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A tribute to Corrado Segre

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In this work we review some papers by Corrado Segre published during the eighties of the XIX century, when he was just above twenty[1]. We believe that doing so may be interesting from the historical point of view as well as helpful to recognizing a link between methods of research used in those years (scarcely present in contemporary literature) and a number of results rediscovered (often without knowing it) in the current century. We thus try to reconstruct the origin of the path that has led to the modern theory of vector bundles on an algebraic curve.

§1. Corrado Segre's programme

To appreciate the innovative character of the ideas put forward by the very young Segre, it is convenient to recall that in those years it was harshly debated upon the usefulness of studying Hyperspace Geometry. Some authors mantained that addressing the geometry of the hyperspaces was an unfruitful intellectual game not certainly helpful to understand the "real" geometry in two or three dimensions. On the other side, Veronese and Bertini at first, and then C.Segre were perfectly aware that not only the study of the geometry of hyperspaces would shed new light on the geometry of curves and surfaces of ordinary space, but also that these latter could be viewed - and this is certainly innovative - as points (defined by a number of parameters) belonging to new algebraic varieties that could not be placed in ordinary space.

The first and probably most inspiring result in this trend of ideas is due to Veronese([V] p.208):

Every rational curve of degree n is a projection of a unique curve C_n of the n-dimensional space $\mathbf{P}^n$ whose (affine) coordinate functions, in terms of a parameter t, are simply:

$$C_n:\ \begin{cases} x_i = t^i \\ i=1,2,\ldots,n \end{cases}$$

Thus, for example, if we consider the rational cubics in the plane, we see that each of so many different patterns is but a different "shadow" of the twisted[2] cubic in $\mathbf{P}^3$. Having in mind Plato's myth of the cavern, we could think of the plane or the three-space as the wall of the cavern on which

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[1] C.Segre was born in 1863

[2] "twisted" means "not contained in a hyperplane".

the shadows are cast of objects "living" in hyperspaces.

This result can easily be extended by Veronese:

Every rational curve C of degree n, twisted in $\mathbf{P}^r$, *is a projection of the curve* $C_n \subset \mathbf{P}^n$.

In particular it is $r \leq n$. Veronese called C_n the *normal model* of C. The fact that every projective variety $X \subset \mathbf{P}^r$ of degree d admits a normal model (now called linearly normal), that is the fact that X is a projection of a well-defined variety $Y \subset \mathbf{P}^N$ of degree d not obtainable as a projection of a variety of the same degree placed in a higher dimensional space, was altogether clear to Segre, as it was clear the importance of calculating this maximum dimension N. This fact leads to the problem of calculating the dimension of $H^0(X,\mathcal{O}(1))$, i.e. to the Riemann-Roch problem.

Once he has determined the number N, Segre sets himself the task of finding all the normal models of the varieties of a given type up to projective transformations of the space $\mathbf{P}^n$. After this second step has been carried out, he can conclude that every given variety is the shadow of some of these normal models. The properties of the variety can be inferred by the geometry of the normal model and the way it is projected into $\mathbf{P}^r$.

§2. Rational ruled surfaces

Segre tries out at once (1884) his difficult programme in the case of the rational ruled surfaces, where he obtains a complete result [S1]:

Every rational ruled surface, not a cone, of degree n, twisted in $\mathbf{P}^3$ *(or* $\mathbf{P}^r$*) has, as its normal model, one of the surfaces* $F_m \subset \mathbf{P}^{n+1}$, $m=1,2,\ldots,[\frac{n}{2}]$, *where* F_m *is defined by the equations:*

$$F_m: \begin{cases} x_i = t^i & x_{m+1+j} = ut^j \\ i=1,2,\ldots,m & j=0,1,\ldots,n-m \end{cases}$$

The surfaces F_m, now called Hirzebruch surfaces, can be projectively characterized as the surfaces made of the lines joining two corresponding curves C_n and C_{n-m}, placed in two spaces $\mathbf{P}^n$ and $\mathbf{P}^{n-m}$, which in turn are skewly embedded in $\mathbf{P}^{n+1}$.

We now wish to sketch the proof that Segre gives of the above result, which also contains a proof, for the rank two case, of the celebrated theorem - now known as *Grothendieck's theorem*[3] [Gr] - according to which every algebraic vector bundle over $\mathbf{P}^1$ splits as a direct sum of line bundles.

Let $S \subset \mathbf{P}^r$ be a rational ruled surface of degree n. Then its hyperplane section $C = S \cap H$ will be

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[3]this theorem has a long history (see e.g.[OSS]).In an algebraic form it had also been proved by Dedekind and Weber [DW].

a rational curve of degree n, twisted in H. Otherwise one could find a hyperplane H' containing C and a point $p \in S \setminus C$, in which case H' would contain the fiber r through p (since it would contain p and $x = C \cap r$). $H' \cap S$ would then have degree at least $n+1$, and thus $H' \supset S$, a contradiction. From Veronese's theorem it then follows $n \geq r-1$. The opposite inequality, i.e. the fact that every rational ruled surface in $\mathbf{P}^r$ of degree n ($r<n+1$) can always be obtained as a projection of a similar ruled surface in $\mathbf{P}^{n+1}$, is thought by Segre to be evident. In our opinion the assertion should have been given some justification. In fact Bertini in his comprehensive book about Hyperspace Geometry[Ber] gives of this assertion a rather complicated proof (which is actually a proof by Segre himself relative to the case of ruled surfaces on an elliptic curve adapted to the rational case, see Appendix I).

