ON SINGULARITIES OF $\mathcal{M}_{\mathbb{P}^3}(c_1, c_2)$

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Let $\mathcal{M}_{\mathbb{P}^3}(c_1, c_2)$ be the moduli space of stable rank-2 vector bundles on $\mathbb{P}^3$ with Chern classes c_1, c_2. We prove the following results: (1) Let k, β, γ be three integers such that $k > 0$, $0 \leq \beta < \gamma$, $\gamma \geq 2$, $k\gamma - (k+1)\beta > 0$; then the moduli space $\mathcal{M}_{\mathbb{P}^3}(0, k\gamma^2 - (k+1)\beta^2)$ is singular (the case $k = 2$, $\beta = 0$ was previously proved by M. Maggesi).

(2) Let k, β, γ be three integers, with β and γ odd, such that $k > 0$, $0 < \beta < \gamma$, $\gamma \geq 5$, $k\gamma - (k+1)\beta + 1 > 0$; then the moduli space $\mathcal{M}_{\mathbb{P}^3}(-1, k(\gamma/2)^2 - (k+1)(\beta/2)^2) + 1/4)$ is singular.

In particular $\mathcal{M}_{\mathbb{P}^3}(0, 5)$, $\mathcal{M}_{\mathbb{P}^3}(-1, 6)$ are singular.

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0. Introduction

The first examples of singular moduli spaces of stable vector bundles on a projective space were found by the authors in [1], where it is shown that the symplectic special instanton bundles on $\mathbb{P}^5$ with second Chern class $c_2 = 3, 4$ correspond to singular points of their moduli space. Later R. M. Miró-Roig [10] detected an example in the case of rank-3 vector bundles on $\mathbb{P}^3$.

We denote by $\mathcal{M}_{\mathbb{P}^3}(c_1, c_2)$ the moduli space of stable rank-2 vector bundles on $\mathbb{P}^3$ with Chern classes c_1, c_2.

In spite of the vast literature concerned with rank-2 bundles on $\mathbb{P}^3$, there were no examples in this case until recently, when M. Maggesi [9] has proved that pulling back some particular instanton bundles with second Chern class $c_2 = 2$ by a finite morphism $\mathbb{P}^3 \to \mathbb{P}^3$ one always obtain singular points in the corresponding moduli spaces. More precisely for any integer $d \geq 2$ the moduli space $\mathcal{M}_{\mathbb{P}^3}(0, 2d^2)$ is singular; $d = 2$ gives $\mathcal{M}_{\mathbb{P}^3}(0, 8)$.

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The above result suggests that singularities must be expected in $\mathcal{M}_{\mathbb{P}^3}(c_1, c_2)$ with only a few exceptions.

The aim of the present paper is in fact to prove that $\mathcal{M}_{\mathbb{P}^3}(c_1, c_2)$ is singular for a very large class of values of (c_1, c_2) (including odd c_1's). The idea is to replace the usual pull-back by the more general construction of "pulling back over $\mathbb{C}^4 \backslash 0$ ", introduced in [7] and developed in [2]. The results are the following:

(1) Let k, β, γ be three integers such that $k > 0$, $0 \leq \beta < \gamma$, $\gamma \geq 2$, $k\gamma - (k+1)\beta > 0$; then the moduli space $\mathcal{M}_{\mathbb{P}^3}(0, k\gamma^2 - (k+1)\beta^2)$ is singular. In particular taking $k = 2$, $\gamma = 2$, $\beta = 1$, we obtain that $\mathcal{M}_{\mathbb{P}^3}(0,5)$ is singular; in this case the singular points we have detected lie in the closure of the open set consisting of instanton bundles [12].

(2) Let k, β, γ be three integers, with β and γ odd, such that $k > 0$, $0 < \beta < \gamma$, $\gamma \geq 5$, $k\gamma - (k+1)\beta + 1 > 0$; then the moduli space $\mathcal{M}_{\mathbb{P}^3}(-1, k(\gamma/2)^2 - (k+1)(\beta/2)^2) + 1/4)$ is singular. In particular $\mathcal{M}_{\mathbb{P}^3}(-1,6)$ is singular (take $k = 2$, $\gamma = 5$, $\beta = 3$).

1. Let $\mathbb{P}^3 = \mathbb{P}^3(V)$ be the projective space of hyperplanes in V.

In particular $H^0(\mathbb{P}^3, \mathcal{O}(1)) \simeq V$.

A special instanton bundle is a bundle $E \in \mathcal{M}_{\mathbb{P}^3}(0, k)$ such that $h^1(E(-2)) = 0$ and $h^0(E(1)) = 2$.

