraic geometry.

*Supported by CRDF, grant RM1–206, and INTAS, grant INTAS–OPEN–97–1570.
**Partially supported by MURST funds.
Received January 9, 2002. Accepted February 6, 2003.
The authors are grateful to A. N. Tyurin for fruitful advice and thank G. Trautmann for useful comments and his simpler proof of Theorem 3. The first author thanks the Italian GNSAGA–INDAM for financial support. The second author is a member of the European network EAGER.

An invariant theoretical description of \( MI(k) \)

Our first goal is to describe the moduli space \( MI(k) \) in terms of invariant theory. The group \( SL_{2k+2} \) acts canonically on the space \( C^{2k+2} \). Let \( \omega \) be a nondegenerated 2-form on \( C^{2k+2} \) and \( Sp_{2k+2} \) the stabilizer of \( \omega \) in the group \( SL_{2k+2} \). The 2-form \( \omega \) defines canonically the \( Sp_{2k+2} \)-isomorphism \( C^{(2k+2)*} \cong C^{2k+2} \). We have the canonical actions of the group \( SL_4 \times SL_k \times Sp_{2k+2} \) on the spaces \( C^4, C^{k^2}, C^{2k+2}, C^k, C^k \times C^k, \ldots \).

We have the canonical quadratic \( SL_4 \times SL_k \times Sp_{2k+2} \)-morphism

\[
\gamma : C^{4*} \otimes C^{k^2} \otimes C^{2k+2} \rightarrow S^2C^{4*} \otimes \wedge^2 C^{k^2}.
\]

\( \gamma(A) \) is the symmetrization in the two indices corresponding to \( C^{4*} \) and the full contraction in the indices corresponding to \( C^{2k+2} \) of the tensor product \( A \otimes A \otimes \omega \). Also consider the canonical bilinear

\[
\beta : C^{4*} \otimes C^{k^2} \otimes C^{2k+2} \times C^{2k+2} \rightarrow C^{4*} \otimes C^{k^2},
\]

and

\[
\varepsilon : C^{4*} \otimes C^{k^2} \otimes C^{2k+2} \times C^4 \otimes C^k \rightarrow C^{2k+2}.
\]

Consider the following conditions for an element \( A \in C^{4*} \otimes C^{k^2} \otimes C^{2k+2} \):

\( (E_1) \) \( \varepsilon(A, f \otimes b) \neq 0 \) for all \( 0 \neq f \in C^4, 0 \neq b \in C^k \),

\( (E_2) \) \( \gamma(A) = 0 \),

\( (E_3) \) \( \beta(A, h) \neq 0 \) for all \( 0 \neq h \in C^{2k+2} \).

An element \( A \in C^{4*} \otimes C^{k^2} \otimes C^{2k+2} \) defines the sheaf morphism \( O^{2k+2} \rightarrow O(1)^k \). \( f_A \) is the composition \( C^{2k+2} \otimes O \rightarrow H^0(O(1)) \otimes C^{k^2} \otimes O \rightarrow C^{k^2} \otimes O(1) \), where \( H^0(O(1)) = C^{4*} \); the left map is given by \( A \), and the right map is the evaluation of \( H^0(O(1)) \) at points of \( P^1 \). The morphism \( f_A \) and the symplectic structure over \( O^{2k+2} \) define the sequence

\[
O(-1)^k \xrightarrow{f_A} O^{2k+2} \xrightarrow{f_A} O(1)^k.
\]

The condition \( (E_1) \) means that \( f_A \) is surjective or that \( \text{Ker} f_A \) is locally free. The condition \( (E_2) \) means that the above sequence is a complex. Therefore, \( (E_1) \) and \( (E_2) \) together mean that \( 1 \) is a monad according to [BH]. The condition \( (E_3) \) means moreover that the cohomology bundle \( E \) of the monad is a stable vector bundle. It is well known (see e.g., [AO1], Th. 2.8) that the conditions \( (E_1) \) and \( (E_2) \) imply \( (E_3) \).