In any event, Segre reduces himself to classifying the rational ruled surfaces S, not cones, of degree n in $\mathbf{P}^{n+1}$(that otherwise would be cones over a C_n). Now, if C is an irreducible unisecant curve of degree $m \leq n$, then C is normal, i.e. it generates a $\mathbf{P}^m$. Indeed, if it were in a space $\mathbf{P}^\mu$ with $\mu < m$ "*siccome tutte le generatrici dovrebbero tagliare quella curva, si potrebbe per lo spazio stesso e per $n-\mu$ punti della superficie posti fuori di esso e su generatrici diverse far passare un iperpiano il quale conterrebbe le $n-\mu$ generatrici passanti per quei punti ed inoltre la curva di ordine m e quindi taglierebbe la superficie in una curva composta di ordine $n+m-\mu>n$ il che non può essere se quell'iperpiano non contiene tutta la superficie*"[4].

Once this has been established, Segre considers the unisecant curve C_m of least possible degree m contained in S. Since there always exists a hyperplane containing $[\frac{n+1}{2}]$ generatrices and S is not a cone, such hyperplane will also cut S along a unisecant curve of degree $\leq n-[\frac{n+1}{2}]$. *It follows* $m \leq [\frac{n}{2}]$. He then considers a hyperplane H passing through m distinct generatrices of S and $n-2m+1$ other points located on generatrices different from the previous ones. It is easy to see that, being $m \leq [\frac{n}{2}]$, such a hyperplane actually exists and moreover *does not contain generatrices other than the m already given*. Otherwise H would cut the curve C_m in at least $m+1$ points and would thus contain the whole C_m. But then, H would also contain the generatrices through the additional $n-2m+1$ points. It follows that H would contain C_m and $m+(n-2m+1)=n-m+1$ generatrices and hence would intersect S in a curve of degree $n+1$, contrary to S being twisted in $\mathbf{P}^{n+1}$. Thus H cuts S along the m generatrices as well as along a further irreducible curve C_{n-m} of degree $n-m$. From this one can easily conclude that $C_m \cap C_{n-m} = \emptyset$, hence *S is projectively equivalent to a* F_m.

[4] [S1] p.267, "*since all the generatrices should cut that curve, we could have a hyperplane containing that same space and $n-\mu$ points of the surface placed outside it, and on different generatrices. Such hyperplane would contain the $n-\mu$ generatrices passing through those points and, moreover, the curve of degree m; hence it would cut the surface along a reducible curve of degree $n+m-\mu>n$, which is impossible because the hyperplanes does not contain the whole surface.*"

The fact that the above theorem is equivalent to the splitting theorem of vector bundles is immediately clear if we set up a "dictionary" that allows us to translate the language of projective geometry into that of vector bundles.

§3. A useful dictionary

A ruled variety $S \subset \mathbf{P}^n$ with fibers $\mathbf{P}^s$ of genus g will be thought of as being defined by a morphism from a curve X of genus g into the grassmannian of the subspaces of $\mathbf{P}^n$ of dimension s:

$$\phi: X \to Gr(\mathbf{P}^s, \mathbf{P}^n)$$

such that ϕ is birational onto its image and $S = \bigcup_{x \in X} S_x$ where $S_x = \phi(x)$ is a vector subspace of $\mathbf{P}^n$ of dimension s which we shall call the *fiber* over x (or, with classic terminology, the *generatrix* over x).

We can canonically associate to S a holomorphic (algebraic) bundle over X of rank s+1, given by the preimage under ϕ, of the tautological bundle on the grassmannian. More specifically, such bundle, denoted by E_S, is given as a point set by

$$E_S = \{(x,y) | y \in E_x\} \subset X x \mathbf{C}^{n+1}$$

where, after a choice of coordinates, we have set $\mathbf{P}^n = \mathbf{P}(\mathbf{C}^{n+1})$ e $S_x = \mathbf{P}(E_x)$, E_x being a vector subspace of $\mathbf{C}^{n+1}$ of dimension s+1.

