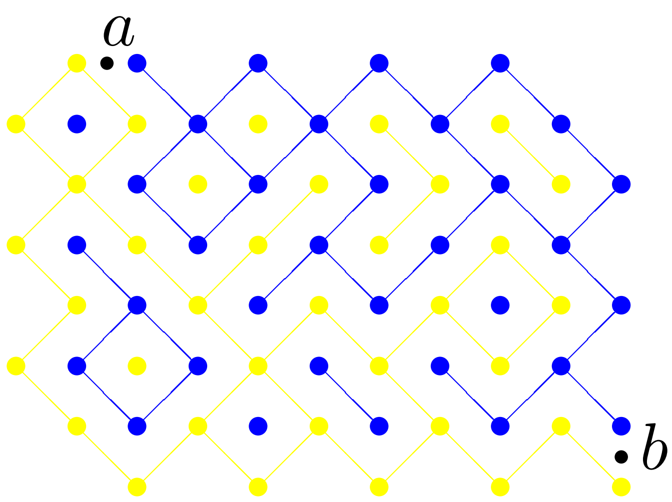
Fortuin-Kasteleyn random cluster model at criticality



Collections of blue edges, clusters are max subgraphs connected by blue edges

Dobrushin-type boundary conditions:

arc ba wired, arc ab dual-wired

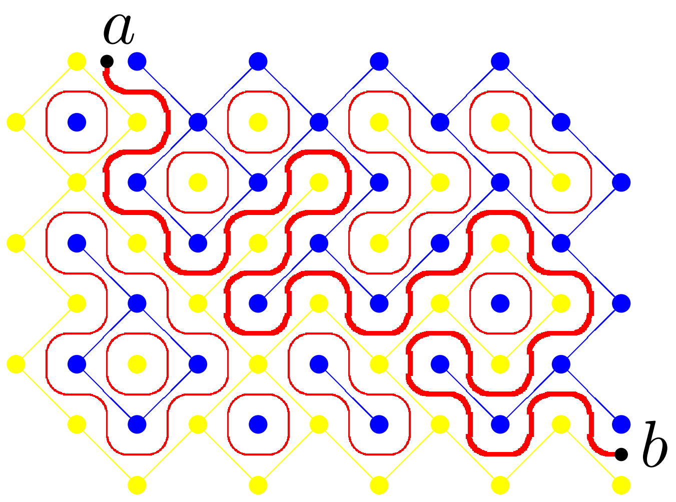
$$\mathbf{Prob} \asymp \left(\frac{1-x}{x}\right)^{\#\ \text{blue edges}} q^{\#\ \text{clusters}}$$

For $x = x_c(q) = 1/(\sqrt{q}+1)$ self-dual

Random cluster representation of q-state Potts model: $q = 2$ **FK Ising model**, $q = 1$ bond percolation on the square lattice, $q = 0$ uniform spanning tree.

Conjecture [Rohde-Schramm,...]. *Interface has conformally invariant SLE as a scaling limit for $q \in [0,4]$ and $x = x_c(q)$.*

Fortuin-Kasteleyn random cluster model at criticality: loop representation



Draw loops on the medial lattice, separating clusters from dual clusters

Dobrushin-type boundary conditions lead to an interface $\gamma : a \leftrightarrow b$

$$\mathbf{Prob} \asymp \left(\frac{1-x}{x}\right)^{\#\text{ blue edges}} q^{\#\text{ clusters}}$$

For $x = x_c(q) = 1/(\sqrt{q}+1)$ self-dual and $\mathbf{Prob} \asymp (\sqrt{q})^{\#\text{ loops}}$

Random cluster representation of q-state Potts model: $q = 2$ **FK Ising model**, $q = 1$ bond percolation on the square lattice, $q = 0$ uniform spanning tree.

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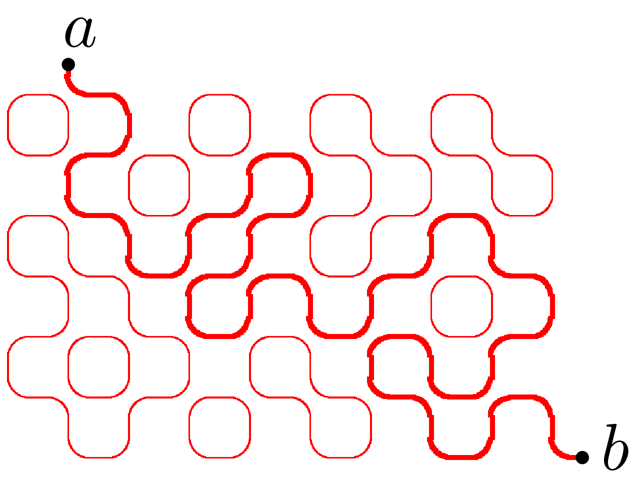
Fortuin-Kasteleyn random cluster model at criticality: loop representation