Set

\[
I = \{ A \in C^{4*} \otimes C^{k^2} \otimes C^{2k+2} | \text{the condition } (E_i) \text{ holds for } A \},
\]

\[
I = I_1 \cap I_2 \cap I_3 = I_1 \cap I_2.
\]

and consider the canonical mapping \( \pi : I \rightarrow I/G \), where \( G = SL_k \times Sp_{2k+2} \times C^* \) and \( I/G \) is the set of \( G \)-orbits in \( I \).
Remark 1. In [CO] it was proved that there exists a structure of an affine variety on $I/G$ such that the mapping $\pi$ is the invariant-theoretical factorization. Moreover, the factor $I/G$ is the geometrical factor.

Lemma 2. For any $A \in I$ we have

$$\dim(T_{\pi(A)}MI(k)) = \dim(T_AI) - 3k^2 - 5k - 3.$$ 

Therefore, $\dim(T_{\pi(A)}MI(k)) \geq 8k - 3$ and

$$\dim(T_{\pi(A)}MI(k)) = 8k - 3 \text{ if and only if } \dim(T_AI) = 3k^2 + 13k.$$ 

This is a well known result (see [O], Pr. 1.4 for example). For the convenience of the reader here is the sketch of the proof. Let $K$ be the kernel of $f_A$ in (1). From (1) we get the two sequences:

$$0 \rightarrow \wedge^2(\mathcal{O}(-1)^k) \rightarrow K(-1)^k \rightarrow S^2K \rightarrow S^2E \rightarrow 0$$

and

$$0 \rightarrow S^2K \rightarrow S^2(\mathcal{O}^{2k+2}) \rightarrow \mathcal{O}(1)^{k(2k+2)} \xrightarrow{g_A} \wedge^2(\mathcal{O}(1)^k) \rightarrow 0.$$ 

From the first sequence it follows that $h^1(S^2E) = h^1(S^2K) - k^2$.

From the second sequence it follows that $h^1(S^2K) = \dim \ker(H^0(g_A)) - (2k+3)(k+1)$. Now observe that $H^0(g_A)$ is $d\gamma|_A$, hence $\ker(H^0(g_A))$ can be identified with $T_AI$ and this concludes the proof. □

Theorem 3. Suppose that $E$ is an instanton bundle on $\mathbb{P}^3$ and $H$ is a plane. Then $h^0(E|_H) \leq 1$.

Proof. (Trautmann) From the sequence

$$0 \rightarrow E(-2) \rightarrow E(-1) \rightarrow E|_H(-1) \rightarrow 0$$

we have $H^0(E|_H(-1)) = 0$. If $s$ is any section of $E|_H$, then its cokernel is the ideal sheaf $I_Z$ of a 0-dimensional subscheme $Z$ in $H$ because if $Z$ contains a divisorial component, then $H^0(E|_H(-1)) \neq 0$. Obviously, $H^0(I_Z) = 0$ hence $s$ must span $H^0(E|_H)$. □

Definition 1. $W(E) = \{H \in \mathbb{P}^{3*} \mid h^0(E|_H) \neq 0\}$ is called the variety (scheme) of unstable planes of $E$. Its scheme structure is defined as the degeneracy locus of the mapping

$$H^1(E(-1)) \otimes \mathcal{O} \rightarrow H^1(E) \otimes \mathcal{O}(1)$$

over $\mathbb{P}^{3*}$ (Theorem 3 shows that this map drops rank at most by one).

For an element $A \in \mathbb{C}^{3*} \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2}$ define the subvariety

$$X_A = \{(\mathcal{F}, \mathcal{B}) \in \mathbb{P}^{3*} \times \mathbb{P}^{k-1*} \mid f^* \otimes b^* \in \text{Im}(\beta(A, \cdot))\}.$$ 

Lemma 4. Let $q_1$ be the projection of $\mathbb{P}^{3*} \times \mathbb{P}^{k-1*}$ on $\mathbb{P}^{3*}$. We have $W(E) = q_1(X_A)$ and the fiber of the projection $X_A \rightarrow q_1(X_A)$ over $H$ is isomorphic to $\mathbb{P}(H^0(E|_H))$. 

Lemma 8. Suppose $A^0 \in I$ and $\dim(W(A^0)) > 3k^2 + 13k$; then there exists $0 \neq S^0 \in S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k$ such that $\xi(A^0, S^0) = 0$, where

$$\xi : \mathbb{C}^{k*} \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2} \times S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \longrightarrow \mathbb{C}^4 \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2}$$

is the canonical bilinear $\text{SL}_4 \times \text{SL}_k \times \text{Sp}_{2k+2}$-morphism.