According to [3, Sec. 3] a special instanton bundle with $c_2 = k$ is the cohomology bundle of a monad

$$H \otimes \mathcal{O}(-1) \xrightarrow{M} H^* \otimes \Omega^1(1) \xrightarrow{B'} K \otimes \mathcal{O}\,, \tag{1.1}$$

where H (resp. K) is a complex vector space of dimension k (resp. $2k-2$) and the map B' factors through the diagram

$$\begin{array}{ccc} H^* \otimes \Omega^1(1) & & \\ \big\downarrow e & \searrow B' & \\ H^* \otimes V \otimes \mathcal{O} & \underset{\tilde{B}'}{\cdots\longrightarrow} & K \otimes \mathcal{O} \end{array}$$

In the above diagram the map e is obtained by the dual Euler sequence and $\tilde{B}'$ has the matrix representation

$$\begin{bmatrix} x & x' & & & & & \\ y & y' & x & x' & & & \\ & & y & y' & \cdots & & \\ & & & & & y & y' \end{bmatrix}^t,$$

where $\{x, x', y, y'\}$ is a basis of V^* .

Let $\{a, b, c, d\}$ be a basis of V dual to $\{x', -x, y', -y\}$. Let us write $V = U \oplus U$ so that (a, c) and (b, d) are coordinates in $U \oplus 0$ and $0 \oplus U$ respectively.

It is straightforward to check that the above diagram completes to the following:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H^* \otimes \Omega^1(1) & & & & \\
 & & & & \downarrow{\scriptstyle e} & & & & \\
0 & \longrightarrow & T \otimes \mathcal{O} & \longrightarrow & H^* \otimes V \otimes \mathcal{O} & \xrightarrow{\tilde{B}'} & K \otimes \mathcal{O} & \longrightarrow & 0 \\
 & & & A \searrow & \downarrow & & & & \\
 & & & & H^* \otimes \mathcal{O}(1) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where T is a complex vector space of dimension $2k+2$ and the matrix representation of A is

$$\begin{pmatrix}
a & b & & & c & d & & \\
 & a & b & & & c & d & \\
 & & \ddots & \ddots & & & \ddots & \ddots \\
 & & & a \;\; b & & & & c \;\; d
\end{pmatrix}.$$

Consider now the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{K} & \xrightarrow{f} & H^* \otimes \Omega^1(1) & \xrightarrow{B'} & K \otimes \mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow & & \| & & \\
0 & \longrightarrow & T \otimes \mathcal{O} & \longrightarrow & H^* \otimes V \otimes \mathcal{O} & \longrightarrow & K \otimes \mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle A} & & \downarrow & & & & \\
 & & H^* \otimes \mathcal{O}(1) & = & H^* \otimes \mathcal{O}(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From (1.1) it follows that $H \subset H^0(\mathcal{K}(1)) =: H'$.

H' corresponds to the linear syzygies of A and the natural composition

$$H' \otimes \mathcal{O}(-1) \xrightarrow{v} \mathcal{K} \xrightarrow{u} T \otimes \mathcal{O}$$

has the matrix representation

$$\begin{pmatrix}
-c & -d & & & \\
 & -c & \ddots & & \\
 & & \ddots & -d & \\
 & & & -c & -d \\
a & b & & & \\
 & a & \ddots & & \\
 & & \ddots & b & \\
 & & & a & b
\end{pmatrix}.$$

According to [3, Sec. 3.4] the map M of the monad (1.1) factors through the diagram

$$\begin{array}{ccccc} & & H^*\otimes\wedge^2V\otimes\mathcal{O}(-1) & & \\ & \nearrow R & \uparrow M' & \searrow e' & \\ H'\otimes\mathcal{O}(-1) & \xleftarrow{\alpha} & H\otimes\mathcal{O}(-1) & \xrightarrow{M} & H^*\otimes\Omega^1(1)\,, \\ & \searrow v & \downarrow v\circ\alpha & f\nearrow & \\ & & \mathcal{K} & & \end{array}$$

where e' is the natural map, α is represented by the Hankel matrix

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{k+1} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{k+2} \\ \vdots & \vdots & & \vdots \\ \alpha_{k-1} & \alpha_k & \dots & \alpha_{2k} \end{pmatrix}$$

and R has the matrix form

$$\begin{bmatrix} x\wedge x' & (x\wedge y'-x'\wedge y) & y\wedge y' & & & \\ & \ddots & & \ddots & & \ddots \\ & & & x\wedge x' & (x\wedge y'-x'\wedge y) & y\wedge y' \end{bmatrix}.$$

Hence the composition

$$H\otimes\mathcal{O}(-1)\xrightarrow{w\circ v\circ\alpha}T\otimes\mathcal{O}\xrightarrow{A}H^*\otimes\mathcal{O}(1)$$

gives a symplectic monad. The Proposition 3.5 of [3] translates easily to the following statement.

