Failure of n-uniqueness: a family of examples

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In this paper, the connections between model theory and the theory of infinite permutation groups (see [11]) are used to study the n-existence and the n-uniqueness for n-amalgamation problems of stable theories. We show that, for any $n \geq 2$, there exists a stable theory having $(k+1)$-existence and k-uniqueness, for every $k \leq n$, but has neither $(n+2)$-existence nor $(n+1)$-uniqueness. In particular, this generalizes the example, for $n = 2$, due to E.Hrushovski given in [3].

1 Introduction

Considerable work (e.g. [1], [3], [4], [9], [13]) has explored higher amalgamation properties for stable and simple theories. In this paper we analyze uniqueness and existence properties for a countable family of stable theories. In contrast to previous methods our approach uses group-theoretic techniques. We begin by giving some basic definitions.

Let T be a complete and simple L-theory with quantifier elimination. We denote by $\mathcal{C}_T$ the category of algebraically closed substructures of models of T with embeddings as morphisms. Also, given $n \in \mathbb{N}$, we denote by $P(n)$ the partially ordered set of all subsets of $\{1,\ldots,n\}$ and by $P(n)^-$ the set $P(n) \setminus \{1,\ldots,n\}$.

An *n-amalgamation problem* over $\mathrm{acl}(\emptyset)$ is a functor $a : P(n)^- \to \mathcal{C}_T$ such that

- (i)**:** $a(\emptyset) = \mathrm{acl}(\emptyset)$;
- (ii)**:** whenever $s_1, s_2, s_3 \in P(n)^-$ and $(s_1 \cap s_2) \subset s_3$, the algebraically closed sets $a(s_1), a(s_2)$ are independent over $a(s_1 \cap s_2)$ within $a(s_3)$;
- (iii)**:** $a(s) = \mathrm{acl}\{a(i) \mid i \in s\}$, for every $s \in P(n)^-$.

In here we denote by $\mathrm{acl}(A)$ the algebraic closure of A in T^{eq}. We recall that the objects of $P(n)^-$ (viewed as a category) are simply the elements of $P(n)^-$. Also, the morphisms of $P(n)^-$ are the inclusions $\iota_{s,t} : s \hookrightarrow t$, for every $s, t \in P(n)^-$ with $s \subseteq t$. In particular, an n-amalgamation problem assigns a morphism

$$a_{s,t} : a(s) \to a(t),$$

to every $s, t \in P(n)^-$ with $s \subseteq t$. The morphism $a_{s,t}$ is called *transition map* and, by functoriality, we have

$$a_{s_2,s_3} \circ a_{s_1,s_2} = a_{s_1,s_3},$$

for every $s_1, s_2, s_3 \in P(n)^-$ with $s_1 \subseteq s_2 \subseteq s_3$. By definition, the morphisms in $\mathcal{C}_T$ are the embeddings, that is, $a_{s,t}$ is the restriction of an automorphism to the algebraically closed substructure $a(s)$.

A *solution* of a is a functor $\bar{a} : P(n) \to \mathcal{C}_T$ extending a to the full power set $P(n)$ and satisfying the conditions $(i), (ii), (iii)$ (i.e. including the case $s = \{1, \dots, n\}$). In particular, in order to find a solution of a, we need to determine n embeddings

$$f_i : a(\{1, \dots, n\} \setminus \{i\}) \longrightarrow a(\{1, \dots, n\}) = \mathrm{acl}(\{a(i) \mid i \in \{1, \dots, n\}\}),$$

(for $1 \leq i \leq n$) compatible with a, that is,

$$f_i \circ a_{s,\{1,\dots,n\}\setminus\{i\}} = f_j \circ a_{s,\{1,\dots,n\}\setminus\{j\}}$$

for every $i, j \in \{1, \dots, n\}$ and $s \subseteq \{1, \dots, n\} \setminus \{i, j\}$.

The theory T is said to have *n-existence* (over $\mathrm{acl}(\emptyset)$) if every n-amalgamation problem over $\mathrm{acl}(\emptyset)$ has at least one solution. Similarly, we shall say that the theory T has *n-uniqueness* (over $\mathrm{acl}(\emptyset)$) if every n-amalgamation problem over $\mathrm{acl}(\emptyset)$ has at most one solution up to isomorphism (for more details see [9] and [12]).

It is a well known fact that every simple theory has 2-existence, by the presence of non-forking extensions. Moreover, if the theory is stable, then, by stationarity of strong types, 2-uniqueness holds. Consequentially, also 3-existence holds (for a proof see Lemma 3.1 of [9]). However, 3-uniqueness and 4-existence can fail for a general stable theory. Indeed, in [3], the authors thank E. Hrushovski for supplying an example of a stable theory which has neither 4-existence nor 3-uniqueness. The example is the following. Its construction involves a finite cover (for more details about finite covers see [5]).

Example 1.1 Let Ω be a countable set, $[\Omega]^2$ the set of 2-subsets of Ω, and $C = [\Omega]^2 \times \mathbb{Z}/2\mathbb{Z}$. Also let $E \subseteq \Omega \times [\Omega]^2$ be the membership relation, and let P be the subset of C^3 such that $((w_1, \delta_1), (w_2, \delta_2), (w_3, \delta_3))$ lies in P if and only if there are distinct $c_1, c_2, c_3 \in \Omega$ such that $w_1 = \{c_2, c_3\}, w_2 = \{c_1, c_3\}, w_3 = \{c_1, c_2\}$ and $\delta_1 + \delta_2 + \delta_3 = 0$. Now let M be the model with the 3-sorted universe $\Omega, [\Omega]^2, C$ and equipped with relations E, P and projection on the first coordinate $\pi : C \to [\Omega]^2$. Since M is a reduct of $(\Omega, \mathbb{Z}/2\mathbb{Z})^{\mathrm{eq}}$, we get that $T = \mathrm{Th}(M)$ is stable. It is shown in [3] that T has neither 4-existence nor 3-uniqueness.

In this paper we generalize this example. We summarize our main results in the following theorem.