Moreover, a hyperplane $\mathbf{P}^{n-1} = \mathbf{P}(H)$ of $\mathbf{P}^n$ defines (noncanonically) a holomorphic section of E_S^* and (canonically) an element of $\mathbf{P}(\Gamma(X, E_S^*))$. Indeed, fixing a basis in $\mathbf{C}^{n+1}/H$ which is of dimension one, determines the projection $p: \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}/H \simeq \mathbf{C}$ and hence the global section $s_H: X \to E_S^*$ defined by

$$s_H(x)(y) = p(y) \quad x \in X,\ y \in E_x \subset \mathbf{C}^{n+1}$$

that is zero at all points $x \in X$ such that $E_x \subset H$.It is also immediately seen that such sections generate the fibers at all points. Of course, if we change the basis of $\mathbf{C}^{n+1}/H$, the section will be multiplied by a non-zero constant, and we will have a natural inclusion

$$\mathbf{P}^{n*} \to \mathbf{P}(\Gamma(X, E_S^*)) \tag{1}$$

It is now easy to verify that if S is a projection of S' from a point outside S', then E_S will be isomorphic to $E_{S'}$ and moreover if E^* is a bundle generated by its sections, i.e. if there is a morphism of bundles

$$X x \mathbf{C}^{n+1} \to E^*$$

which is surjective on each fiber, then $\bigcup_{x \in X} \mathbf{P}(E_x) \subset \mathbf{P}(\mathbf{C}^{n+1})$ defines a ruled variety S

whose associated bundle is E. Thus S is linearly normal if and only if tha map (1) is an isomorphism and, moreover, denoted by N the dimension of the space where the normal model "lives", $N = \dim H^0(X, E_S^*) - 1$. We note also that the sections s_H allow to compute the *Chern classes* of E_S^*; indeed the locus of points where $s+1$ generic sections of E_S^* become dependent dually corresponds to the set of fibers S_x that meet the space P^{n-s-1}, intersection of the $s+1$ hyperplanes corresponding to these sections, and thus

$$\deg S = \deg E_S^* \tag{2}$$

Ruled subvarieties of S will then correspond to fiber subbundles of E_S (and viceversa) whose degrees will still verify (2) and, in particular, *unisecant curves* will correspond to line subbundles of E, generic hyperplane sections - not containing whole fibers - will correspond to subbundles of E with trivial quotient, and finally, cones in $\mathbf{P}^n$ having a $\mathbf{P}^k$ as vertex and a ruled variety S in $\mathbf{P}^{n-k-1}$ as basis will correspond to bundles of the form $1^{k+1} \oplus E_S$ (denoting by $1 = X \times C$ the trivial bundle). Two ruled subvarieties S_1 and S_2 added in $\mathbf{C}^{n+1}$ fiber by fiber will give rise to a map

$$E_{S_1} \oplus E_{S_2} \to E_S \tag{3}$$

which will be injective (as a map between sheaves) if the fibers S_{1_x} and S_{2_x} do not meet generically, whereas it will be an isomorphism if S_{1_x} and S_{2_x} generate S_x for every x and $S_1 \cap S_2 = \emptyset$.

In particular, if S is a ruled surface, C_1 and C_2 distinct unisecant of S and L_1, L_2 the associated line bundles, then

$$0 \to L_1 \oplus L_2 \to E_S \to T \to 0$$

where we have denoted by T a torsion sheaf with support in $C_1 \cap C_2$. We thus find, upon computing the Chern classes, the simple intersection formula (already known to Segre):

$$\deg C_1 \cap C_2 = (\deg C_1 + \deg C_2) - \deg S \tag{4}$$

We finally observe that if we project $S \subset \mathbf{P}^n$ in $S' \subset \mathbf{P}^{n-1}$ from a point $p \in S_{x_0}$, the projection E_x in E'_x will be an isomorphism for $x \neq x_0$ having in x_0 cokernel of dimension one. We thus obtain the exact sequence (of sheaves):

$$0 \to E_S \to E_S' \to \mathcal{O}_{x_0} \to 0 \tag{5}$$

where $\mathcal{O}_{x_0}$ is the skyscraper sheaf in x_0 with fiber C. Moreover, if we perform a projection with center a fiber S_{x_0} of S we obtain

$$E_{S'} \simeq E_S(x_0)$$

The transformation (5) is called by many authors (e.g.. [M],[MN],[T]) an *elementary transformation.*

§4. Bundles on $\mathbf{P}^1$ of higher rank

In a paper of 1885-86[S2], Segre considers the case of bundles of $\mathbf{P}^1$ of rank $i+1$, with special attention to the case $i=2$. He obtains the Riemann-Roch formula, which can be stated by saying that if E is generated by its global sections and has degree n and rank $i+1$, then

$$h^0(\mathbf{P}^1,E)-1=n+i$$

Using the same methods and arguments given for the $i=1$ case, he studies, when $i=2$, the rational ruled surfaces and the unisecant curves of minimal degree contained in the surface S being able to prove that there always three curves of degrees m_1,m_2,m_3 generating the plane S_x for every x. From this we deduce that

$$E=L_1\oplus L_2\oplus L_3$$

The case of arbitrary rank, which was to be addressed in the dissertation of his disciple A.Bellatalla[Bel], is summarized thus: "*i ragionamenti qui fatti pel caso i=2 si estenderanno facilmente ad i qualunque e l'analogia permetterà di prevederne senz'altro i risultati, sicchè non ne farò più oggetto di un nuovo lavoro*"[5].