Proof. From $\dim(T_{A^0} I) > 3k^2 + 13k$ it follows that the differential $(d\gamma|_{A^0})^*$ is noninjective. The differential $(d\gamma|_{A^0})^*$ is noninjective iff $(d\gamma|_{A^0})$ is nonsurjective. Therefore, the assumption of Lemma 8 is equivalent to $H^2(S^2 E^0) \neq 0$. The second symmetric power of the left-hand side of (1) gives $H^2(S^2 E^0) \simeq H^2(S^2(Ker f_{A^0}))$. The second symmetric power of the right hand side of (1) gives

$$H^2(S^2(Ker f_{A^0})) \simeq \text{Coker} \left[ H^0(O(1)) \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2} \xrightarrow{\Phi} H^0(O(2)) \otimes \wedge^2(\mathbb{C}^{k*}) \right].$$

Lemma 8 follows because the dual of $\Phi$ can be identified with $\xi(A^0, \cdot).$
Algebraic lemmas

In this section we prove some algebraic lemmas which we use in the proof of our main result.

**Lemma 9.** Suppose $R$ is a nonzero block-matrix:

\[
R = \begin{bmatrix} R^1 \\ R^2 \end{bmatrix},
\]

where $R^1$ is a skew-symmetric matrix of size $k \times k$; then there exists a column $v_0$ of height $k$ such that

\[
Rv_0 = \begin{bmatrix} \lambda_1 u_0 \\ \lambda_2 u_0 \end{bmatrix} \neq 0
\]

for some column $u_0$ of height $k$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

**Proof.** Suppose that $\det(R^1) \neq 0$. In this case set $v_0 \in \ker(R^2 - \mu_0 R^1)$, where $\mu_0$ is a root of the equation $\det(R^2 - \mu R^1) = 0$.

Suppose that $\det(R^1) = 0$. One can assume that

\[
R^1 = \begin{bmatrix} R_{11}^1 \\ 0 \\ 0 \end{bmatrix}, \quad R^2 = \begin{bmatrix} R_{11}^2 \\ R_{21}^2 \\ R_{22}^2 \end{bmatrix},
\]

where $R_{11}^1$ is a skew-symmetric matrix of size $k' \times k'$, $k' < k$, $\det(R_{11}^1) \neq 0$ and $R_{21}^2$ is a skew-symmetric matrix of size $k' \times k'$. If $R_{12}^2 \neq 0$ or $R_{22}^2 \neq 0$, then we set $v_0 = \begin{bmatrix} 0 \\ v'_0 \end{bmatrix}$ for some $v'_0$ such that $R_{12}^2 v'_0 \neq 0$ or $R_{22}^2 v'_0 \neq 0$. If $R_{12}^2 = 0$ and $R_{22}^2 = 0$, then $R_{21}^2 = 0$ and we set $v_0 = \begin{bmatrix} v'_0 \\ 0 \end{bmatrix}$, where

\[
\begin{bmatrix} R_{11}^1 \\ R_{21}^2 \end{bmatrix} v'_0 = \begin{bmatrix} \lambda_1 u'_0 \\ \lambda_2 u'_0 \end{bmatrix} \neq 0. \quad \square
\]

Consider the linear spaces $\mathbb{C}^4$ and $\mathbb{C}^k$. Let $f_1, \ldots, f_4$ be the standard basis of $\mathbb{C}^4$ and let $f_1^*, \ldots, f_4^*$ be the dual basis of the dual space $\mathbb{C}^{4*}$. Let $b_1, \ldots, b_k$ be the standard basis of $\mathbb{C}^k$ and let $b_1^*, \ldots, b_k^*$ be the dual basis of the dual space $\mathbb{C}^{k*}$. The group $\text{SL}_4$ acts canonically on the space $\mathbb{C}^4$ and the group $\text{SL}_k$ acts canonically on the space $\mathbb{C}^k$. So the actions of the group $\text{SL}_4 \times \text{SL}_k$ are defined on the spaces $\mathbb{C}^4, \mathbb{C}^{4*}, \mathbb{C}^k, \mathbb{C}^{k*}, \mathbb{C}^4 \otimes \mathbb{C}^k, \ldots$.