Dense loop collections on
the square lattice
Dobrushin-type boundary conditions:
besides loops an interface $\gamma: a \leftrightarrow b$



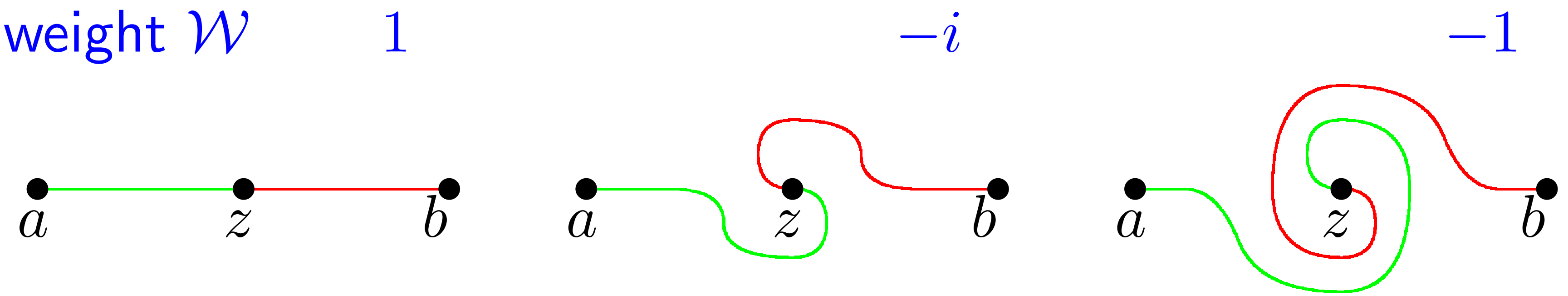
For self-dual x (conjecturally critical) $\mathbf{Prob} \asymp (\sqrt{q})^{\#\ \text{loops}}$

Random cluster representation of q-state Potts model: $q = 2$ **FK Ising model**, $q = 1$ bond percolation on the square lattice, $q = 0$ uniform spanning tree.

Conjecture [Rohde-Schramm,...]. *Interface has conformally invariant SLE as a scaling limit for* $q \in [0, 4]$.

FK Ising preholomorphic observable: $\boldsymbol{F(z) := \frac{1}{\sqrt{2\delta}} \mathbb{E}\, \chi_{z\in\gamma} \cdot \mathcal{W}}$

- Fermionic weight $\mathcal{W} \;:=\; \exp\left(-i\,\frac{1}{2}\,\textbf{winding}(\gamma, b \to z)/2\right)$



Note: through a given edge interface always goes in the same direction, so complex weight is uniquely defined up to sign.

Theorem. *For FK Ising when lattice mesh* $\delta \to 0$

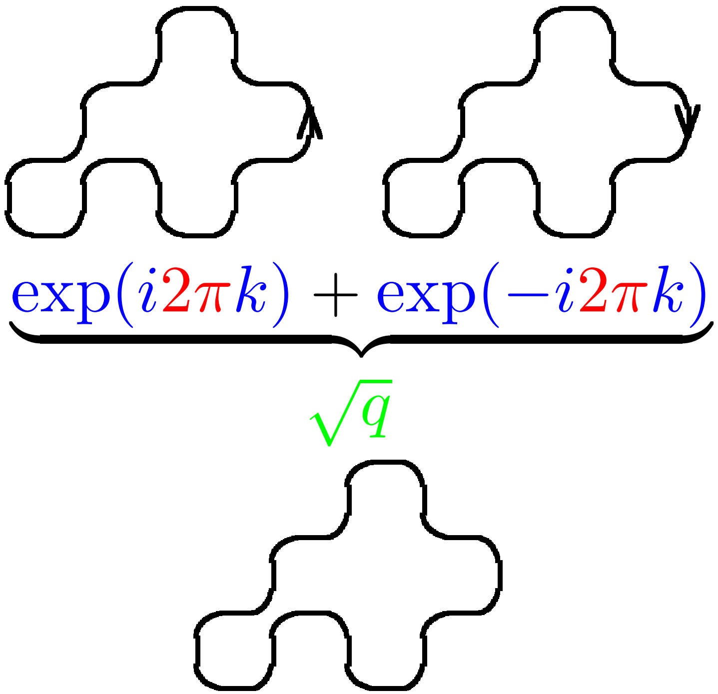
$$\boldsymbol{F(z) \rightrightarrows \sqrt{\Phi'(z)}\ \ \text{inside}\ \Omega,}$$

where Φ *maps conformally* Ω *to a horizontal width* 1 *strip,* $a, b \mapsto \text{ends}$. *The limit is conformally covariant.*

Where complex weights come from? [cf. Baxter]