Proposition 1.1. *Let E be a special instanton bundle belonging to $\mathcal{M}_{\mathbb{P}^3(V)}(0,k)$. One can choose a splitting $V=U\oplus U$ and homogeneous coordinates (a,b) on $U\oplus 0$, (c,d) on $0\oplus U$ such that E is the cohomology bundle of the monad*

$$\mathbb{C}^k\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{B}U^{k+1}\otimes\mathcal{O}_{\mathbb{P}^3}\xrightarrow{A}\mathbb{C}^k\otimes\mathcal{O}_{\mathbb{P}^3}(1)\,, \tag{1.2}$$

where A (resp. B) is a $k\times(2k+2)$ (resp. $(2k+2)\times k$) matrix given by

$$A=\begin{pmatrix} a & b & & & c & d & & \\ & a & b & & & c & d & \\ & & \ddots & \ddots & & & \ddots & \ddots \\ & & & a\ \ b & & & & c\ \ d \end{pmatrix} \tag{1.3}$$

$$B=\begin{pmatrix} -c & -d & & \\ & -c & \ddots & \\ & & \ddots & -d \\ & & & -c\ \ -d \\ a & b & & \\ & a & \ddots & \\ & & \ddots & b \\ & & & a\ \ b \end{pmatrix}\begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{k+1} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{k+2} \\ \vdots & \vdots & & \vdots \\ \alpha_{k-1} & \alpha_k & \dots & \alpha_{2k} \end{pmatrix} \tag{1.4}$$

and $\alpha_0, \dots, \alpha_{2k}$ *are constant coefficients subject to the condition*

$$J := \det \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_k \\ \alpha_1 & \alpha_2 & \dots & \alpha_{k+1} \\ \vdots & \vdots & & \vdots \\ \alpha_k & \alpha_{k+1} & \dots & \alpha_{2k} \end{pmatrix} \neq 0 . \tag{1.5}$$

Conversely every cohomology bundle of a monad (1.2) *satisfying the above conditions is a special instanton bundle* $E \in \mathcal{M}_{\mathbb{P}^3}(0, k)$.

Let us recall the following classical:

Lemma 1.2. *The complex polynomial* J *defined in* (1.5) *is irreducible.*

The condition

$$rk \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_k \\ \alpha_1 & \alpha_2 & \dots & \alpha_{k+1} \\ \vdots & \vdots & & \vdots \\ \alpha_k & \alpha_{k+1} & \dots & \alpha_{2k} \end{pmatrix} \leq 1$$

defines a rational normal curve C in the projective space $\mathbb{P}^{2k}$, hence the condition

$$rk \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_k \\ \alpha_1 & \alpha_2 & \dots & \alpha_{k+1} \\ \vdots & \vdots & & \vdots \\ \alpha_k & \alpha_{k+1} & \dots & \alpha_{2k} \end{pmatrix} \leq p$$

defines the locus spanned by the $(p-1)$-dimensional linear subspaces which are p-secants to C. It follows that $J = 0$ defines the locus spanned by the $(k-1)$-dimensional linear subspaces which are k-secants to C, which is well known to be irreducible.

From now on, let us fix homogeneous coordinates (a, b, c, d) on $\mathbb{P}(V) = \mathbb{P}(U \oplus U)$ as above. There is a natural action of $SL(2) \cong SL(U)$ on the matrices A and B through the transformations

$$\begin{pmatrix} a \\ c \end{pmatrix} \to g \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \to g \begin{pmatrix} b \\ d \end{pmatrix} \quad (g \in SL(2)).$$

It follows that $SL(2) \cong SL(U)$ acts on the monad (1.2), hence on its cohomology E. More precisely, for $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ let

$$Q_g = \begin{pmatrix} xI_{(k+1)} & yI_{(k+1)} \\ zI_{(k+1)} & wI_{(k+1)} \end{pmatrix}$$

It is easy to check that $g^*A = AQ_g$, $g^*B = Q_g^{-1}B$, which implies that the monads (A, B) and (g^*A, g^*B) are equivalent. Moreover, the monad (1.2) is $SL(U)$-invariant ($SL(U)$ acts trivially on $\mathbb{C}^k$ and diagonally on U^{k+1}).

We recall that the minimal resolution of a vector bundle E on the projective space is by definition the sheafification of the minimal resolution of the module $\oplus_{t\in\mathbf{Z}}H^0(E(t))$.

Lemma 1.3. *The bundle E defined by the monad (1.2) admits the following $SL(U)$-invariant minimal resolution:*

$$0 \to S^{k-1}U \otimes \mathcal{O}_{\mathbb{P}^3}(-k-2) \to (S^kU)^2 \otimes \mathcal{O}_{\mathbb{P}^3}(-k-1) \to [S^{k+1}U \otimes \mathcal{O}_{\mathbb{P}^3}(-k)] \oplus [\mathbb{C}^2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)] \to E \to 0 \ . \tag{1.6}$$

Proof. We prove first that the minimal resolution of E has the form

$$0 \to W_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-k-2) \to W_1 \otimes \mathcal{O}_{\mathbb{P}^3}(-k-1) \to [W_0 \otimes \mathcal{O}_{\mathbb{P}^3}(-k)] \oplus [\mathbb{C}^2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)] \to E \to 0\,, \tag{1.7}$$

where $\dim W_0 = k+2$, $\dim W_1 = 2k+2$, $\dim W_2 = k$.