§5. Rank two bundles on an elliptic curve

The methods used in the rational case still apply, with little change, to the elliptic case, essentially because if $X\subset\mathbf{P}^n$ is an elliptic curve, then X is non-special. It is easy to see that if $S\subset\mathbf{P}^N$ is a twisted elliptic ruled surface of degree n, then its hyperplane section C will also be twisted in $\mathbf{P}^{N-1}$ and thus, being non-special it will result $n-1\geq N-1$. On the other hand, if S is not a cone, it cannot be $n=N$; otherwise a generic hyperplane H of $\mathbf{P}^N$ containing a generatrix would cut S along a further unisecant elliptic curve C_{n-1} of degree $n-1$, which would generate a space L of dimension at most $n-2$. Now the linear system of hyperplanes through L would cut S along the fixed C_{n-1} and a variable line so that S would turn out to be a rational ruled surface, against the assumption. It is thus $N\leq n-1$. Also, Segre succeeds in constructing explicitly (see Appendix I) starting with S in $\mathbf{P}^N$ (if $N<n-1$) another elliptic ruled surface $S'\subset\mathbf{P}^{n-1}$ of degree n projecting itself onto S.

Having solved in this way the Riemann-Roch problem, Segre begins to classify the elliptic ruled surfaces of degree n in $\mathbf{P}^{n-1}$, not cones, investigating first the possible degrees of the unisecants, and then, from these possible numerical invariants, obtaining explicit examples of elliptic ruled surfaces with the given numerical invariants.

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[5] [S2] p.96, "*The argument given for the case $i=2$ will easily extend to the case of arbitrary i, where, by analogy, we can anticipate the results. Hence I shall not address this case in future works*".

It is worth noting that this analysis leads to a new aspect (not present in the rational case): the existence of *indecomposable* ruled surfaces, that is, such that every two distinct unisecant curves have nonempty intersection. In other words, we believe for the first time, *indecomposable bundles of rank two make their appearance.* In order to construct the easiest (but easily extendible) such example let us consider a smooth plane cubic Γ on which we set an inflection point O, zero element of Γ thought of as an algebraic group. Let $A \in \Gamma, A \neq O$ be a point and $\tau_A : \Gamma \to \Gamma$ the *translation* taking a point P to the point $P \oplus A$ (where $\oplus$ denotes addition in Γ). Embedding two copies of Γ in $\mathbf{P}^5$ in such a way as to make them skew, we can construct a ruled surface $S_A \subset \mathbf{P}^5$ by joining with a line the images in $\mathbf{P}^5$of every pair of points P and $\tau(P)$. It is easy to see that the bundle E_{S_A} associated to such surface splits, and in addition

$$E_{S_A} = \mathcal{O}(-3O) \oplus \mathcal{O}(-3A)$$

Moreover the surface S_A contains only two cubic unisecants and ∞^2 quartic unisecants forming an algebraic system of degree 2. In other words two generic curves of the system meet transversally at two points. Projecting now in $\mathbf{P}^4$ the surface S_A from a point P_O chosen on the generatrix through O, outside the two planes, we obtain a surface $S'_A \subset \mathbf{P}^4$ of degree 5 containing an algebraic system of ∞^1 unisecants of minimal degree 3 obtained by projecting the quartics of S_A passing through P_O. The surface S'_A does not contain conics, since otherwise these would be projections of either a conic of S_A (there is no such a conic) or a cubic of S_A passing through P_O (there is no such a cubic by the choice of P_0). The surface S'_A, being smooth and normal, is thus indecomposable since otherwise if C_1 and C_2 were unisecants of degree n_1 and n_2 in S'_A, with $C_1 \cap C_2 = \emptyset$, then by (4) $n_1 + n_2 = 5$ and hence $n_1 \leq 2$ or $n_2 \leq 2$.

It is also easy to see that once the points O, P_0 and the embeddings of the two planes in $\mathbf{P}^5$ are fixed, the surfaces S'_A, obtained by different values of A, are not projectively equivalent in $\mathbf{P}^4$. On the other hand, we can choose coordinates in order that we can set arbitrarily on a given surface the points O, P_0 and the two planes. In other words there is, in $\mathbf{P}^4$, a family $S'_A (A \in \Gamma \setminus \{0\})$ of indecomposable ruled surfaces of degree 5. For $A = O$ the above construction "degenerates" in a decomposable surface generated by a plane cubic and a double line. However, making a different choice for the zero element of Γ, we obtain that the indecomposable surfaces are parametrized by the points of Γ, and any other surface is projectively equivalent to one of those. Therefore, counting the moduli, we see that all corresponding bundles can be obtained from a fixed one by tensoring by a line bundle of degree zero.

Translating this into modern language the indecomposable bundle $E^*_A := E_{S'_A}$ is given by the sequence

$$0 \to \mathcal{O}(-3O) \oplus \mathcal{O}(-3A) \to E^*_A \to \mathcal{O}_O \to 0$$

from which we derive, with easy calculations, the extension

$$0 \to \mathcal{O} \to E_A(-2O) \to \mathcal{O}(3A-2O) \to 0$$

Hence we see that the indecomposable bundles of rank 2 and degree 1 can all be obtained as extensions of the type

$$0 \to \mathcal{O} \to ? \to \mathcal{O}(P) \to 0$$

for some $P \in C$. It is now easy to find out (see [Ha] p.377) that the above bundles can all be gotten from the particular extension with $P=O$ by tensoring by a suitable line bundle. This is in accordance with the corollary on p.434 of [A], where the case of rank>2 is also treated.