Set $\mathbf{2\cos(2\pi k) = \sqrt{q}}$. Orient loops
$\Leftrightarrow$ height function changing by ± 1
whenever crossing a loop *(think of a geographic map with contour lines)*
New $\mathbb{C}$ partition function (local!):

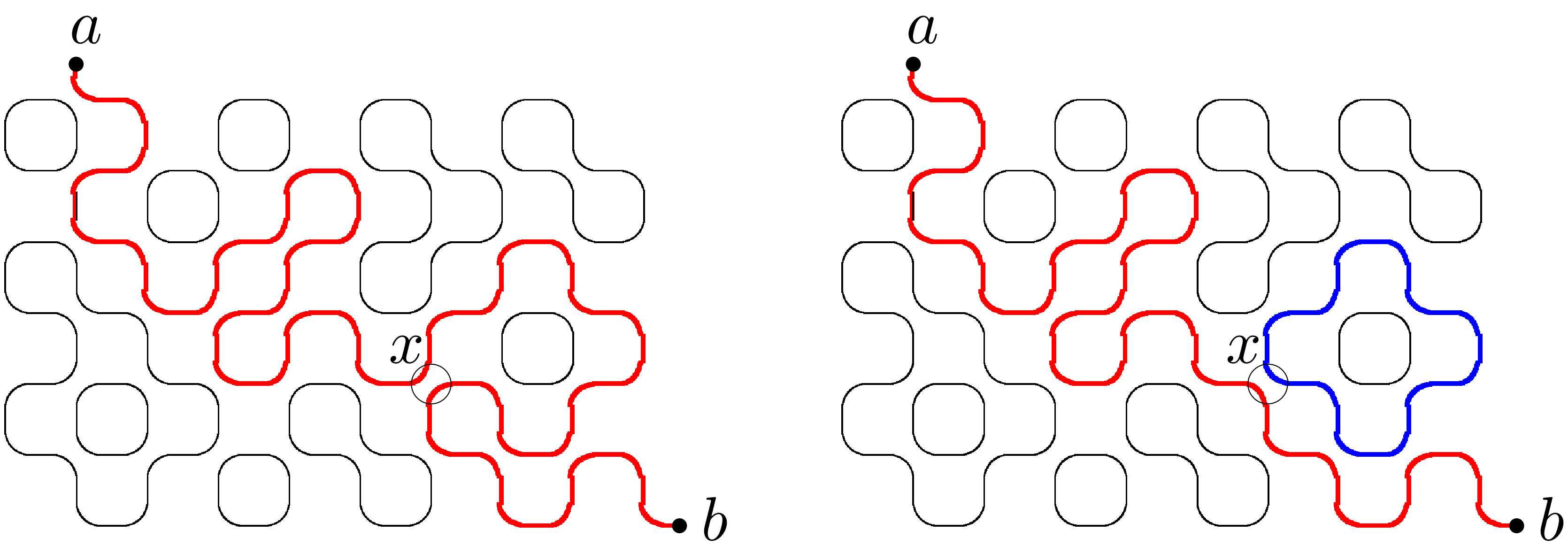
$$\boldsymbol{Z^{\mathbb{C}} = \sum \prod_{\text{sites}} \exp(i \text{ winding} \cdot k)}$$



Forgetting orientation projects onto the original model: $\operatorname{Proj}\left(Z^{\mathbb{C}}\right) = Z$

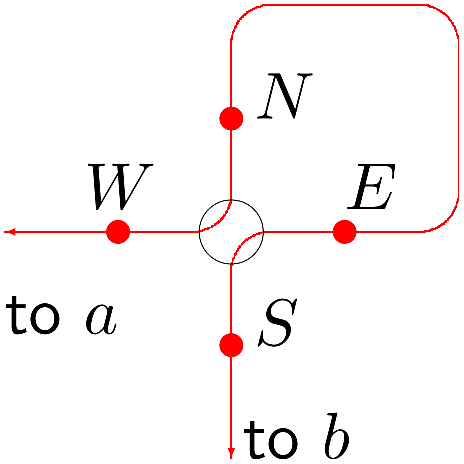
Orient interface $b \longrightarrow z$ and $a \longrightarrow z$ $\Leftrightarrow$ $+2$ monodromy at z
Can rewrite our observable as $\boldsymbol{F(z) = Z_{+2 \text{ monodromy at z}}}$
Note: being attached to $\partial\Omega$, γ is weighted differently from loops

