A VANISHING THEOREM FOR THE IDEAL SHEAF OF CODIMENSION TWO SUBVARIETIES OF $\mathbf{P}^n$ (*)

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SOMMARIO. - *Sia* $X \subset \mathbf{P}^n$ *una varietà di codimensione* 2. *Proviamo che* $H^q(\mathcal{I}_X(t)) = 0$ *per* $n \geq q + 4t + 3$ *e* $1 \leq q \leq n-2$.

SUMMARY. - *Let* $X \subset \mathbf{P}^n$ *be a 2-codimensional variety. we prove that* $H^q(\mathcal{I}_X(t)) = 0$ *for* $n \geq q + 4t + 3$ *and* $1 \leq q \leq n-2$.

Let X be a smooth codimension two subvariety of $\mathbf{P}^n(\mathbf{C})$ $(n \geq 6)$, and let $\mathcal{I}_X$ be its ideal sheaf. In this short note we prove the following

THEOREM. $H^q(\mathcal{I}_X(t)) = 0$ *for* $n \geq q+4t+3$ *and* $1 \leq q \leq n-2$.

For $q = 1$ the theorem says that X is t-normal if $n \geq 4t+4$. As well known, the research for bounds on the t-normality is important because X is projectively normal (i.e. t-normal for all t) if and only if it is a complete intersection. The theorem slightly improves our result in [AO2] except when $t = 1$ or $t = 2$, $q = 1$.

REMARK. Let $K_X = \mathcal{O}_X(a)$ be the canonical bundle [L]. Then by Serre duality the theorem is true also for $2 \leq q \leq n-1$ and $4(t-a)+q \geq 3$.

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The proof of the theorem is essentially the same proof given in [AO2] with the difference that we begin the induction one step before, using Barth vanishing instead of Zak theorem on linear normality.

Let us state the basic results that we need. Let $Y \subset \mathbf{P}^n$ be smooth of dimension d.

FACT 1 (*Barth vanishing* [B]). $H^q(\mathcal{I}_Y) = 0$ for $q \leq 2d - n + 1$.

For $q = 1$ the vanishing above is equivalent to the connectedness of Y for $d \geq \frac{n}{2}$.

FACT 2 (*Ein* [E]). *Let N_Y be the normal bundle of Y in $\mathbf{P}^n$ and N_Y^* its dual. Let $q \leq 2d - n$ and $d \geq \frac{2}{3}n$. Suppose that for some integer t we have $H^q(N_Y^*(j)) = 0$ for $1 \leq j \leq t$. Then $H^q(\mathcal{I}_Y(j)) = 0$ for $1 \leq j \leq t$.*

FACT 3 (*Holme-Schneider* [HS]). *Let $d = n-2$ and let $K_Y = \mathcal{O}_Y(a)$ be the canonical bundle. If $a \leq n+1$ then Y is a complete intersection.*

Now we sketch the proof of the theorem. As in [AO1] we construct a filtration of subvarieties $X = X_1 \supset X_2 \supset X_3 \supset \ldots$ in the following way: if p is a point in $\mathbf{P}^n$ we define $X_{k+1} = \{x \in X_k | p \in T_x X_k\}$. By a theorem of Mather [M] for a generic choice of p all X_k are smooth of dimension $n - 2k$. For an account of Mather's proof see also [AO3].

X_k is the zero locus (as scheme) of a section of the bundle $N_{X_k, X_{k-1}}(-1)$ [AO1]. We set

$$I_k = \mathcal{I}_{X_k, \mathbf{P}^n} \quad \mathcal{I}_k = \mathcal{I}_{X_k, X_{k-1}} \quad N_k = N_{X_k, \mathbf{P}^n} \quad \mathcal{N}_k = N_{X_k, X_{k-1}} \, .$$

Then $\mathcal{N}_k(-1)|_{X_{k+1}} \simeq \mathcal{N}_{k+1}$.