§6. The ruled surfaces of arbitrary genus

The Riemann-Roch problem for the vector bundles of arbitrary rank on a curve of genus p is conclusively solved by Segre in [S5] (1887) following a different route. In the course of a letter exchange with Schubert, Segre comes to know of a interesting counting formula which combined with a formula of Zeuthen (now known as Hurwitz's theorem) allows to compute the genus of a multisecant curve γ, drawn on a variety $S \subset \mathbf{P}^n$ - ruled in linear spaces of dimension s parametrized by a curve of genus p - in terms of its degree and the projective characters of S. From that formula Segre can ingeniously deduce (see Appendix II) the dimension of the projective space where the normal model of S "lives", which could be written

$$h^0(E_S^*) \geq n-(s+1)(g-1) \tag{6}$$

where he distinguishes the cases (special and non-special) when relation (6) is an inequality or an inequality ($h^1(E_S^*) \neq 0$, $h^1(E_S^*)=0$). Subsequently, the same genus formula, together with "Castelnuovo's lemma", will lead Segre to a new proof, projective and hyperspatial as well, of the Riemann-Roch theorem for the curves.

After he has determined the dimension of the space where the normal model of a ruled surface of genus p "lives", Segre approaches the study of normal models in a research published in two papers of the Mathematische Annalen ([S4],[S6]) in the years 1887 and 1889. The study and the results are very deep, although the proofs - as Segre himself warns - may sometimes lack rigour. Only in recent times the assumptions have been precisely stated and the assertions carefully proved. The principal aim of the above mentioned research is that of determining, for a ruled surface $S \subset \mathbf{P}^N$, of degree n and genus p ($N=n-2p+1$) the family $\mathcal{C}_m$ of its unisecant curves of a fixed degree m. Segre realizes that only under suitable assumptions of genericity is possible to describe the families $\mathcal{C}_m$, hence he supposes these assumptions to be true, without stating them in precise form. Thus, set $d_m=2m-n-p+1$, he can make the following statements:

$$\textit{if } d_m \geq 0, \textit{ then } \dim \mathcal{C}_m = d_m \tag{7}$$

$$\textit{if } d_m < 0, \text{ then } \mathcal{C}_m = \emptyset \tag{8}$$

the index of the family $\mathcal{C}_m$, *i.e. the number of curves of* $\mathcal{C}_m$ (9)
passing through $d_m \geq 0$ *generic points of* S, *is* 2^p

in particular:

$$\textit{if } d_m = 0, \textit{ then } \mathcal{C}_m \textit{ is made of } 2^p \textit{ unisecants of the minimum degree } \frac{(n+p-1)}{2} \tag{10}$$

If the surface is not sufficiently general, the dimensions d_m can (in particular instances) increase. The method used by Segre to construct normal ruled surfaces in $\mathbf{P}^N$ and to study their families of unisecants $\mathcal{C}_m$ consists, as in the example of the elliptic quintic ruled surface of $\mathbf{P}^4$, of starting with decomposable ruled surfaces (easy to work with) embedded in a space $\mathbf{P}^{N+x}$, and then projecting them in $\mathbf{P}^N$ from x of their points. An accurate but simple analysis allows to compare the unisecants of the ruled surface of $\mathbf{P}^{N+x}$ with those of its projection. If the points from which one projects are generic, one can describe the families $\mathcal{C}_m$, since in the case of decomposable ruled surfaces the computation of the dimensions d_m, reduces, essentially, to the Riemann-Roch theorem on the base curve X. (A modern account of this method is given in [Gh]).

The computation of the index 2^p is based, instead, on a counting formula of Castelnuovo ([C]) whose correctness Segre himself doubted: "*La démonstration ingénieuse, que ce géomètre y donne de cette importante formule, pourrait laisser sur sa validité absolue des doutes, qui se réfléchiraient sur le n° présent et plus loin sur les n°° 20 et 21 de ces Recherches; cependant les confirmations qu'on trove de ces résultats me portent à penser qu'ils sont absolument vrais*"[6].

It is surprising that these results have remained unknown for nearly a hundred years until - starting in 1950 - various authors, such as Gunning, Nagata, Maruyama, Atiyah and others, have rediscovered, without being aware of, Segre's results (or, rather, some of them, not even the deepest ones). Thus, for example, (7) implies that, on a ruled surface S of genus p, the minimal self-intersection of one of its unisecants C_0 is never greater than p:

$$C_0^2 \leq p$$

as it is easily seen by noting that the self-intersection of a unisecant curve C of degree m on the surface S of degree n is given by

$$C^2 = 2m - n.$$

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[6] [S6], *footnote n.16*, "*The ingenious proof that this geometer*[Castelnuovo] *has given of such important formula, can leave some doubts about its validity. These will be reflected on the results of the present section and of* §§ 20-21 *of this paper. However, the confirmations that one has found of these results lead me to think that they are absolutely true*".