Theorem 1.2 *For any $n \geq 2$, there exists a stable theory T_n such that T_n has $(k+1)$-existence and k-uniqueness for any $k \leq n$, but T_n has neither $(n+2)$-existence nor $(n+1)$-uniqueness.*

Also in Proposition 6.2 we prove that, for $n = 2$, the stable theory T_2 given in Theorem 1.2 coincides with the theory in Example 1.1.

All the material we present is expressed in a purely algebraic terminology. Indeed, the problem of n-uniqueness for a theory has also a natural formulation in terms of permutation groups, as is shown in [9, Proposition 3.5]. We adopt this approach here.

In Section 2, we introduce certain permutation modules which will be used to construct the automorphism groups of the countable $\aleph_0$-categorical structures M_n on which is based Theorem 1.2.

As is clear from the definition, the study of amalgamation problems requires a precise understanding of the algebraic closure in T^{eq}. Since the structures M_n are countable and $\aleph_0$-categorical, the algebraic closure can be rephrased with group theoretic terminology: it can be determined by studying certain closed subgroups of the automorphism group of M_n. This is done in Section 3 and Section 4.

2 The $\mathrm{Sym}(\Omega)$-submodule structure of $\mathbb{F}^{[\Omega]^n}$

We begin by reviewing some definitions and basic facts about permutation groups and permutation modules.

If C is a set, then the symmetric group $\mathrm{Sym}(C)$ on C can be considered as a topological group. The open sets in this topology are arbitrary unions of cosets of pointwise stabilizers of finite subsets of C. A subgroup Γ of $\mathrm{Sym}(C)$ is closed if and only if each element of $\mathrm{Sym}(C)$ which preserves all the orbits of Γ on C^n, for all $n \in \mathbb{N}$, is in Γ. It is well known that closed subgroups in this topology are precisely automorphism groups of first-order structures on C, see [2, Theorem 5.7] or [11].

Throughout the sequel we denote by $\mathbb{F}$ a field, $\mathbb{F}_2$ the integers modulo 2, Ω a countable set and $[\Omega]^n$ the set of n-subsets of Ω.

The natural action of the symmetric group $\mathrm{Sym}(\Omega)$ on $[\Omega]^n$ turns $\mathbb{F}[\Omega]^n$, the vector space over $\mathbb{F}$ with basis consisting of the elements of $[\Omega]^n$, into a $\mathrm{Sym}(\Omega)$-module. We will characterize the submodules of $\mathbb{F}[\Omega]^n$ in terms of certain $\mathrm{Sym}(\Omega)$-homomorphisms. The following definition is based on concepts first introduced in [10].

Definition 2.1 ([6], Def. 3.4) If $0 \le j \le n$, then the map $\beta_{n,j} : \mathbb{F}[\Omega]^n \to \mathbb{F}[\Omega]^j$, given by

$$\beta_{n,j}(\omega) = \sum_{\omega' \in [\omega]^j} \omega' \qquad (\text{for } \omega \in [\Omega]^n)$$

and extended linearly to $\mathbb{F}[\Omega]^n$, is a $\mathrm{Sym}(\Omega)$-homomorphism (in here we denote by $[\omega]^j$ the set of j-subsets of ω).

It is shown in [6] (see also [10]) that the submodules of $\mathbb{F}[\Omega]^n$ are completely determined by the maps $\beta_{n,j}$. Indeed, it is proved in [6, Corollary 3.17] that every submodule U of $\mathbb{F}[\Omega]^n$ is an intersection of kernels of β-maps, i.e. $U = \cap_{j\in S} \ker \beta_{n,j}$ for some subset S of $\{0, \dots, n\}$.

Using the controvariant Pontriagin duality we have that the dual module of $\mathbb{F}[\Omega]^n$ is $\mathbb{F}^{[\Omega]^n}$, i.e. the set of functions from $[\Omega]^n$ to $\mathbb{F}$. We recall that $\mathbb{F}^{[\Omega]^n}$ has a natural faithful action on $[\Omega]^n \times \mathbb{F}$ given by $(w, \delta)^f = (w, f(w) + \delta)$. Hence, $\mathbb{F}^{[\Omega]^n}$, endowed with the relative topology, becomes a topological $\mathrm{Sym}(\Omega)$-module and a profinite subgroup of $\mathrm{Sym}([\Omega]^n \times \mathbb{F})$. Also, given any map $\beta_{n,j} : \mathbb{F}[\Omega]^n \to \mathbb{F}[\Omega]^j$, there is a natural dual continuous $\mathrm{Sym}(\Omega)$-homomorphism $\beta^*_{n,j} : \mathbb{F}^{[\Omega]^j} \to \mathbb{F}^{[\Omega]^n}$ defined by

$$(\beta^*_{n,j} f)(\omega) = \sum_{x \in [\omega]^j} f(x).$$

Now, the lattice of the closed submodules of $\mathbb{F}^{[\Omega]^n}$ is the dual of the lattice of the submodules of $\mathbb{F}[\Omega]^n$. We point out that using the algorithm described in [6, Section 5], the lattice of the closed submodules of $\mathbb{F}^{[\Omega]^n}$ can be easily computed. Here we record the following fact that we are frequently going to use.

Proposition 2.2 *For $n \ge 1$, we have* $\mathrm{im}\,\beta^*_{n,n-1} = \ker \beta^*_{n+1,n}$.

P r o o f. The submodule $\mathrm{im}\,\beta_{n+1,n}$ of $\mathbb{F}[\Omega]^n$ is of the form $\cap_{j\in S} \ker \beta_{n,j}$, for some subset S of $\{0, \dots, n\}$. By [6, Proposition 3.19], we have that $\mathrm{im}\,\beta_{n+1,n} \subseteq \ker \beta_{n,j}$ if and only if 2 divides $n + 1 - j$. Therefore $S = \{j \mid 2 \text{ divides } n + 1 - j\}$.

Also by [6, Proposition 4.1], we have that if 2 divides $n + 1 - j$, then $\ker \beta_{n,n-1} \subseteq \ker \beta_{n,j}$. This yields $\mathrm{im}\,\beta_{n+1,n} = \cap_{j\in S} \ker \beta_{n,j} = \ker \beta_{n,n-1}$. In particular, the sequence

$$\mathbb{F}[\Omega]^{n+1} \xrightarrow{\beta_{n+1,n}} \mathbb{F}[\Omega]^n \xrightarrow{\beta_{n,n-1}} \mathbb{F}[\Omega]^{n-1}$$

is exact.