Consider the linear space $S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k$. For an element $S \in S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k$ define

\[
\text{rk}(S) = \dim(\text{Im}(\rho(S, \cdot))),
\]

where

\[
\rho : S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \times \mathbb{C}^{4*} \otimes \mathbb{C}^{k*} \longrightarrow \mathbb{C}^4 \otimes \mathbb{C}^k
\]

is the canonical bilinear $\text{SL}_4 \times \text{SL}_k$-morphism. Note that $\text{rk}(S)$ is an even number.

**Lemma 10.** Suppose $2 \leq k \leq 5$ and consider $S \in S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k$ such that $2 \leq \text{rk}(S) \leq 2k - 2$. Then one of the following conditions holds:
\[ \rho(S, B^* \otimes b^* \otimes b') = f_0 \otimes b_0^* \neq 0 \quad \text{for some} \quad B^* \in \mathbb{C}^{4*} \otimes \mathbb{C}^{k*}, \quad f_0 \in \mathbb{C}^{4^*}, \quad b_0^* \in \mathbb{C}^{k^*}. \]

(2) \( \text{rk} (S) = 6 \) and there exists \( 0 \neq f^* \in \mathbb{C}^{4^*} \) such that \( \rho(S, f^* \otimes b^*) = 0 \) for all \( b^* \in \mathbb{C}^{k^*}. \)

(3) \( \text{rk} (S) = 8 \) and \( \dim(Z_S) \geq 2 \), where
\[
Z_S = \{ (f^*, b^*) \in P^{3*} \times P^{k-1*} \mid \rho(S, f^* \otimes b^*) = 0 \},
\]
\[
P^{3*} = P^{C^{4*}}, P^{k-1*} = P^{C^{k*}}.
\]

Proof. Consider the coordinate expression of \( S \) in the bases \( \{ f_i \} \) and \( \{ b_i \} \):
\[
S = \sigma_{ij} f_i f_j \otimes b_i \wedge b_j.
\]

We get a block matrix \( \sigma \) defined by
\[
\sigma = (\sigma_{ij})_{1 \leq i, j \leq k} = \begin{bmatrix}
0 & \sigma_{12} & \ldots & \sigma_{1k} \\
\sigma_{21} & 0 & \ldots & \sigma_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{k1} & \sigma_{k2} & \ldots & 0
\end{bmatrix},
\]

where \( \sigma_{ij} = (\sigma_{ij})_{1 \leq l, p \leq 4} \) is a symmetric matrix of size \( 4 \times 4 \), \( \sigma_{ij} = -\sigma_{ji} \).

Rewrite the coordinate expression of \( S \) as
\[
S = \tilde{\sigma}_{ij} f_i f_j \otimes b_i \wedge b_j.
\]

Then we get a second block matrix \( \tilde{\sigma} \) defined by
\[
\tilde{\sigma} = (\tilde{\sigma}_{ij})_{1 \leq i, j \leq 4} = \begin{bmatrix}
\tilde{\sigma}_{11} & \tilde{\sigma}_{12} & \tilde{\sigma}_{13} & \tilde{\sigma}_{14} \\
\tilde{\sigma}_{21} & \tilde{\sigma}_{22} & \tilde{\sigma}_{23} & \tilde{\sigma}_{24} \\
\tilde{\sigma}_{31} & \tilde{\sigma}_{32} & \tilde{\sigma}_{33} & \tilde{\sigma}_{34} \\
\tilde{\sigma}_{41} & \tilde{\sigma}_{42} & \tilde{\sigma}_{43} & \tilde{\sigma}_{44}
\end{bmatrix},
\]

where \( \tilde{\sigma}_{ij} = (\tilde{\sigma}_{ij})_{1 \leq l, p \leq k} \) is a skew-symmetric matrix of size \( k \times k \), \( \tilde{\sigma}_{ij} = \tilde{\sigma}_{ji} \). Let \( r \) be the maximal rank of full contractions of \( S \otimes b^* \otimes b'^* \) over all \( b^*, b'^* \in \mathbb{C}^{k^*} \). Transform the basis \( \{ b_i \} \) and obtain
\[
r = \text{rk}(\sigma_{12}).
\]