This is the form in which Nagata[N] gives this theorem without quoting Segre's paper. In the language of vector bundles, an equivalent formulation of the above fact can be given by saying that if E is a bundle of rank 2 and degree n on a curve X of genus p and L_0 is the line subbundle of E of maximal degree, then

$$\deg L_0 \geq [\frac{n-p+1}{2}].$$

This fact is an immediate consequence of (7) if E is generated by its global sections, case to which we can however reduce ourselves by tensoring by a suitable line bundle. It is in this form that we find Segre's results in Gunning[Gu] and others, among whom Stuhler[St], Lange-Narasimhan[LN], Lange[L1], at the end of the seventies. A study of the family of unisecants $\mathcal{C}_m$, together with a critical revision of Segre's results and a clarification of the assumptions of generality that he makes, can be found in Ghione[Gh], where the number 2^p is also computed (see also [GhLa]) without using Castelnuovo's counting formula. This number has also been computed by Hirschowitz[Hi] in 1984 and by Lange[L2] in 1985. We also wish to note that the general ruled surfaces, in the sense of Segre, give rise to vector bundles having the property that the subbundle of maximal degree has degree $[\frac{n-p+1}{2}]$, therefore being particular instances of *stable bundles in the sense of Mumford.*

A bundle E for which the inequality

$$\deg L \leq \frac{\deg E - p + 1}{2}$$

holds for every line subbundle $L \subset E$ is called *strongly stable* [Hi],[T]. The fact that the generic bundle is strongly stable and an extended version to the case of arbitrary rank have been profitably used by some authors.

We remark also that the family of unisecants $\mathcal{C}_m$ of the surface S corresponds, using our dictionary, to (Grothendieck) scheme of the quotients of E_S^* of rank 1 and degree m. Of these schemes Segre had in a sense given the dimension d_m and - for $d_m = 0$ - the number of points.

To conclude we would like to observe that Segre's interest for the theory of ruled varieties was also motivated by the study of algebraic curves from a projective point of view. Indeed, a linear series g_1^k on a curve $X \subset \mathbf{P}^n$ defines a rational ruled variety, made of the linear spaces generated by the divisors of the series, containing X as k-secant. Conversely, the existence of such variety S guarantees the existence of the g_1^k on X. Starting from this observation E.Mezzetti and G.Sacchiero ([MS]) could study, in recent times, those components of the Hilbert scheme of the curves of $\mathbf{P}^n$ arising from the consideration of the k-gonal curves.

In view of the previous considerations we would like to reflect on how the irregular flow of the mathematical (and cultural) fashions, with its whirlpools, can bury, for decades, entire theories, outstanding results and methods, to see them come up again out of necessity but without apparent link with their origin.

Appendix I: Riemann-Roch for rational and elliptic ruled surfaces

In this appendix we want to give the proof of Riemann-Roch theorem for an elliptic ruled surface following Segre. Really the construction is relative to the rational case such as Bertini([Ber] pag.356) proposes it, but it is essentially the same(with minor changes) as the proof of Segre in the elliptic case[S3].

We need the following lemma: *Let $C_n \subset \mathbf{P}^n$ be the Veronese curve, set a point $O \notin C_n$ and let Γ be the cone over C_n with vertex O. If $P_1,\ldots,P_{n+1}$ are independent points of Γ, there exists a rational normal curve contained in Γ which contains them.* Indeed, let H be the hyperplane generated by $P_1,\ldots,P_n$ and set $C := \Gamma \cap H$. We embed the ambient space in $\mathbf{P}^{n+1}$ in such a way that H is contained in a hyperplane H' of $\mathbf{P}^{n+1}$ not containing O but containing the normal model C' of C. C is the projection of C' from a point $O' \in H'$. Let Γ' be the cone over C' with vertex O. Γ' projects itself from O' onto Γ and the points $P_1,\ldots,P_{n+1}$ are the projections of the independent points $P_1',\ldots P_{n+1}'$ of Γ' which generate the hyperplane H'' of $\mathbf{P}^{n+1}$. The projection of the rational curve $H'' \cap \Gamma'$ of degree n is the curve that we looked for.

Now, let S be a rational ruled surface of degree n generating a space $\mathbf{P}^{n+1-k}$. We want to explicitly construct a new rational ruled surface of degree n generating a space $\mathbf{P}^{n+1}$ and projecting itself onto S. For the convenience of the reader we join a picture (fig. 1) in which the indexes denote the sequence of the various steps.