Now the Pontriagin duality is an exact controvariant functor on the sequences of the form $A \to B \to C$. This says that $\mathrm{im}\,\beta^*_{n,n-1} = \ker \beta^*_{n+1,n}$. □

3 Closed submodules of finite index in $\mathbb{F}_2^{[\Omega]^n}$

If A is a finite subset of Ω, then we write simply $\mathrm{Sym}(\Omega \setminus A)$ for the subgroup of $\mathrm{Sym}(\Omega)$ fixing pointwise A. In this section we study the closed $\mathrm{Sym}(\Omega \setminus A)$-submodules of $\mathbb{F}_2^{[\Omega]^{n-1}}$ of finite index. We start by considering the case $A = \emptyset$.

Lemma 3.1 *If $n \ge 1$, then $\mathbb{F}_2^{[\Omega]^n}$ has no proper closed* $\mathrm{Sym}(\Omega)$*-submodule of finite index.*

P r o o f. Let K be a closed submodule of $\mathbb{F}_2^{[\Omega]^n}$ of finite index. Then, $\mathbb{F}_2^{[\Omega]^n}/K$ is a finite $\mathrm{Sym}(\Omega)$-module. Since $\mathrm{Sym}(\Omega)$ has no proper subgroup of finite index, we get that $\mathrm{Sym}(\Omega)$ centralizes $\mathbb{F}_2^{[\Omega]^n}/K$. It follows that $f^\sigma - f \in K$, for every $\sigma \in \mathrm{Sym}(\Omega)$.

Let L be the annihilator of K in $\mathbb{F}_2[\Omega]^n$, i.e. $L = \{w \in \mathbb{F}_2[\Omega]^n \mid g(w) = 0 \text{ for every } g \in K\}$. Since K is a closed $\mathrm{Sym}(\Omega)$-submodule, the set L is a $\mathrm{Sym}(\Omega)$-submodule of $\mathbb{F}_2[\Omega]^n$. Now, let f be in $\mathbb{F}_2^{[\Omega]^n}$, σ in $\mathrm{Sym}(\Omega)$ and w in L. We get

$$0 = (f^\sigma - f)(w) = f^\sigma(w) - f(w) = f(w^{\sigma^{-1}} - w).$$

This says that $w^{\sigma^{-1}} - w$ is annihilated by every element of $\mathbb{F}_2^{[\Omega]^n}$. Therefore, $w^{\sigma^{-1}} - w = 0$ and σ centralizes w. This shows that $\mathrm{Sym}(\Omega)$ centralizes L. Since $n \geq 1$, the only element of $\mathbb{F}_2[\Omega]^n$ centralized by $\mathrm{Sym}(\Omega)$ is the zero vector. Hence $L = 0$ and, by the Pontriagin duality, $K = \mathbb{F}_2^{[\Omega]^n}$. □

In the forthcoming analysis we shall denote finite subsets of Ω by capital letters, while the elements of $[\Omega]^n$ will be generally denoted by lower cases.

Now, let A be a finite subset of Ω. To describe the closed $\mathrm{Sym}(\Omega \setminus A)$-submodules of $\mathbb{F}_2^{[\Omega]^{n-1}}$ of finite index we have to introduce some notation. Let B be a subset of A. We denote by $V_{B,A}$ the $\mathrm{Sym}(\Omega \setminus A)$-submodule of $\mathbb{F}_2^{[\Omega]^{n-1}}$ defined by

$$V_{B,A} = \{f \in \mathbb{F}_2^{[\Omega]^{n-1}} \mid f(w) = 0 \;\forall\, w \in [\Omega]^{n-1} \text{ with } w \cap A \neq B\} \tag{1}$$

and we denote by V_A the $\mathrm{Sym}(\Omega \setminus A)$-submodule of $\mathbb{F}_2^{[\Omega]^{n-1}}$ defined by

$$V_A = \bigoplus_{B \subseteq A, |B| < n-1} V_{B,A}. \tag{2}$$

In the following lemma we describe the elements of V_A.

Lemma 3.2 *Let A be a finite subset of Ω. Then*

$$V_A = \{f \in \mathbb{F}_2^{[\Omega]^{n-1}} \mid f(w) = 0 \textit{ for every } w \in [A]^{n-1}\}. \tag{3}$$

P r o o f. We denote by W the vector space on the right hand side of Equation (3). We start by proving that $V_A \subseteq W$. Let B be a subset of A with $|B| < n-1$ and f be in $V_{B,A}$. Consider w in $[A]^{n-1}$. Since $|B| < n-1$, $|w| = n-1$ and $w \subseteq A$, we have $w \cap A = w \neq B$. By Equation (1), we get $f(w) = 0$. This implies $f \in W$ and so $V_{B,A} \subseteq W$. Thence, by Equation (2), we obtain $V_A \subseteq W$.

Conversely, we prove that $W \subseteq V_A$. Let f be in W. For every subset B of A with $|B| < n-1$ define

$$f_B(w) = \begin{cases} f(w) & \text{if } w \cap A = B, \\ 0 & \text{if } w \cap A \neq B. \end{cases}$$

Clearly, $f_B \in \mathbb{F}_2^{[\Omega]^{n-1}}$ and, by Equation (1), $f_B \in V_{B,A}$. Let w be in $[\Omega]^{n-1}$ with $w \not\subseteq A$. Since $|w \cap A| < n-1$, we have

$$\left(\sum_{B \subseteq A, |B| < n-1} f_B\right)(w) = \sum_{B \subseteq A, |B| < n-1} f_B(w) = f_{w \cap A}(w) = f(w).$$

Similarly, let w be in $[\Omega]^{n-1}$ with $w \subseteq A$ (that is, $w \in [A]^{n-1}$). As $f \in W$, we have $f(w) = 0$. Also, by definition of f_B, we obtain $f_B(w) = 0$. This shows that $f = \sum_{B \subseteq A, |B| < n-1} f_B$. By Equation (2), it follows that $f \in V_A$. □