We have
\[
2k - 2 \geq \text{rk}(S) = \text{rk}(\sigma) = \text{rk}(\tilde{\sigma}) \geq 2 \text{rk}(\sigma_{12}) = 2r.
\]

Therefore one of the following cases holds:

(a) \( r = 1 \) or \( 2 \),
(b) \( r = 3, \text{rk}(\sigma) = 6, \) and \( k \geq 4 \),
(c) \( r = 4, \text{rk}(\sigma) = 8, \) and \( k = 5 \),
(d) \( r = 3, \text{rk}(\sigma) = 8, \) and \( k = 5 \).
Transform the basis \( \{ f_i \} \) and obtain

\[
\sigma_{lp}^{12} = \begin{cases} 
1 & \text{if } 1 \leq l = p \leq r, \\
0 & \text{if } l \neq p \text{ or } l = p > r.
\end{cases}
\]  

(3)

From (2) it follows that \( \sigma_{lp}^{ij} = 0 \) for \( l, p > r \), whence

\[
\tilde{\sigma}^{ij} = 0 \quad \text{for } i, j > r.
\]  

(4)

(a) Consider the case (a).

In this case we prove that the condition (1) holds, i.e., we prove that there exists a column \( f^0 \) of height 4 and columns \( b^0, B^{*01}, \ldots, B^{*04} \) of height \( k \) such that

\[
\tilde{\sigma} \begin{bmatrix} B^{*01} \\ \vdots \\ B^{*04} \end{bmatrix} = \begin{bmatrix} f_1^0 b^0 \\ \vdots \\ f_4^0 b^0 \end{bmatrix} \neq 0.
\]

Suppose that \( \tilde{\sigma}^{ij} \neq 0 \) for some \( 1 \leq i \leq 2 \) and \( 3 \leq j \leq 4 \). In this case we set \( B^{*0k} = 0 \) for all \( k \neq j \) and choose \( B^{*0j} \) by using (4) and Lemma 9.

Suppose that \( \tilde{\sigma}^{ij} = 0 \) for all \( 1 \leq i \leq 2 \) and \( 3 \leq j \leq 4 \). We have \( \tilde{\sigma}^{lp} \neq 0 \) for some \( 1 \leq l \leq 2 \) and \( 1 \leq p \leq 2 \). In this case we set \( B^{*0k} = 0 \) for all \( k \neq p \) and choose \( B^{*0p} \) by using (4) and Lemma 9.

(b) Consider the case (b).

In this case we prove that the condition (2) holds, i.e., we prove that there exists a column \( f^{*0} \) of height 4 such that

\[
\sigma \begin{bmatrix} b_1^* f^{*0} \\ \vdots \\ b_k^* f^{*0} \end{bmatrix} = 0
\]  

(5)

for any column \( b^* \) of height \( k \).

From the condition \( \text{rk}(\sigma) = 6 \) and 2 it follows that

\[
\sigma^{ij} = \begin{bmatrix} 
\sigma_{11}^{ij} & \sigma_{12}^{ij} & \sigma_{13}^{ij} & 0 \\
\sigma_{21}^{ij} & \sigma_{22}^{ij} & \sigma_{23}^{ij} & 0 \\
\sigma_{31}^{ij} & \sigma_{32}^{ij} & \sigma_{33}^{ij} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

From this, for

\[
 f^{*0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

it easily follows (5).
(c) Consider the case (c).
In this case we prove that the condition (3) holds. We have

$$Z_S = \{ (\bar{f}, \bar{b}) = \left( \begin{array}{c} f_1^* \\ \vdots \\ f_5^* \\ b_1^* \\ \vdots \\ b_5^* \end{array} \right), \sigma \left( \begin{array}{c} b_1^* f^* \\ \vdots \\ b_5^* f^* \end{array} \right) = 0 \}.$$ 