1. In the space $H_0 = \mathbf{P}^{n+1-k} \supset S$ we choose a generic hyperplane $H_1 = \mathbf{P}^{n-k}$ such that $C_1 := H_1 \cap S$ is an irreducible, rational curve of degree n.
2. We can obtain C_1 projecting a rational normal curve $V_2 \subset \mathbf{P}^n =: H_2$ from a space $O_2 = \mathbf{P}^{k-1}$. Let $K_2 = \mathbf{P}^{n+1}$ be the space generated by H_0 and O_2, and let Γ_2 be the projecting cone.
3. An hyperplane $H_3 = \mathbf{P}^{n-1}$ such that $O_2 \in H_3 \subset H_2$ cuts V_2 in the independent points $P_1, P_2, \ldots, P_n$.
4. H_3 projects itself onto the hyperplane $H_4 = \mathbf{P}^{n-k-1}$ of H_1 which cuts C_1 in the points $P_1', P_2', \ldots, P_n'$ obtained by projecting $P_1, P_2, \ldots P_n$.
5. Let now $H_5 = \mathbf{P}^{n-k}$ be an hyperplane of H_0 containing H_4, it will meet the surface in the curve C_5 which have in common with C_1 the points $P_1', P_2', \ldots, P_n'$.
6. Let $\Gamma_6 \subset K_2$ be the cone over C_5 with vertex O_2, this cone of degree n generates a space $H_6 = \mathbf{P}^n$ and contains the n independent points $P_1,\ldots,P_n$. We choose on Γ_6 another generic point P_{n+1}. By the previous lemma, there exists a rational curve V_6 of degree n containing $P_1,\ldots,P_{n+1}$ which projects itself from O_2 onto C_5.

7. Let A be a point on V_2 and A' its projection on C_1. Let B' be the corresponding point on C_5 by means of the generatrix of S through A', and B be the point which projects itself on B' from O_2. Joining A and B with a line in $K_2 = \mathbb{P}^{n+1}$ we get a rational ruled surface of degree n which generates K_2 and projects itself onto S.

The construction works with minor changes in the case of genus 1([S3] n.4).

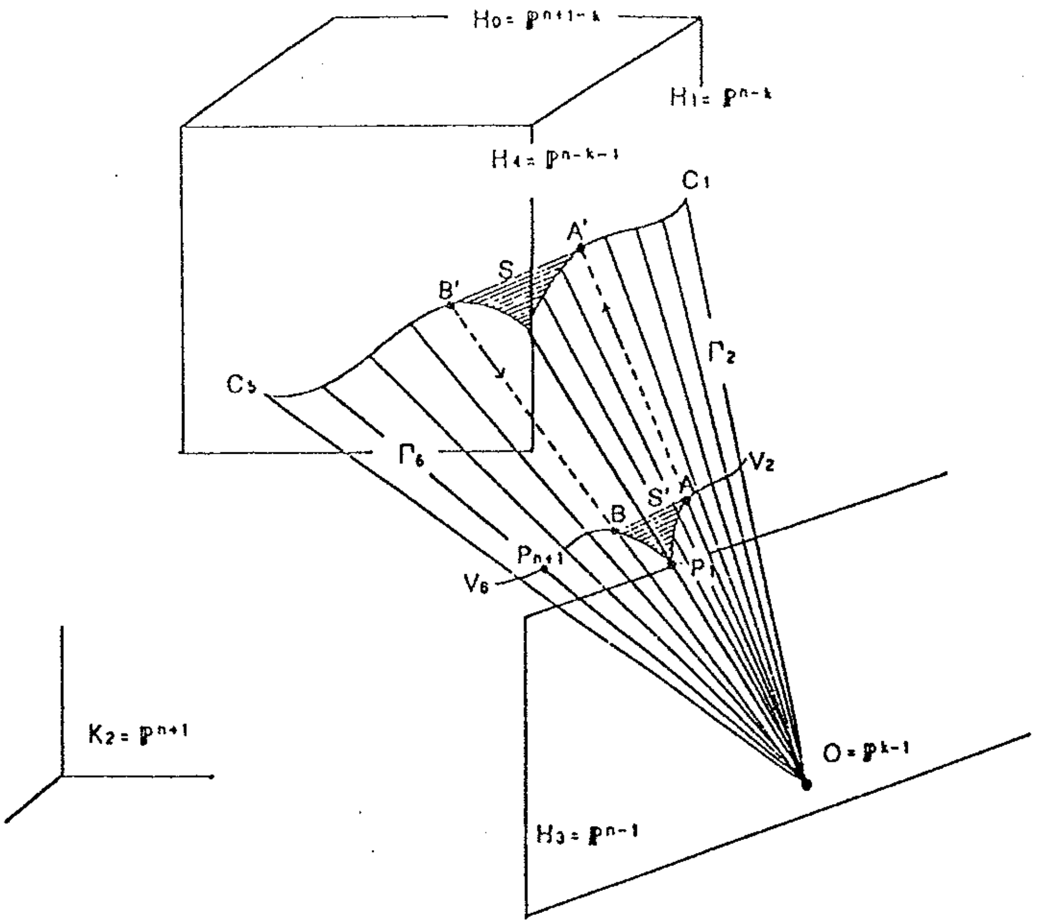


Fig. 1

Appendix II: Riemann-Roch for ruled varieties

The proof of Segre of the Riemann-Roch theorem for ruled varieties in projective spaces over a curve X of genus p is based on a counting formula that Schubert communicated to him in a letter. A modern proof with an analysis of the assumptions under which the result holds is in [GS].