Lemma 3.3 *Let A be a finite subset of Ω. For each $B \subseteq A$, the* $\mathrm{Sym}(\Omega \setminus A)$*-modules $V_{B,A}$ are closed submodules of $\mathbb{F}_2^{[\Omega]^{n-1}}$. Moreover,*

$$\mathbb{F}_2^{[\Omega]^{n-1}} = \bigoplus_{B \subseteq A, |B| \leq n-1} V_{B,A} \tag{4}$$

and each $V_{B,A}$ is $\mathrm{Sym}(\Omega \setminus A)$*-isomorphic to $\mathbb{F}_2^{[\Omega \setminus A]^{n-1-|B|}}$.*

Proof. Since $V_{B,A}$ is an intersection of pointwise stabilizers of finite sets of $[\Omega]^{n-1} \times \mathbb{F}_2$, it is closed in $\mathbb{F}_2^{[\Omega]^{n-1}}$. It is straightforward to verify the remaining statements. □

Lemma 3.4 *Let A be a finite subset of Ω. The module V_A has finite index in $\mathbb{F}_2^{[\Omega]^{n-1}}$. Also, if V is a closed* $\mathrm{Sym}(\Omega \setminus A)$*-submodule of $\mathbb{F}_2^{[\Omega]^{n-1}}$ of finite index, then $V_A \subseteq V$.*

Proof. By Equations (2) and (4), we have that $\mathbb{F}_2^{[\Omega]^{n-1}}/V_A$ is isomorphic to $\oplus_{|B|=n-1} V_{B,A}$, which has dimension $\binom{|A|}{n-1}$. Therefore V_A has finite index in $\mathbb{F}_2^{[\Omega]^{n-1}}$.

Let V be a closed $\mathrm{Sym}(\Omega \setminus A)$-submodule of $\mathbb{F}_2^{[\Omega]^{n-1}}$ of finite index. Let $B \subseteq A$ with $|B| < n-1$. By Lemma 3.3, $V_{B,A}$ is $\mathrm{Sym}(\Omega \setminus A)$-isomorphic to $\mathbb{F}_2^{[\Omega \setminus A]^{n-1-|B|}}$. Since $[V_{B,A} : V_{B,A} \cap V] = [V_{B,A} + V : V]$ is finite, we have that $V_{B,A} \cap V$ has finite index in $V_{B,A}$. Now, by Lemma 3.1, the module $V_{B,A}$ does not have any proper closed $\mathrm{Sym}(\Omega \setminus A)$-submodule of finite index. Therefore $V_{B,A} = V_{B,A} \cap V$ and $V_{B,A} \subseteq V$. By definition of V_A in Equation (2), we get $V_A \subseteq V$. □

In the following lemma we describe the elements of $V_A + \ker \beta^*_{n,n-1}$.

Lemma 3.5 *Let A be a finite subset of Ω. We have $V_A + \ker \beta^*_{n,n-1} = \{f \in \mathbb{F}_2^{[\Omega]^{n-1}} \mid (\beta^*_{n,n-1} f)(w) = 0$ for every $w \in [A]^n\}$.*

Proof. If $n = 1$, then the equality is clear. So assume $n \geq 2$.

By Lemma 3.2, the elements of V_A are the functions $f \in \mathbb{F}_2^{[\Omega]^{n-1}}$ vanishing on each element of $[A]^{n-1}$. Now, if $f_1 \in V_A$, $f_2 \in \ker \beta^*_{n,n-1}$ and $w \in [A]^n$, then

$$(\beta^*_{n,n-1}(f_1 + f_2))(w) = (\beta^*_{n,n-1} f_1)(w) = \sum_{w' \in [w]^{n-1}} f_1(w') = 0.$$

Therefore, it remains to prove that if $f \in \mathbb{F}_2^{[\Omega]^{n-1}}$ and $(\beta^*_{n,n-1} f)(w) = 0$ for every $w \in [A]^n$, then $f \in V_A + \ker \beta^*_{n,n-1}$. Let a be a fixed element of A and let $g \in \mathbb{F}_2^{[\Omega]^{n-2}}$ be the function defined by

$$g(\omega) = \begin{cases} f(\omega \cup \{a\}) & \text{if } \omega \subseteq A \text{ and } a \notin \omega, \\ 0 & \text{otherwise}. \end{cases}$$

Set $f_2 = \beta^*_{n-1,n-2} g$. By Proposition 2.2, we have that $f_2 \in \mathrm{im}\, \beta^*_{n-1,n-2} = \ker \beta^*_{n,n-1}$. Set $f_1 = f - f_2$. We claim that f_1 lies in V_A, from which the lemma follows. By Lemma 3.2, it suffices to prove that $f_1(w') = 0$ for every $w' \in [A]^{n-1}$. Let w' be in $[A]^{n-1}$. Assume $a \in w'$. By the definition of g, we have

$$f_2(w') = (\beta^*_{n-1,n-2} g)(w') = \sum_{\omega \in [w']^{n-2}} g(\omega) = g(w' \setminus \{a\}) = f(w')$$

and $f_1(w') = 0$. Now assume $a \notin w'$. By the definition of g and by the hypothesis on f, we have

$$\begin{aligned} f_2(w') &= (\beta^*_{n-1,n-2} g)(w') = \sum_{\omega \in [w']^{n-2}} g(\omega) = \sum_{\omega \in [w']^{n-2}} f(\omega \cup \{a\}) \\ &= \sum_{x \in [w' \cup \{a\}]^{n-1}} f(x) + f(w') = (\beta^*_{n,n-1} f)(w' \cup \{a\}) + f(w') = f(w'), \end{aligned}$$

and $f_1(w') = 0$. □

Definition 3.6 We write W_A for $\beta^*_{n,n-1}(V_A)$, with V_A as in Equation (2).

Now, using the previous lemmas we describe the closed $\mathrm{Sym}(\Omega \setminus A)$-submodules of $\mathrm{im}\,\beta^*_{n,n-1}$ of finite index.