We denote by S the graded module $\mathbb{C}[a,b,c,d]$. It is sufficient to check that the following sequence of free S-modules is exact

$$0 \to W_2 \otimes S(-k-2) \xrightarrow{\delta} W_1 \otimes S(-k-1) \xrightarrow{\gamma} [W_0 \otimes S(-k)] \oplus [\mathbb{C}^{k+2} \otimes S(-1)] \xrightarrow{\beta} U^{k+1} \otimes S \xrightarrow{A} \mathbb{C}^k \otimes S(1)\,, \tag{1.8}$$

where A has been defined in (1.3) and β, γ, δ are described below. β is defined by the blocks $\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ where B_i's are $(k+1)\times(k+2)$ blocks whose nonzero entries are

$$(B_1)_{ij} := -c \text{ for } 1 \le i = j \le k+1,\ -d \text{ for } 1 \le i = j-1 \le k+1$$

$$(B_2)_{ij} := (-1)^{i+j} a^{i-j} b^{k-i+1} c^{j-1} \text{ for } 1 \le j \le i \le k+1$$

$$(B_3)_{ij} := a \text{ for } 1 \le i = j \le k+1,\ b \text{ for } 1 \le i = j-1 \le k+1$$

$$(B_4)_{ij} := (-1)^{i+j} b^{k-j+2} c^{i-1} d^{j-i-1} \text{ for } 1 \le i \le j-1 \le k+1$$

γ is defined by the blocks $\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ where C_i's are $(k+2)\times(k+1)$ blocks whose nonzero entries are

$$(C_1)_{ij} := (-1)^{i+j} b^{k-j+1} c^{i-1} d^{j-i} \text{ for } 1 \le i \le j \le k+1$$

$$(C_2)_{ij} := (-1)^{i+j} a^{i-j-1} b^{k-i+2} c^{j-1} \text{ for } 1 \le j \le i-1 \le k+1$$

$$(C_3)_{ij} := c \text{ for } 1 \le i \le j \le k+1,\ a \text{ for } 1 \le i-1 \le j \le k+1$$

$$(C_4)_{ij} := -d \text{ for } 1 \le i \le j \le k+1,\ -b \text{ for } 1 \le i-1 \le j \le k+1$$

δ is defined by the blocks $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ where D_i's are $(k+1)\times k$ blocks whose nonzero entries are

$$(D_1)_{ij} := d \text{ for } 1 \le i = j \le k, \; b \text{ for } 1 \le i-1 = j \le k$$

$$(D_2)_{ij} := c \text{ for } 1 \le i = j \le k, \; a \text{ for } 1 \le i-1 = j \le k$$

It is easy to check that the above sequence is a complex. In order to prove the exactness one checks that the ranks of the matrices A, β, γ, δ are constant on every point of $\mathbb{P}^3$ and precisely their values are

$$rk\ A = k, \quad rk\ \beta = k+2, \quad rk\ \gamma = k+2, \quad rk\ \delta = k$$

In fact it is easy to extract suitable minors of the right order from the above matrices whose values are equal to some powers of the indeterminates. Now the exactness follows by the criterion of Buchsbaum and Eisenbud [4, Theorem 20.9]. In this special case a direct argument can be supplied. In fact the exactness of the sheafified sequence of (1.8) follows because at each step of the sequence the Image is a sub-bundle of the Kernel of the same rank. Then the sheafified $\tilde{\ker}\,\beta$ satisfies $H^1(\tilde{\ker}\,\beta(t)) = 0 \quad \forall t \in \mathbf{Z}$.

The minimal resolution is clearly $SL(U)$-invariant; hence the vector spaces W_2, W_1 and W_0 are representations of $SL(U)$(e.g. W_2 is the subspace of degree $k+2$ elements in $Tor^2_S\left(\bigoplus_{t\in\mathbf{Z}} H^0(\mathcal{K}(t)), \mathbb{C}\right)$, where $SL(U)$ acts, see [5]); (1.6) follows by computing the cohomology groups of suitable twists of E as $SL(U)$-representations from (1.7) and comparing them with the analogous results obtained from (1.2). □

From (1.6) we find

Lemma 1.4. *$H^0E(t) = \mathbb{C}^k \otimes S^{t-1}(U \oplus U)$ for $1 \le t \le k-1$ as $SL(U)$-representations; moreover, for $t \ge k$ $H^0E(t)$ is a direct sum of symmetric powers $S^m(U)$ with $0 \le m \le t+1$.*

Our next aim is the computation of H^1 End $E(-2)$ for a bundle E defined by the monad (1.2.) Following [8, Sec. 4] H^1 End $E(-2)$ is the kernel of a linear homomorphism

$$\delta : \wedge^2 H \to \wedge^2 H^*$$

where $H = H^1E(-1)$ is a k-dimensional complex vector space. In our case $H = \mathbb{C}^k$ as $SL(U)$-representation (from (1.2)), and δ is a morphism of representations; moreover, $I = I(\alpha_0, \ldots, \alpha_{2k}) = \det \delta$ is a polynomial in the coefficients $\alpha_0 \ldots \alpha_{2k}$ appearing in the matrix (1.4). Hence

Lemma 1.5. *H^1 End $E(-2) = \mathbb{C}^r$ as $SL(U)$-representation, $0 \le r \le \binom{k}{2}$. Moreover, H^1 End $E(-2) = 0$ if and only if $I(\alpha_0, \ldots, \alpha_{2k}) \ne 0$.*