Consider the matrix

$$\tilde{\sigma} = \begin{bmatrix} 0 & E_4 & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ -E_4 & 0 & \sigma_{23} & \sigma_{24} & \sigma_{25} \end{bmatrix},$$

where $E_4$ is the identity matrix of size $4 \times 4$. The 8 rows of the matrix $\tilde{\sigma}$ are the first 8 rows of the matrix $\sigma$. Since $\text{rk} (\sigma) = 8 = \text{rk} (\tilde{\sigma})$ and for a matrix $P$ of size $20 \times p$ we have:

$$\sigma P = 0 \text{ iff } \tilde{\sigma} P = 0. \quad (6)$$

For $3 \leq i \leq 5$ consider the following matrix $P_i$ of size $20 \times 4$:

$$P_i = \begin{bmatrix} \sigma_{i1} \\ P_{i1} \\ P_{i2} \\ P_{i3} \end{bmatrix},$$

where $P_{ii} = -E_4$ and $P_{ij} = 0$ for $j \neq i$. We see that $\tilde{\sigma} \cdot P_i = 0$.

From (6) it follows that $\sigma \cdot P_i = 0$ or

$$\sigma^{ji} = \sigma_{ij} \sigma^{2i} - \sigma_{2j} \sigma^{1i}, \quad 3 \leq j \leq 5.$$

From this we obtain

$$0 = \sigma^{ji} + \sigma^{ij} = \sigma_{ij} \sigma^{2i} - \sigma_{2j} \sigma^{1i} + \sigma_{1i} \sigma^{2j} - \sigma_{2i} \sigma^{1j}$$

$$= [\sigma_{1j}, \sigma_{2i}] + [\sigma_{1i}, \sigma_{2j}], \quad 3 \leq i, j \leq 5.$$

One can rewrite these equations into the following compact form:

$$[t_1 \sigma_{13} + t_2 \sigma_{14} + t_3 \sigma_{15}, t_1 \sigma_{23} + t_2 \sigma_{24} + t_3 \sigma_{25}] = 0 \quad (7)$$

for all $t_1, t_2, t_3 \in \mathbb{C}$.

Claim 11. For every $(b_1^*, b_2^*, b_3^*) \neq (0, 0, 0)$ there exists $(b_1^*, b_2^*)$ and a nonzero column $f^*$ of height 4 such that

$$\sigma \left( \begin{array}{c} b_1^* f^* \\ \vdots \\ b_5^* f^* \end{array} \right) = 0.$$
Proof of Claim 11. From (7) it follows that the symmetric matrices
\[ b_3^* \sigma^{13} + b_4^* \sigma^{14} + b_5^* \sigma^{15}, \quad b_3^* \sigma^{23} + b_4^* \sigma^{24} + b_5^* \sigma^{25} \]
commute. Therefore they have a common eigenvector \( f^* \) with the eigenvalues \(-b_2^*, b_1^*\), respectively. We have
\[
\tilde{\sigma} \begin{bmatrix} b_1^* f^* \\ \vdots \\ b_5^* f^* \end{bmatrix} = 0
\]
and from this and (6) Claim 11 follows. \(\square\)

From Claim 11 it follows that \( \dim(Z_S) \geq 2 \).

(d) Consider the case (d).
In this case we prove that the condition (3) holds, i.e., we prove that \( \dim(Z_S) \geq 2 \).

Claim 12. Suppose \( N \subset PC^{5*} \) is a line in general position; then there exists \( 0 \neq f^{*0} \in C^4, \ 0 \neq b^{*0} \in N \) such that \( \rho(S, f^{*0} \otimes b^{*0}) = 0 \).