Proposition 3.7 *Let A be a finite subset of Ω. The module W_A is the unique minimal closed $\mathrm{Sym}(\Omega \setminus A)$-submodule of $\mathrm{im}\,\beta^*_{n,n-1}$ of finite index. Furthermore, $W_A = \{g \in \mathrm{im}\,\beta^*_{n,n-1} \mid g(w) = 0$ for every $w \in [A]^n\}$.*

P r o o f. Let W be a closed $\mathrm{Sym}(\Omega \setminus A)$-submodule of $\mathrm{im}\,\beta^*_{n,n-1}$ of finite index. By the first isomorphism theorem W is the image via $\beta^*_{n,n-1}$ of some closed $\mathrm{Sym}(\Omega \setminus A)$-submodule V of $\mathbb{F}_2^{[\Omega]^{n-1}}$ of finite index. Now, by Lemma 3.4, we get $V_A \subseteq V$. So $\beta^*_{n,n-1}(V_A) \subseteq \beta^*_{n,n-1}(V) = W$. Hence, $W_A = \beta^*_{n,n-1}(V_A)$ is the unique minimal closed $\mathrm{Sym}(\Omega \setminus A)$-submodule of $\mathrm{im}\,\beta^*_{n,n-1}$ of finite index.

Now, from Lemma 3.5 the rest of the proposition is immediate. □

4 The infinite family of examples

Before introducing our examples, we need to set some auxiliary notation.

Definition 4.1 Let M be a structure and A, B subsets of M. We denote by $\overline{\mathrm{Aut}(A/B)}$ the subgroup of $\mathrm{Aut}(M)$ fixing setwise A and fixing pointwise B. The setwise stabilizer of A in $\mathrm{Aut}(M)$ will be denoted by $\mathrm{Aut}(M)_{\{A\}}$, while the permutation group induced by $\overline{\mathrm{Aut}(A/B)}$ on A will be denoted by $\mathrm{Aut}(A/B)$.

Let $n \geq 2$ be an integer and Ω be a countable set.

Definition 4.2 We consider M_n the multisorted structure with sorts Ω, $[\Omega]^n$ and $[\Omega]^n \times \mathbb{F}_2$ and with automorphism group $\mathrm{im}\,\beta^*_{n,n-1} \rtimes \mathrm{Sym}(\Omega)$. Note that this is well-defined as $\mathrm{im}\,\beta^*_{n,n-1}$ is a closed submodule of $\mathbb{F}_2^{[\Omega]^n}$.

Moreover, the theory $T_n = \mathrm{Th}(M_n)$ is stable (see Section 6).

In the next paragraph we introduce some notation that would be useful to describe the algebraically closed sets of M_n.

Denote by $\pi : [\Omega]^n \times \mathbb{F}_2 \to [\Omega]^n$ the projection on the first coordinate. Given A a finite subset of M_n, we have that A is of the form $A_1 \cup A_2 \cup A_3$, where A_1 belongs to the sort Ω, A_2 belongs to the sort $[\Omega]^n$ and A_3 belongs to the sort $[\Omega]^n \times \mathbb{F}_2$. Consider $\tilde{A}_2 \subseteq \Omega$ the union of the elements in A_2 and $\tilde{A}_3 \subseteq \Omega$ the union of the elements in $\pi(A_3)$. We define the *support* of A, written $\mathrm{supp}(A)$, to be the subset $A_1 \cup \tilde{A}_2 \cup \tilde{A}_3$ of Ω. Finally, we define $\mathrm{cl}(A)$ to be the subset of M_n

$$\mathrm{cl}(A) := \mathrm{supp}(A) \cup [\mathrm{supp}(A)]^n \cup ([\mathrm{supp}(A)]^n \times \mathbb{F}_2)$$

In the rest of this section we describe the algebraically closed sets in the structure M_n. Here we consider structures *up to interdefinability*, which allows us to identify an $\aleph_0$-categorical structure with its automorphism group. So we identify two substructures A_1, A_2 of a structure M, if $\mathrm{Aut}(A_1) = \mathrm{Aut}(A_2)$. If M is an $\aleph_0$-categorical structure and $A \subset M$, we denote the algebraic closure $\mathrm{acl}^{\mathrm{eq}}(A)$ of A simply by $\mathrm{acl}(A)$, i.e. the union of the finite $\mathrm{Aut}(M/A)$-invariant sets of M^{eq}. We recall that definable subsets of $\mathrm{acl}(A)$ correspond, up to interdefinability, to closed subgroups of $\mathrm{Aut}(M/A)$ of finite index, see [8, Section 4.1] or Theorem 4.1 in the article *"The structure of totally categorical structures"* by W. Hodges [11, page 116].

Similarly, if $A \subset M$, we denote the definable closure $\mathrm{dcl}^{\mathrm{eq}}(A)$ of A simply by $\mathrm{dcl}(A)$, i.e. the set of the points of M^{eq} fixed by $\mathrm{Aut}(M/A)$.

Lemma 4.3 *Let A be a finite set of M_n. Then*

$$\mathrm{Aut}(M_n/\,\mathrm{cl}(A)) = W_{\mathrm{supp}(A)} \rtimes \mathrm{Sym}(\Omega \setminus \mathrm{supp}(A))$$

(where $W_{\mathrm{supp}(A)}$ is the closed $\mathrm{Sym}(\Omega \setminus \mathrm{supp}(A))$-submodule of $\mathrm{im}\,\beta^*_{n,n-1}$ in Definition 3.6). Moreover, $\mathrm{Aut}(M_n/\,\mathrm{cl}(A))$ is the unique minimal closed subgroup of finite index of $\mathrm{Aut}(M_n/A)$.