Remark 1.6. Both the cases $H^1\mathrm{End}\, E(-2) = 0$ and $H^1\mathrm{End}\, E(-2) \neq 0$ really occur. Let $J = J(\alpha_0, \dots, \alpha_{2k})$ be the polynomial (1.5); the first case corresponds to solutions of $I(\alpha_0, \dots, \alpha_{2k}) \neq 0$, $J(\alpha_0, \dots, \alpha_{2k}) \neq 0$; the second to solutions of $I(\alpha_0, \dots, \alpha_{2k}) = 0$, $J(\alpha_0, \dots, \alpha_{2k}) \neq 0$, which exist because the polynomial J is irreducible by the lemma. In fact if $J = 0$ when $I = 0$ then it follows from the NullstellenSatz that I divides a power of J. Since J is irreducible it follows that I is equal (up to a constant) to a power of J, which is impossible for degree reasons. Indeed the formula in Remark 2 at page 90 of [8] gives deg $I = k(k-1)$ while deg $J = k + 1$.

2. Let $0 \leq \beta < \gamma$ be two integers; let $f_1, \dots, f_4$ be homogeneous polynomials in the variables a, b, c, d without common zeroes of degree $\gamma - \beta$, $\gamma - \beta$, $\gamma + \beta$, $\gamma + \beta$ respectively. Let us take into account the diagram:

$$\begin{array}{ccc} \mathbb{C}^4 \backslash 0 & \xrightarrow{\omega} & \mathbb{C}^4 \backslash 0 \\ \eta \downarrow & & \eta \downarrow \\ \mathbb{P}^3 & & \mathbb{P}^3 \end{array}$$

where ω is defined by $f_1, \dots, f_4$.

According to [2, 7], from any bundle E defined by the monad (1.2) we construct a rank-2 bundle $E_{\beta,\gamma}$ such that $\eta^* E_{\beta,\gamma} = \omega^* \eta^* E$. The bundle $E_{\beta,\gamma}$ is the cohomology of the monad

$$\mathbb{C}^k \otimes \mathcal{O}_{\mathbb{P}^3}(-\gamma) \xrightarrow{A_{\beta,\gamma}} \mathcal{U}^{k+1} \xrightarrow{B_{\beta,\gamma}} \mathbb{C}^k \otimes \mathcal{O}_{\mathbb{P}^3}(\gamma)\,, \tag{2.1}$$

where $\mathcal{U} = \mathcal{O}_{\mathbb{P}^3}(-\beta) \oplus \mathcal{O}_{\mathbb{P}^3}(\beta)$, and $A_{\beta,\gamma}$, $B_{\beta,\gamma}$ are obtained from the matrices A, B in (1.2) replacing a, b, c, d by f_1, f_2, f_3, f_4 respectively.
In particular the Chern classes of $E_{\beta,\gamma}$ are $c_1 = 0, c_2 = k\gamma^2 - (k+1)\beta^2$.
Of course $E_{\beta,\gamma}$ depends on $f_1, \dots, f_4$ but for simplicity we omit this fact in the notations.

The cohomology groups of $E_{\beta,\gamma}(t)$ can be computed by [2, Sec. 2] in particular, Theorem 2.6 of [2] can be rephrased as follows ($SL(U)$ acting on $\mathbb{P}^3$ here plays the same role of $SL(W)$ acting on $\mathbb{P}^5$ there).

Theorem 2.1. *Let $H^i E(t) = T^t(U)$ where T^t is a representation of $SL(U)$. Then*

$$h^i E_{\beta,\gamma}(t) = \sum_{h \in \mathbf{Z}} \sum_{j=0}^{4} (-1)^j h^0 \left[\bigwedge^j (\mathcal{U}^2(-\gamma)) \otimes T^h(\mathcal{U}) \otimes \mathcal{O}_{\mathbb{P}^3}(t - h\gamma) \right].$$

In practice the groups $h^i E_{\beta,\gamma}(t)$ can be computed as follows. Let s_p the dimension of the the degree p summand of the artinian algebra $S = \mathbb{C}[a, b, c, d]/(f_1, f_2, f_3, f_4)$; for $h \in Z$ we write

$$T^h(\mathcal{U})(h\gamma) = \bigoplus_s \mathcal{O}_{\mathbb{P}^3}(\mu_{s,h})\,. \tag{2.2}$$

Let

$$b_q(E) = \sharp\{(s,h) : \mu_{s,h} = q\}\,. \tag{2.3}$$

Then

$$h^i E_{\beta,\gamma}(t) = \sum_{p+q=t} s_p b_q(E)\,. \tag{2.4}$$

Moreover,

$$s_p = \sum_{j=0}^{4} (-1)^j h^0 \left[\bigwedge^j \mathcal{U}^2 \otimes \mathcal{O}_{\mathbb{P}^3}(p - j\gamma) \right]. \tag{2.5}$$

The formulae (2.4), (2.5) are easy adaptations of Theorem 2.1.