Proof of Claim 12. One can assume that \( N = \langle b_1^*, b_2^* \rangle \), where \( b_i^* \) are basic vectors of \( C^{5*} \). We have to prove that there exists a column \( f^{*0} \) of height 4 and \( \lambda_1, \lambda_2 \in C \), \( (\lambda_1, \lambda_2) \neq (0, 0) \) such that
\[
\sigma \begin{bmatrix} \lambda_1 f^{*0} \\ \lambda_2 f^{*0} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0. \tag{8}
\]
Consider the 4th and 8th rows of the matrix \( \sigma \):
\[
\text{row}_4(\sigma) = (0, \ldots, 0, \sigma_{13}^{13}, \sigma_{13}^{14}, \ldots, \sigma_{15}^{13}, \sigma_{15}^{14}), \\
\text{row}_8(\sigma) = (0, \ldots, 0, \sigma_{23}^{23}, \sigma_{23}^{24}, \ldots, \sigma_{25}^{23}, \sigma_{25}^{24}).
\]
We want to show that \( \text{row}_4(\sigma) \) and \( \text{row}_8(\sigma) \) are linearly dependent. Suppose that \( \text{row}_4(\sigma) \) and \( \text{row}_8(\sigma) \) are linearly independent. Then the first 8 rows of the matrix \( \sigma \) are linearly independent. Since \( \text{rk}(\sigma) = 8 \), we see that every row of \( \sigma \) is a linear combination of the first 8 rows. From \( \text{row}_4(\sigma) \neq 0 \) it follows that \( \sigma_{4i}^{1i} \neq 0 \) for some \( 3 \leq i \leq 5, 1 \leq j \leq 4 \). Since \( \sigma_{4i}^{1i} = -\sigma_{4i}^{2i} \neq 0 \), we see that the \((4(i-1)+j)\)th row
\[
\text{row}_{4(i-1)+j}(\sigma) = (\sigma_{4i}^{11}, \sigma_{4i}^{12}, \sigma_{4i}^{13}, \sigma_{4i}^{14}, \sigma_{4i}^{21}, \sigma_{4i}^{22}, \sigma_{4i}^{23}, \sigma_{4i}^{24}, \ldots)
\]
of the matrix \( \sigma \) is not a linear combination of the first 8 rows. This contradiction proves that \( \text{row}_4(\sigma) \) and \( \text{row}_8(\sigma) \) are linearly dependent.

Finally, to obtain (8) we take
\[
f^{*0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
\]
and \( \lambda_1, \lambda_2 \) such that \( (\lambda_1, \lambda_2) \neq (0, 0) \) and \( \lambda_1 \text{row}_4(\sigma) + \lambda_2 \text{row}_8(\sigma) = 0 \). \(\square\)

From Claim 12 it follows that \( \dim(Z_S) \geq 3 > 2 \). \(\square\)
The proof of Theorem 1

We suppose that there exists $A^0 \in I$ such that $\dim(T_{\pi(A^0)}MI(k)) > 8k - 3$ and obtain a contradiction.

From Corollary 7 it follows that

$$\dim(X_{A^0}) \leq 1$$

and by Lemma 2 we have $\dim(T_{A^0}I) > 3k^2 + 13k$. Hence, by Lemma 8 there exists $0 \neq S^0 \in S^2 C^4 \otimes \wedge^2 C^k$ such that

$$\xi(A^0, S^0) = 0.$$

Claim 13. (1) Consider the following composition of linear mappings

$$\rho(S^0, \cdot) \circ \beta(A^0, \cdot) : C^{2k+2} \longrightarrow C^4 \otimes C^k, \quad h \mapsto \rho(S^0, \beta(A^0, h)).$$

Then we have $\rho(S^0, \cdot) \circ \beta(A^0, \cdot) = 0$. (2) Consider the following composition of linear mappings

$$\varepsilon(A^0, \cdot) \circ \rho(S^0, \cdot) : C^{4*} \otimes C^{k*} \longrightarrow C^{2k+2}, \quad B^* \mapsto \varepsilon(A^0, \rho(S^0, B^*)).$$

Then we have $\varepsilon(A^0, \cdot) \circ \rho(S^0, \cdot) = 0$.

Proof of Claim 13. Consider the following nontrivial trilinear $SL_4 \times SL_k \times Sp_{2k+2}$-morphism:

$$\tau : C^{4*} \otimes C^{k*} \otimes C^{2k+2} \times S^2 C^4 \otimes \wedge^2 C^k \times C^{2k+2} \longrightarrow C^4 \otimes C^k,$$

$$(A, S, h) \mapsto \kappa(\xi(A, S), h),$$

where

$$\kappa : C^{4*} \otimes C^{k*} \otimes C^{2k+2} \times C^{2k+2} \longrightarrow C^4 \otimes C^k$$

is the canonical bilinear $SL_4 \times SL_k \times Sp_{2k+2}$-morphism. Note that

$$\tau(A^0, S^0, h) = \kappa(\xi(A^0, S^0), h) \equiv 0. \quad (11)$$

The $SL_4 \times SL_k \times Sp_{2k+2}$-module

$$(C^{4*} \otimes C^{k*} \otimes C^{2k+2}) \otimes (S^2 C^4 \otimes \wedge^2 C^k) \otimes C^{2k+2}$$

contains the irreducible $SL_4 \times SL_k \times Sp_{2k+2}$-module $C^4 \otimes C^k$ with multiplicity 1. Therefore, there exists a unique, up to a scalar factor, nontrivial trilinear $SL_4 \times SL_k \times Sp_{2k+2}$-morphism

$$C^{4*} \otimes C^{k*} \otimes C^{2k+2} \times (S^2 C^4 \otimes \wedge^2 C^k) \times C^{2k+2} \longrightarrow C^4 \otimes C^k.$$