Proof. Set $\Gamma = \operatorname{Aut}(M_n/\operatorname{cl}(A))$. We first prove that $\Gamma = W_{\operatorname{supp}(A)} \rtimes \operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$. By definition of the multisorted structure M_n, we have $\operatorname{Aut} M_n = \operatorname{im} \beta^*_{n,n-1} \rtimes \operatorname{Sym}(\Omega)$. Therefore, an element of Γ is an ordered pair of the form $g\sigma$, where $g \in \operatorname{im} \beta^*_{n,n-1}$ and $\sigma \in \operatorname{Sym}(\Omega)$. The action of $g\sigma$ on the elements belonging to the sorts Ω and $[\Omega]^n$ is given by the permutation σ. Also, the action of $g\sigma$ on the element (w, x) belonging to the sort $[\Omega]^n \times \mathbb{F}_2$ is given by

$$(w, x)^{g\sigma} = (w^\sigma, x + g(w)).$$

This implies that the automorphism $g\sigma$ fixes the elements in $\operatorname{supp}(A)$ and in $[\operatorname{supp}(A)]^n$ (in the sorts Ω and $[\Omega]^n$) if and only if $\sigma \in \operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$. Also, the automorphism $g\sigma$ fixes the elements in $[\operatorname{supp}(A)]^n \times \mathbb{F}_2$ (in the sort $[\Omega]^n \times \mathbb{F}_2$) if and only if $g(w) = 0$ for every $w \in [\operatorname{supp}(A)]^n$. Hence, by the description of the elements of $W_{\operatorname{supp}(A)}$ in Proposition 3.7, we have $g\sigma \in \Gamma$ if and only if $g\sigma \in W_{\operatorname{supp}(A)} \rtimes \operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$.

We claim that Γ is the unique minimal closed subgroup of $\operatorname{Aut}(M_n/A)$ of finite index. Note that Γ is a closed subgroup of $\operatorname{Aut}(M_n/A)$ of finite index.

Now, let H be a closed subgroup of $\operatorname{Aut}(M_n/A)$ of finite index. Up to replacing H with $H \cap \Gamma$, we may assume that $H \subseteq \Gamma$. Let $\mu : \Gamma \to \operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$ be the natural projection. Since μ is a surjective continuous closed map and $\operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$ has no proper subgroup of finite index, we get that $\mu(H) = \operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$. This yields that $H \cap W_{\operatorname{supp}(A)}$ is a closed $\operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$-submodule of $W_{\operatorname{supp}(A)}$ of finite index. Now Proposition 3.7 shows that $H \cap W_{\operatorname{supp}(A)} = W_{\operatorname{supp}(A)}$. So $W_{\operatorname{supp}(A)} \subseteq H$ and $H = \Gamma$. □

In the following we denote by acl_{M_n} the acl in M_n.

Proposition 4.4 *Let A be a finite set of M_n. Then $\operatorname{acl}_{M_n}(A) = \operatorname{cl}(A)$.*

Proof. Let $\bar{b}$ be an m-tuple in M_n and A be a finite set of M_n. We first claim that $\operatorname{Aut}(M_n/\bar{b}) \geq \operatorname{Aut}(M_n/\operatorname{cl}(A))$ if and only if the underlying set of $\bar{b}$ is conteined in $\operatorname{cl}(A)$. One direction is obvious. Suppose that $\operatorname{Aut}(M_n/\bar{b}) \geq \operatorname{Aut}(M_n/\operatorname{cl}(A))$ for some finite $A \subset M_n$. Then by Lemma 4.3 we have that $\operatorname{Aut}(M_n/\operatorname{cl}(\operatorname{cl}(A), \bar{b})$ is a closed subgroup of finite index in $\operatorname{Aut}(M_n/\operatorname{cl}(A), \bar{b}) = \operatorname{Aut}(M_n/\operatorname{cl}(A))$. Hence $\operatorname{Aut}(M_n/\operatorname{cl}(\operatorname{cl}(A), \bar{b})$ is a closed subgroup of finite index in $\operatorname{Aut}(M_n/A)$. By uniqueness of the minimal closed subgroup of finite index of $\operatorname{Aut}(M_n/A)$ we get that $W_{\operatorname{supp}(A)} \rtimes \operatorname{Sym}(\Omega \setminus \operatorname{supp}(A))$ is equal to $W_{\operatorname{supp}(\operatorname{cl}(A),\bar{b})} \rtimes \operatorname{Sym}(\Omega \setminus \operatorname{supp}(\operatorname{cl}(A), \bar{b}))$ and, since $\operatorname{supp}(\operatorname{cl}(A), \bar{b}) = \operatorname{supp}(A, \bar{b})$, this is possible if and only if $\operatorname{supp}(\bar{b}) \subseteq \operatorname{supp}(A)$, which proves the claim.

By definition, $\operatorname{acl}_{M_n}(A)$ is the union of the finite orbits on M_n of $\operatorname{Aut}(M_n/A)$. Let $c \in \operatorname{acl}_{M_n}(A)$. Then $\operatorname{Aut}(M_n/A, c)$ is a closed subgroup of finite index in $\operatorname{Aut}(M_n/A)$. Hence, by Lemma 4.3, $\operatorname{Aut}(M_n/A, c) \geq \operatorname{Aut}(M_n/\operatorname{cl}(A)$. By the above argument we have that $c \in \operatorname{cl}(A)$.

Let $c \in \operatorname{cl}(A)$, then $\operatorname{Aut}(M_n/A) \geq \operatorname{Aut}(M_n/A, c) \geq \operatorname{Aut}(M_n/\operatorname{cl}(A))$. Hence the index of $\operatorname{Aut}(M_n/A, c)$ in $\operatorname{Aut}(M_n/A)$ is finite. □

Let $c^{\mathrm{eq}} \in M_n^{\mathrm{eq}}$. Then c^{eq} is a 0-definable equivalence class of a tuple b of elements in M_n. We denote by $\int(c^{\mathrm{eq}})$ the union of elements in M_n of c^{eq}. Similarly if $A \subseteq M_n^{\mathrm{eq}}$, we denote by $\int(A)$ the set of elements in M_n $\bigcup_{c^{\mathrm{eq}} \in A} \int(c^{\mathrm{eq}})$.

Proposition 4.5 *Let A be a finite set of M_n. Then $\int(\operatorname{acl}(A)) = \operatorname{cl}(A)$. In particular $\operatorname{acl}(\emptyset) = \emptyset$.*

Proof. Fix an enumeration $\bar{b}$ of $\operatorname{acl}_{M_n}(A)$ and set $\Gamma = \operatorname{Aut}(M_n/\operatorname{acl}_{M_n}(A))$. Consider the trivial relation $R = \{(b^\alpha, b^\alpha) : \alpha \in \operatorname{Aut}(M_n)\}$. Since R is an $\operatorname{Aut}(M_n)$-orbit, R is a 0-definable equivalence relation in M_n. Consider the R-equivalence class of $\bar{b}$. The pointwise stabilizer of $\bar{b}$ in $\operatorname{Aut}(M_n)$ is Γ which, by Lemma 4.3 and Proposition 4.4, has finite index in $\operatorname{Aut}(M_n/A)$ and so $\bar{b} \in \operatorname{acl}(A)$.