Proof: discrete analyticity by local rearrangement

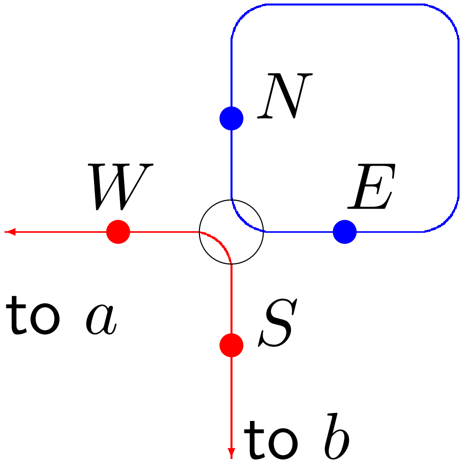


If a-b interface passes through x,
changing connections at x creates two configurations.
Additional loop on the right $\Rightarrow$ weights differ by a factor of $\sqrt{q} = \sqrt{2}$

Proof: discrete relation $\boldsymbol{F(N) + F(S) = F(E) + F(W)}$



$X\lambda^2$	$F(N)$	0
X	$F(S)$	$X\sqrt{2}$
$X\lambda$	$F(W)$	$X\lambda\sqrt{2}$
$X\bar{\lambda}$	$F(E)$	0



$\lambda = \exp(-i\pi/4)$ is the weight per $\pi/2$ turn. Two configurations together contribute equally to both sides of the relation:

$$X\lambda^2 + X + X\sqrt{2} = X\bar{\lambda} + X\lambda + X\lambda\sqrt{2}$$

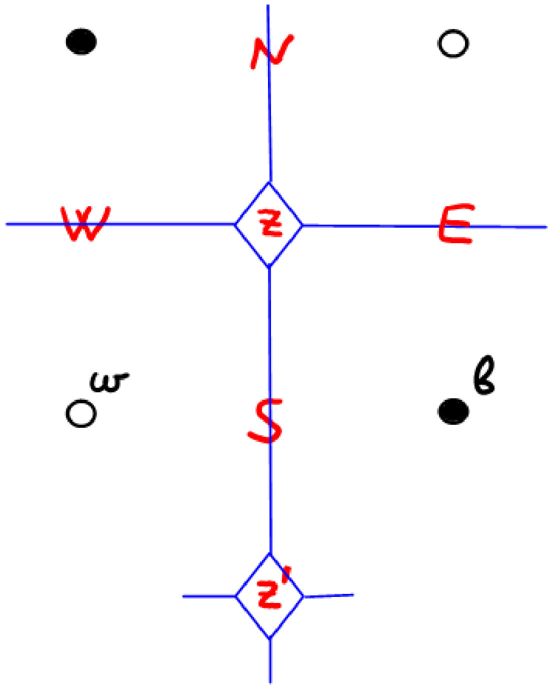
$$i + 1 + \sqrt{2} = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\sqrt{2} \qquad \square$$

Proof: $\boldsymbol{F(N) + F(S) = F(E) + F(W)}$ $\Rightarrow$ **s-Hol**

Interface always passes an edge always in the same direction, so complex weight is uniquely defined up to sign.

For example $\mathcal{W}(S)$ is proportional to

$$\pm\frac{1}{\sqrt{(z-S)}} = \pm\frac{1}{\sqrt{i(w-b)}}$$



Thus $F(E) \perp F(W)$ and $F(N) \perp F(S)$ and they give the same complex number, which we denote by $F(z)$, in two orthogonal bases.

Conclusion: (same for isoradial)

$$F(S) = \mathrm{Proj}\left(F(z), \pm\frac{1}{\sqrt{i(w-b)}}\right) = \mathrm{Proj}\left(F(z'), \pm\frac{1}{\sqrt{i(w-b)}}\right)$$

Proof: Riemann-(Hilbert-Privalov) boundary value problem

When z is on the boundary, winding of the interface $b \to z$ is uniquely determined, same as for $\partial\Omega$. So weight $\mathcal{W} = \tau^{-1/2}$.

Thus F solves the discrete version of the covariant Riemann BVP

$$\mathrm{Im}\left(\boldsymbol{F(z)\cdot\tau^{1/2}}\right) = \mathbf{0}, \quad \textbf{where } \boldsymbol{\tau} \textbf{ is the tangent to } \boldsymbol{\partial\Omega}.$$

Plus the interface **always** passes through a and b, winding is unique, so $|\boldsymbol{F(a)}| = |\boldsymbol{F(b)}| = \mathbf{1}/\sqrt{\mathbf{2}\boldsymbol{\delta}}$.

The continuum case is solved by $F = \sqrt{\Phi'}$,
where $\Phi : \Omega \to \text{infinite horizontal strip}, a, b \mapsto \text{ends}.$

Check: on $\partial\Omega$
$\mathrm{Im}\Phi = \text{const} \Rightarrow d\Phi \in \mathbb{R}_+ \Rightarrow \Phi' \cdot dz \in \mathbb{R}_+ \Rightarrow \mathrm{Im}\left(\sqrt{\Phi'}\cdot\tau^{1/2}\right) = 0$