Therefore,

$$\rho(S, \cdot) \circ \beta(A, \cdot)(h) \equiv c_1 \tau(A, S, h) \quad (12)$$
for some $c_1 \in \mathbb{C}$ and
\[ (\varepsilon(A, \cdot) \circ \rho(S, \cdot))^*(h) \equiv c_2 \tau(A, S, h) \] \tag{13}
for some $c_2 \in \mathbb{C}$.

From (12) and (11) we have
\[ (\rho(S^0, \cdot) \circ \beta(A^0, \cdot))(h) = c_1 \tau(A^0, S^0, h) = 0. \]
This gives us statement (1). From (13) and (11) we have
\[ (\varepsilon(A^0, \cdot) \circ \rho(S^0, \cdot))^*(h) = c_2 \tau(A^0, S^0, h) = 0. \]
From this statement (2) follows. \( \square \)

From Claim 13 (1) we have
\[ \text{Im}(\beta(A^0, \cdot)) \subset \text{Ker}(\rho(S^0, \cdot)). \] \tag{14}
On the other hand, by (E3) we have \( \text{rk}(\beta(A^0, \cdot)) = 2k + 2 \) and with (14) this gives us
\[ \text{rk}(\rho(S^0, \cdot)) \leq 2k - 2. \] \tag{15}

From (15) it follows that one of the conditions (1)–(3) of Lemma 10 holds for \( S = S^0 \).

I. Consider the case when the condition (1) of Lemma 10 holds for \( S = S^0 \).

By the condition (1) of Lemma 10 there exists \( B^{*0} \in \mathbb{C}^{k*} \otimes \mathbb{C}^{k*} \) such that \( \rho(S^0, B^{*0}) = f^0 \otimes b^0 \neq 0 \). Thus, we have \( \varepsilon(A^0, f^0 \otimes b^0) = \varepsilon(A^0, \rho(S^0, B^{*0})) = 0 \) by Claim 13 (2) and, therefore, \( A^0 \notin I_1 \). But this contradicts to \( A^0 \in I \).

II. Consider the case when the condition (2) of Lemma 10 holds for \( S = S^0 \).

From (15) it follows that \( k = 4 \) or \( k = 5 \). By the condition (2) of Lemma 10 we have \( \{f^{*0}\} \times \mathbb{C}^{k*} \subset \text{Ker}(\rho(S^0, \cdot)) \). On the other hand, we have (14) and
\[ \dim(\text{Ker}(\rho(S^0, \cdot))) - \dim(\text{Im}(\beta(A^0, \cdot))) = \begin{cases} 0 & \text{if } k = 4, \\ 2 & \text{if } k = 5. \end{cases} \]
Therefore, \( \text{Im}(\beta(A^0, \cdot)) \supset \{f^{*0}\} \times M \) for some linear subspace \( M \subset \mathbb{C}^{k*} \) of dimension \( \geq 3 \). But this contradicts (9).

III. Consider the case when the condition (3) of Lemma 10 holds for \( S = S^0 \).

From (15) it follows that \( k = 5 \). Thus,
\[ \dim(\text{Im}(\beta(A^0, \cdot))) = 12 = \dim(\text{Ker}(\rho(S^0, \cdot))) \]
and from this together with (14) it follows that \( \text{Im}(\beta(A^0, \cdot)) = \text{Ker}(\rho(S^0, \cdot)) \). Therefore, \( X_{A^0} = Z_{S^0} \). From this and the condition (3) of Lemma 10 we obtain \( \dim(X_{A^0}) = \dim(Z_{S^0}) \geq 2 \). But this again contradicts (9).
References