Let $c^{\mathrm{eq}} \in \operatorname{acl}(A)$, then $\operatorname{Aut}(M_n/A, c^{\mathrm{eq}})$ is a closed subgroup of finite index of $\operatorname{Aut}(M_n/A)$. By Lemma 4.3 $\operatorname{Aut}(M_n/A, c^{\mathrm{eq}})$ contains Γ. Being $\operatorname{Aut}(M_n/A, c^{\mathrm{eq}})$ also open in $\operatorname{Aut}(M_n/A)$ there exists a finite tuple $\bar{b}$ in M_n such that $\operatorname{Aut}(M_n/A, c^{\mathrm{eq}})$ contains the basic open subgroup $\operatorname{Aut}(M_n/A, \bar{b})$. Moreover $c^{\mathrm{eq}} = \bar{b}^{\operatorname{Aut}(M_n/A, c^{\mathrm{eq}})}$. By $\aleph_0$-categoricity the index of $\operatorname{Aut}(M_n/A, \bar{b})$ in $\operatorname{Aut}(M_n/A, c^{\mathrm{eq}})$ is finite. Then, the index of $\operatorname{Aut}(M_n/A, \bar{b})$ in $\operatorname{Aut}(M_n/A)$ is finite and so $\Gamma \leq \operatorname{Aut}(M_n/A, \bar{b})$. Hence by the same argument used in Proposition 4.4, we get that the underlying set in M_n of $\bar{b}$ is contained in $\mathrm{cl}(A) = \mathrm{acl}_{M_n}(A)$. From the fact that $\mathrm{Aut}(M_n/A, c^{\mathrm{eq}}) \leq \mathrm{Aut}(M_n/A)$ and $\bar{b} \in \mathrm{acl}_{M_n}(A)$ it follows immediately that also the underlying set of the $\mathrm{Aut}(M_n/A, c^{\mathrm{eq}})$-orbit $\bar{b}^{\mathrm{Aut}(M_n/A,c^{\mathrm{eq}})}$ is contained in $\mathrm{acl}_{M_n}(A)$. □

Corollary 4.6 *Let A be a finite set of M_n. Then,*

$$\mathrm{Aut}(M_n)_{\{\mathrm{acl}_{M_n}(A)\}} = \mathrm{Aut}(M_n)_{\{\mathrm{acl}(A)\}}.$$

Proof. From Proposition 4.4 and Proposition 4.5 it follows that $\mathrm{Aut}(M_n)_{\{\mathrm{acl}(A)\}} \leq \mathrm{Aut}(M_n)_{\{\mathrm{acl}_{M_n}(A)\}}$. Now, let $g \in \mathrm{Aut}(M_n)_{\{\mathrm{acl}_{M_n}(A)\}}$. Note that $\mathrm{acl}_{M_n}(A^g) = \mathrm{acl}_{M_n}(A)$. Consequently, $\mathrm{acl}(A^g) = \mathrm{acl}(A)$. If $c^{\mathrm{eq}} \in \mathrm{acl}(A)$, then the index of $\mathrm{Aut}(M_n/A, c^{\mathrm{eq}})$ in $\mathrm{Aut}(M_n/A)$ is finite. Therefore, $\mathrm{Aut}(M_n/A^g, (c^{\mathrm{eq}})^g) = g^{-1}\,\mathrm{Aut}(M_n/A, c^{\mathrm{eq}})g$ has finite index in $\mathrm{Aut}(M_n/A^g) = g^{-1}\,\mathrm{Aut}(M_n/A)g$, which implies that $(c^{\mathrm{eq}})^g \in \mathrm{acl}(A^g) = \mathrm{acl}(A)$. □

Proposition 4.7 *Let A be a finite subset of M_n. Then,* $\mathrm{dcl}(\mathrm{acl}_{M_n}(A)) = \mathrm{acl}(A)$.

Proof. Let $c^{\mathrm{eq}} \in \mathrm{acl}(A)$, i.e. the stabilizer of c^{eq} in $\mathrm{Aut}(M_n/A)$ has finite index in $\mathrm{Aut}(M_n/A)$. We need to show that the stabilizer of c^{eq} in $\mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A))$ is equal to $\mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A))$. We have the following disequality:

$$|\,\mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A)) : \mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A), c^{\mathrm{eq}})| \leq |\,\mathrm{Aut}(M_n/A) : \mathrm{Aut}(M_n/A, c^{\mathrm{eq}})|$$

Then $|\,\mathrm{Aut}(M_n/A) : \mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A), c^{\mathrm{eq}})|$ is finite. By Lemma 4.3 and Proposition 4.4 it follows that $\mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A), c^{\mathrm{eq}})$, is equal to $\mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A))$, i.e. $c^{\mathrm{eq}} \in \mathrm{dcl}(\mathrm{acl}_{M_n}(A))$.

Let $c^{\mathrm{eq}} \in \mathrm{dcl}(\mathrm{acl}_{M_n}(A))$. We need to show that $\mathrm{Aut}(M_n/A, c^{\mathrm{eq}})$, has finite index in $\mathrm{Aut}(M_n/A)$. We have that

$$\begin{gathered} |\,\mathrm{Aut}(M_n/A) : \mathrm{Aut}(M_n/\,\mathrm{cl}(A)), c^{\mathrm{eq}})| = \\ |\,\mathrm{Aut}(M_n/A) : \mathrm{Aut}(M_n/A, c^{\mathrm{eq}})|\,|\,\mathrm{Aut}(M_n/A, c^{\mathrm{eq}}) : \mathrm{Aut}(M_n/\,\mathrm{cl}(A), c^{\mathrm{eq}})| \end{gathered} \tag{5}$$