LEMMA 1. *If $n \geq 2k + q + 2$ then $H^q(X_k, \mathcal{N}_k^*(p)) = 0$ for $p \leq 0$.*

Proof. Consider the twisted Koszul complex

$$0 \to \mathcal{O}_{X_k}(-a - n + 2k - 2 + p) \to \mathcal{N}_k^*(p) \to \mathcal{I}_{k+1}(p-1) \to 0 \, .$$

By fact 3 we may suppose $a \geq n + 2$ and then by Kodaira vanishing we have the result.

LEMMA 2. *Let* $n \geq 2k + q + 2$. *If* $H^q(\mathcal{N}_k^*(j)) = 0$ *for* $0 \leq j \leq s$ *then* $H^q(N_k^*(j)) = 0$ *for* $0 \leq j \leq s$.

Proof. We really prove more, i.e. $H^q(N_i^*|_{X_k}(j)) = 0$ for $0 \leq j \leq s$ and for $i = 1, \ldots, k$.

We proceed by induction on i. Suppose that $H^q(\mathcal{N}_k^*(j)) = 0$ for $0 \leq j \leq s$. For $i = 1$ we have $N_1^*|_{X_k}(j) \simeq \mathcal{N}_k^*(j + 1 - k)$ and by the lemma 1 and our hypothesis we have the thesis. Then we use the sequence

$$0 \to N_i^*|_{X_k}(j) \to N_{i+1}^*|_{X_k}(j) \to \mathcal{N}_{i+1}^*|_{X_k}(j) \to 0 \ ,$$

the isomorphism $\mathcal{N}_{i+1}^*|_{X_k}(j) \simeq \mathcal{N}_k^*(j + i + 1 - k)$, the lemma 1 and our hypothesis again.

Now fix n, q, t as in the hypothesis of the theorem. We may suppose $t \geq 1$. Consider the statement

$(*)_k$ $H^{q+i}(I_{k-i}(j - k + 1)) = 0$ for $0 \leq i \leq k - 1$, $k - 1 \leq j \leq t$,

$(*)_{t+1}$ is $H^{q+i}(I_{t+1+i}) = 0$ for $0 \leq i \leq t$ which is true by Barth vanishing exactly for $n \geq q + 4t + 3$.

$(*)_1$ is $H^q(I_1(j)) = 0$ for $0 \leq j \leq t$ which implies the theorem.

So it suffices to prove that $(*)_k \Rightarrow (*)_{k-1}$ for $2 \leq k \leq t + 1$.

Suppose $(*)_k$ true and let $0 \leq i \leq k - 2$. From the sequence

$$0 \to I_{k-i-1}(j - k + 1) \to I_{k-i}(j - k + 1) \to \mathcal{I}_{k-i}(j - k + 1) \to 0$$

we get

$$H^{q+i}(\mathcal{I}_{k-i}(j - k + 1)) = 0 \text{ for } 0 \leq i \leq k - 2,\ k - 1 \leq j \leq t$$

and of course this vanishing is true also for $j = k-2$ by Kodaira vanishing.

Consider now the twisted Koszul complex

$$0 \to \mathcal{O}_{X_{k-i-1}}(-a - n + k - 2i - 2 + j) \to$$

$$\to \mathcal{N}_{k-i-1}^*(j - k + 2) \to \mathcal{I}_{k-i}(j - k + 1) \to 0 \ .$$

By fact 3 and our hypothesis we may suppose that the bundle on the left is a negative line bundle. Then by Kodaira vanishing it follows

$$H^{q+i}(\mathcal{N}^*_{k-i-1}(j-k+2)) = 0 \text{ for } 0 \leq i \leq k-2,\ k-2 \leq j \leq t.$$

By lemma 2 we have

$$H^{q+i}(N^*_{k-i-1}(j-k+2)) = 0 \text{ for } 0 \leq i \leq k-2,\ k-2 \leq j \leq t,$$

and by fact 2 it follows

$$H^{q+i}(I_{k-i-1}(j-k+2)) = 0 \text{ for } 0 \leq i \leq k-2,\ k-2 \leq j \leq t.$$

The last vanishing is true also for $j = k-2$ by Barth vanishing, so that it gives exactly $(*)_{k-1}$. This concludes the proof.

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