The above formulae still hold if we replace $E_{\beta,\gamma}$ (resp E) by End $E_{\beta,\gamma}$ (resp End E). More precisely let H^i End $E(t) = R^t(U)$ where R^t is a representation of $SL(U)$. Then

$$h^i \text{ End } E_{\beta,\gamma}(t) = \sum_{h\in\mathbf{Z}} \sum_{j=0}^{4} (-1)^j h^0 \left[\bigwedge^j (\mathcal{U}^2(-\gamma)) \otimes R^h(\mathcal{U}) \otimes \mathcal{O}_{\mathbb{P}^3}(t - h\gamma) \right].$$

Proposition 2.2. *Let E be defined by the monad* (1.2). *Then $E_{\beta,\gamma}$ is stable if and only if $k\gamma - (k+1)\beta > 0$.*

Proof. We need to show that $h^0 E_{\beta,\gamma} = 0$ iff $k\gamma - (k+1)\beta > 0$. We compute $h^0 E_{\beta,\gamma}$ substituting $t = 0$ in the formula (2.4). Since $h^0 E(h) = 0$ for $h \le 0$ and $s_p = 0$ for $p < 0$, a contribution to the right-hand side of (2.4) can occur only for $q \le 0$. By Lemma 1.4 the integers $\mu_{s,h}$ appearing in (2.3) are strictly positive for $1 \le h \le k-1$, while for $h = k$ we have $T^k(\mathcal{U})(2\gamma) = S^{k+1}(\mathcal{U})(k\gamma) \oplus [\mathbb{C}^k \otimes S^{k-1}(\mathcal{U})(k\gamma)] = \bigoplus_s \mathcal{O}_{\mathbb{P}^3}(\mu_{s,k})$ with $\inf_s\{\mu_{s,k}\} = k\gamma - (k+1)\beta$; for $h > k$ we have $k\gamma - (k+1)\beta < h\gamma - (h+1)\beta \le \inf_s\{\mu_{s,h}\}$; the conclusion follows.

Let us denote by $\mathcal{SI}(k)$ the family of special instanton bundles defined by the monad (1.2) as in Proposition 1.1 for a fixed system of coordinates.

It is clearly a flat family of vector bundles on $\mathbb{P}^3$, parametrized by the scheme $S = \mathbb{C}^{2k+1} \setminus \{J \neq 0\}$. □

Main Theorem I. *Let $0 \le \beta < \gamma$ be two integers ($\gamma \ge 2$), such that $k\gamma - (k+1)\beta > 0$. Let E be a special instanton bundle belonging to $\mathcal{SI}(k)$ such that h^1 End $E(-2) \neq 0$ (such E exists by Remark 1.4). The moduli space $\mathcal{M}_{\mathbb{P}^3}(0, k\gamma^2 - (k+1)\beta^2)$ is singular at the point corresponding to the bundle $E_{\beta,\gamma}$.*

Proof. By Proposition 2.2 we can define a natural map

$$\begin{aligned} \mathcal{SI}(k) &\to \mathcal{M}_{\mathbb{P}^3}(0, k\gamma^2 - (k+1)\beta^2) \\ [F] &\to [F_{\beta,\gamma}] \end{aligned}$$

which is clearly algebraic.

Let E be as in the statement. In any neighborhood of its class $[E] \in \mathcal{SI}(k)$ there is a $[F]$ with h^1 End $F(-2) = 0$. Moreover we can suppose that $\forall t \in \mathbf{Z}$ H^1 End $F(t)$ is a direct summand of H^1 End $E(t)$ as a $SL(U)$-module; this can be checked as follows. The minimal resolutions of E and F contain the same $SL(U)$-modules by (1.6). It follows easily that

$$H^i(E(t)) \simeq H^i(F(t)) \quad \forall i = 0, \dots 3, \quad \forall t \in \mathbf{Z} \text{ (as } SL(U)\text{-modules)} \tag{2.6}$$

By tensoring the resolution of E (resp. F) with the bundle E (resp. F) itself, we get the two sequences

$$\begin{aligned} 0 &\to S^{k-1}U \otimes E(-k-2) \to (S^kU)^2 \otimes E(-k-1) \\ &\to [S^{k+1}U \otimes E(-k)] \oplus [\mathbb{C}^2 \otimes E(-1)] \to \text{End } E \to 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned} 0 &\to S^{k-1}U \otimes F(-k-2) \to (S^kU)^2 \otimes F(-k-1) \\ &\to [S^{k+1}U \otimes F(-k)] \oplus [\mathbb{C}^2 \otimes F(-1)] \to \text{End } F \to 0\,. \end{aligned} \tag{2.8}$$