Thus, let $S \subset \mathbf{P}^N$ be a variety of degree n ruled in projective spaces of dimension s on a curve X of genus p. Let $\gamma \subset S$ be a curve of degree m k-secant($k \geq s+1$), i.e. such that the morphism

$$\pi|_\gamma : \gamma \to X$$

is a k-fold covering of the base X. Now the genus of the curve γ can be computed in terms of these numerical invariants and of a further invariant z which measures essentially the number of divisors of degree s+1 contained in $\gamma_x = \gamma \cap S_x$ and which do not generate the fiber S_x. Thus, under suitable and natural assumptions, the genus g of γ is given by:

$$g = \frac{k-1}{s(s+1)}[(s+1)(m-s) - kn] + kp - \frac{z}{\binom{k-2}{s-1}}$$

We suppose now that there exists on S a curve γ (s+1)-secant such that for every $x \in X$ the s+1 points belonging to $\gamma \cap S_x = \mathbf{P}^s$ are independent. In this case z=0 and the above formula gives for the genus of γ the value:

$$g = m + (s+1)(p-1) + 1 - n$$

It is easy to check that S is normal if and only if γ is normal, that is

$$\chi(\gamma, \mathcal{O}_\gamma(1)) = \chi(S, \mathcal{O}_S(1)) = \chi(X, E_S^*).$$

If we apply the Riemann-Roch theorem to the curve γ we get:

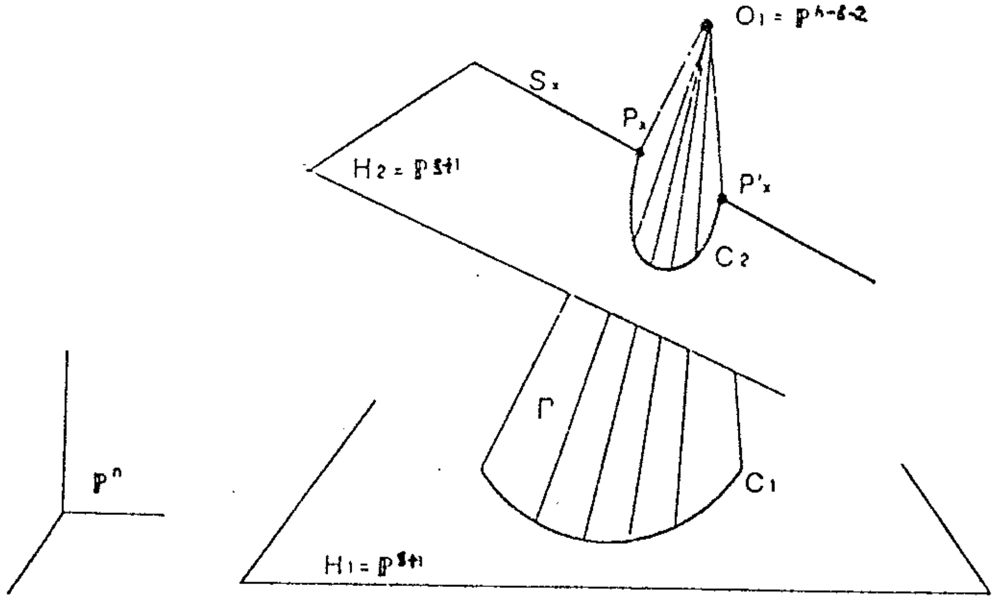
$$\chi(S, \mathcal{O}_S(1)) = m - g + 1$$

and substituting the value of g:

$$\chi(S, \mathcal{O}_S(1)) = n - (s+1)(p-1).$$

Then the key point is to construct the curve γ with the required property. Segre's technique consists in cutting S with a suitable cone $\Gamma \subset \mathbf{P}^N$. We make this construction in the following steps(see fig. 2):

1. We choose in $\mathbf{P}^N$ a space $O_1 = \mathbf{P}^{N-s-2}$ and a rational normal curve C_1 of degree s+1 in such a way that $O_1 \cap S = \emptyset$, C_1 is contained in a space $H_1 = \mathbf{P}^{s+1} \subset \mathbf{P}^N$ and $H_1 \cap O_1 = \emptyset$. Let Γ be the cone over C_1 with vertex O_1. We have dim $\Gamma = N - s$.
2. Let $S_x = \mathbf{P}^s$ ($x \in X$) be any generatrix of the ruled variety S and consider a space $H_2 = \mathbf{P}^{s+1}$ which contains S_x but does not meet the vertex O_1 of the cone. Such a space exists because $O_1 \cap S_x = \emptyset$ and then O_1 and S_x generate an hyperplane of $\mathbf{P}^N$. H_2 cuts the cone Γ in a rational normal curve C_2 of degree s+1. As S_x is an hyperplane of H_2, it cuts C_2 in the s+1 independent points$P_x, P'_x, P''_x \ldots$ It follows that $\Gamma \cap S_x$ consists of s+1 independent points for every x and then $\gamma := \Gamma \cap S$ is the curve that we looked for.



$O_1 = \mathbb{P}^{n-s-2}$
S_x
P_x
P'_x
$H_2 = \mathbb{P}^{s+1}$
C_2
Γ
$\mathbb{P}^n$
C_1
$H_1 = \mathbb{P}^{s+1}$

Fig. 2

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