By (2.6), (2.7) and (2.8) it follows that H^1 End $E(t)$ and H^1 End $F(t)$ can be computed from sequences containing exactly the same $SL(U)$-modules, then a semicontinuity argument shows that $\forall t \in \mathbf{Z}$ H^1 End $F(t)$ is a direct summand of H^1 End $E(t)$ as a $SL(U)$-module. Hence if we write the formulae (2.4)

$$h^1 \text{ End } E_{\beta,\gamma} = \sum_{p+q=0} s_p b_q(\text{End } E)\,, \qquad h^1 \text{ End } F_{\beta,\gamma} = \sum_{p+q=0} s_p b_q(\text{End } F)$$

we see that $b_q(\text{End } E) \geq b_q(\text{End } F)$ for $q \neq 2$, while h^1 End $E(-2) \neq 0$ forces $b_{2\gamma}(\text{End } E)$ to be strictly greater than $b_{2\gamma}(\text{End } F)$. As a consequence

$$h^1 \text{ End } E_{\beta,\gamma} > h^1 \text{ End } F_{\beta,\gamma}$$

which clearly implies that $[E_{\beta,\gamma}]$ is a singular point of $\mathcal{M}_{\mathbb{P}^3}(0, k\gamma^2 - (k+1)\beta^2)$. □

Taking $k = 2, \gamma = 2, \beta = 1$ we obtain:

Corollary. $\mathcal{M}_{\mathbb{P}^3}(0,5)$ *is singular.*

In this particular case the singular points we have detected lie in the closure of the open set consisting of instanton bundles (see [12]). In fact (e.g. by [12]) the bundle $E_{1,2}$ just obtained for $k = 2$ has spectrum $(-1,-1,0,1,1)$. By Theorem 2.14 in [12] $E_{1,2}$ lies in the closure of instanton bundles. Moreover by [12] it follows that no component of $M(0,5)$ containing the special instanton bundles can meet any other component in a point corresponding to the spectrum $(-1,-1,0,1,1)$ and we have such a point which is singular, it follows that any component of $M(0,5)$ containing the special instanton bundles is singular. It would be interesting to know whether the same property is true in the general case.

Let us remark that taking $k = 2, \beta = 0$ we recover the result of [9].

The above result applies to a large set of values of c_2. As a sample one checks on a computer that in the range $5 \le c_2 \le 2000$ there are no more than 300 gaps.

3. In this section we deal with singularities of the moduli spaces $\mathcal{M}_{\mathbb{P}^3}(-1, c_2)$. Let $0 < \beta < \gamma$ be two *odd* integers; let $f_1, \ldots, f_4$ be homogeneous polynomials in the variables a, b, c, d without common zeroes of degree $(\gamma - \beta)/2$, $(\gamma - \beta)/2$, $(\gamma + \beta)/2$, $(\gamma + \beta)/2$ respectively. We can construct the monad

$$\mathbb{C}^k \otimes \mathcal{O}_{\mathbb{P}^3}(-(\gamma+1)/2) \xrightarrow{A_{\beta/2,\gamma/2}} \mathcal{W}^{k+1} \xrightarrow{B_{\beta/2,\gamma/2}} \mathbb{C}^k \otimes \mathcal{O}_{\mathbb{P}^3}((\gamma-1)/2)\,, \tag{3.1}$$

where $\mathcal{W} = \mathcal{O}_{\mathbb{P}^3}(-(\beta+1)/2) \oplus \mathcal{O}_{\mathbb{P}^3}((\beta-1)/2)$, and $A_{\beta/2,\gamma/2}$, $B_{\beta/2,\gamma/2}$ are obtained from the matrices A, B in (1.2) replacing a, b, c, d by f_1, f_2, f_3, f_4 respectively.
The cohomology of the monad (3.1) is a rank-2 bundle $E_{\beta/2,\gamma/2}$ whose Chern classes are $c_1 = -1, c_2 = k(\gamma/2)^2 - (k+1)(\beta/2)^2 + 1/4$.

Let $E_{\beta,\gamma}$ be defined as in Sec. 2 by the monad (2.1), via the homogeneous polynomials $f_1(a^2, b^2, c^2, d^2), \ldots, f_4(a^2, b^2, c^2, d^2)$. Then for $t \in \mathbf{Z}$,

$$\pi^* E_{\beta/2,\gamma/2}(t) = E_{\beta,\gamma}(2t-1)\,, \tag{3.2}$$

where $\pi : \mathbb{P}^3 \to \mathbb{P}^3$ is the finite morphism defined by $\pi(a, b, c, d) = (a^2, b^2, c^2, d^2)$. Moreover

$$\pi^* \mathrm{End}\; E_{\beta/2,\gamma/2}(t) = \mathrm{End}\; E_{\beta,\gamma}(2t) \tag{3.3}$$

In order to compute the cohomology groups of of End $E_{\beta/2,\gamma/2}(t)$ let us consider a more general setting. Let us denote by $\omega : \mathbb{C}^4 \backslash 0 \to \mathbb{C}^4 \backslash 0$ the mapping defined by $f_1, \ldots, f_4$, and by $\eta : \mathbb{C}^4 \backslash 0 \to \mathbb{P}^3$ the projection. Let E,G be bundles on $\mathbb{P}^3$ such that E is endowed with a $SL(U)$-action, and $\eta^* \pi^* G = \omega^* \eta^* E$. Let $H^i E(h) = T^h(U)$, where T^h is a representation of $SL(U)$; let $\mathcal{V} = \mathcal{O}_{\mathbb{P}^3}(-\beta/2) \oplus \mathcal{O}_{\mathbb{P}^3}(\beta/2)$. $\mathcal{V}$ is a $\mathbf{Q}$-bundle, one easily checks that $T^h(\mathcal{V})(h\gamma/2) = \bigoplus_s \mathcal{O}_{\mathbb{P}^3}(\nu_{s,h})$ with $2\nu_{s,h} \in \mathbf{Z}$. As in Sec. 2 we define s_p as the dimension of the degree p summand of the artinian algebra $S = \mathbb{C}[a, b, c, d]/(f_1, f_2, f_3, f_4)$; for $2q \in \mathbf{Z}$ let

$$a_q(E) = \sharp\{(s, h) : \nu_{s,h} = q\}$$

Then

Theorem 3.1. *Let us suppose that $a_q(E) = 0$ for $q \notin \mathbf{Z}$. Then*

$$h^i G(t) = \sum_{p+q=t} s_p a_q(E) \tag{3.4}$$

Proof. Let $F = \pi^* G$. Then by the classical projection formula

$$\begin{aligned} h^i F(2t) &= h^i G(t)\sigma_0 + h^i G(t-1)\sigma_2 + h^i G(t-2)\sigma_4\,, \\ h^i F(2t-1) &= h^i G(t-1)\sigma_1 + h^i G(t-2)\sigma_3\,, \end{aligned} \tag{3.5}$$

where σ_i is the degree i summand of the artinian algebra $\mathbb{C}[a,b,c,d]/(a^2,b^2,c^2,d^2)$. On the other hand by (2.2), (2.3) and (2.4)

$$h^i F(l) = \sum_{p+m=l} r_p b_m(E)\,, \tag{3.6}$$

where

$$b_m(E) = \sharp\{(s,h) : \mu_{s,h} = m\}\,,$$

and r_p is the degree p summand of the artinian algebra $\mathbb{C}[a,b,c,d]/({f_1}^2, {f_2}^2, {f_3}^2, {f_4}^2)$. By construction $\mu_{s,h} = 2\nu_{s,h}$, hence

$$b_{2m}(E) = a_m(E), (m \in \mathbf{Z}) \tag{3.7}$$

and the assumption implies

$$b_{2m-1}(E) = 0, (m \in \mathbf{Z})\,. \tag{3.8}$$

We can view Eqs. (3.5) as a linear system whose unknown are $h^i G(t)$. Since the solutions of the system are clearly unique, in order to verify (3.4) it is enough to check that the right-hand sides of (3.4) for $t \in \mathbf{Z}$ are solutions of (3.5). □

This follows easily from (3.6), (3.7) (3.8) and the identities

$$r_{2p} = s_p\sigma_0 + s_{p-1}\sigma_2 + s_{p-2}\sigma_4\,,$$
$$r_{2p-1} = s_{p-1}\sigma_1 + s_{p-2}\sigma_3 \quad .$$

Let us check for example the first equation of (3.5):

$$\begin{aligned}
&\sum_{p+q=t} s_p a_q(E)\sigma_0 + \sum_{p+q=t-1} s_p a_q(E)\sigma_2 + \sum_{p+q=t-2} s_p a_q(E)\sigma_4 \\
&\quad = \sum_{p+q=t} s_p a_q(E)\sigma_0 + \sum_{p+q=t} s_{p-1} a_q(E)\sigma_2 + \sum_{p+q=t} s_{p-1} a_q(E)\sigma_4 \\
&\quad = \sum_{p+q=t} b_{2q}(E)(s_p\sigma_0 + s_{p-1}\sigma_2 + s_{p-2}\sigma_4) \\
&\quad = \sum_{p+q=t} b_{2q}(E) r_{2p} \\
&\quad = \sum_{2p+2q=2t} r_{2p} b_{2q}(E) = h^i F(2t)\,.
\end{aligned}$$

As a consequence we get

Corollary 3.2. h^1 End $E_{\beta/2,\gamma/2}(t) = \sum_{p+q=t} s_p a_q(\text{End } E)$.

Proof. $a_q(\text{End } E) = 0$ for $q \notin \mathbf{Z}$ follows tensoring by E the sequence (1.6). □

Main Theorem II. *Let $0 < \beta < \gamma$ be two odd integers ($\gamma \geq 5$), such that $k\gamma - (k+1)\beta + 1 > 0$. Let E be a special instanton bundle belonging to $\mathcal{SI}(k)$ such that h^1 End $E(-2) \neq 0$. The moduli space $\mathcal{M}_{\mathbb{P}^3}(-1, k(\gamma/2)^2 - (k+1)(\beta/2)^2) + 1/4)$ is singular at the point corresponding to the bundle $E_{\beta/2,\gamma/2}$.*

The proof is similar to the proof of the main theorem I. The condition $k\gamma - (k+1)\beta + 1 > 0$ ensures by (3.2) the stability of $E_{\beta/2,\gamma/2}$, while the condition $\gamma \geq 5$ implies $s_\gamma \neq 0$ in Lemma 3.2.

Taking $k = 2$, $\gamma = 5$, $\beta = 3$ we obtain:

Corollary. *$\mathcal{M}_{\mathbb{P}^3}(-1, 6)$ is singular.*

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