Since $c^{\mathrm{eq}} \in \mathrm{dcl}(\mathrm{acl}_{M_n}(A))$ we have that $\mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A), c^{\mathrm{eq}}) = \mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A))$. Lemma 4.3 and the equality (5) imply that $|\,\mathrm{Aut}(M_n/A) : \mathrm{Aut}(M_n/A, c^{\mathrm{eq}})|$ is finite. This proves that $c^{\mathrm{eq}} \in \mathrm{acl}(A)$ and the proof is complete. □

Corollary 4.8 *Let A be a finite subset of M_n. Then*

$$\mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A)) = \mathrm{Aut}(M_n/\,\mathrm{acl}(A)).$$

Proof. Let $g \in \mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A))$ and $c^{\mathrm{eq}} \in \mathrm{acl}(A)$. Proposition 4.7 yields that $(c^{\mathrm{eq}})^g = c^{\mathrm{eq}}$, which means that $g \in \mathrm{Aut}(M_n/\,\mathrm{acl}(A))$. It remains to prove that $\mathrm{Aut}(M_n/\,\mathrm{acl}(A)) \leq \mathrm{Aut}(M_n/\,\mathrm{acl}_{M_n}(A))$. Consider the trivial relation R given by $R = \{(b,b) : b \in M_n\}$. This is a 0-definable relation. Let $a \in \mathrm{acl}_{M_n}(A)$. Then $\{a\} \in M_n^{\mathrm{eq}}$ and $\mathrm{Aut}(M_n/A, \{a\}) = \mathrm{Aut}(M_n/A, a)$ is a closed subgroup of finite index in $\mathrm{Aut}(M_n/A)$. Hence, we can consider that $\mathrm{acl}_{M_n}(A) \subseteq \mathrm{acl}(A)$ and the thesis follows at once. □

Remark 4.9 Proposition 4.4 yields that if A is a finite set of M_n, then $\mathrm{acl}_{M_n}(A) = \mathrm{acl}_{M_n}(\mathrm{supp}(A))$. Therefore, from Proposition 4.7 it follows that $\mathrm{acl}(A) = \mathrm{acl}(\mathrm{supp}(A))$.

Proposition 4.10 *Let $A_1, \ldots, A_n$ be finite subsets in the sort Ω. Then*

$$\mathrm{acl}(\mathrm{acl}(A_1), \ldots, \mathrm{acl}(A_n)) = \mathrm{acl}(\bigcup_{i=1}^{n} A_i).$$

P r o o f. Obviously, $\operatorname{acl}(\bigcup_{k=1}^{n} A_k) \subseteq \operatorname{acl}(\operatorname{acl}(A_1), \dots, \operatorname{acl}(A_n))$.

Let $c^{\mathrm{eq}} \in \operatorname{acl}(\operatorname{acl}(A_1), \dots, \operatorname{acl}(A_n))$ and set $G = \operatorname{Aut}(M_n / \operatorname{acl}(A_1), \dots, \operatorname{acl}(A_n))$. Then, the pointwise stabilizer $G_{c^{\mathrm{eq}}}$ has finite index in G. By Corollary 4.8 we have that

$$G = \bigcap_{i=1}^{n} W_{A_i} \rtimes \operatorname{Sym}(\Omega \setminus A_i).$$

Moreover, $G \geq W_{\bigcup_{i=1}^{n} A_i} \rtimes \operatorname{Sym}(\Omega \setminus \bigcup_{i=1}^{n} A_i)$ and G is a closed subgroup in $\operatorname{Aut}(M_n / \bigcup_{i=1}^{n} A_i)$. So, G is a closed subgroup of finite index in $\operatorname{Aut}(M_n / \bigcup_{i=1}^{n} A_i)$ which implies that also $G_{c^{\mathrm{eq}}}$ is of finite index in $\operatorname{Aut}(M_n / \bigcup_{i=1}^{n} A_i)$. Now, $G_{c^{\mathrm{eq}}} = G \cap \operatorname{Aut}(M_n / \bigcup_{i=1}^{n} A_i, c^{\mathrm{eq}})$ and

$$\begin{gathered} |\operatorname{Aut}(M_n / \textstyle\bigcup_{i=1}^{n} A_i) : \operatorname{Aut}(M_n / \textstyle\bigcup_{i=1}^{n} A_i, c^{\mathrm{eq}})| = \\ |\operatorname{Aut}(M_n / \textstyle\bigcup_{i=1}^{n} A_i) : G_{c^{\mathrm{eq}}}| / |\operatorname{Aut}(M_n / \textstyle\bigcup_{i=1}^{n} A_i, c^{\mathrm{eq}}) : G_{c^{\mathrm{eq}}}|, \end{gathered}$$

i.e. $c^{\mathrm{eq}} \in \operatorname{acl}(\bigcup_{i=1}^{n} A_i)$. □

5 k-existence and k-uniqueness for M_n

In this section we prove Theorem 1.2. Note that, up to renaming the elements of Ω, we may assume that $\Omega = \mathbb{N}$. In the sequel we denote by $[k]$ the subset $\{1, \dots, k\}$ of $\mathbb{N}$. Also, given $i \in [k]$, we denote by $[k] - i$ the set $\{1, \dots, k\} \setminus \{i\}$. Finally, we denote the theory $\operatorname{Th}(M_n)$ by T_n.

We start by studying k-uniqueness in T_n. We first single out the following technical lemma which would be used in Proposition 5.2.

Lemma 5.1 *Let k and n be integers, with $k < n$, and $A_1, \dots, A_k$ be subsets of Ω. Then*

$$(\dagger) \qquad \bigcap_{i=1}^{k} \left(V_{A_i} + \ker \beta^*_{n,n-1} \right) = \left(\bigcap_{i=1}^{k} V_{A_i} \right) + \ker \beta^*_{n,n-1}.$$

P r o o f. We denote the left-hand-side of $(\dagger)$ by $V_{1,k}$ and the right-hand-side of $(\dagger)$ by $V_{2,k}$ (where the label k is used in order to remember the number of intersections).

We argue by induction on k. Note that if $k = 0$ or $k = 1$, then there is nothing to prove. Assume $(\dagger)$ holds for k intersections (where $k \geq 1$) and that $k + 1 < n$. In particular, we point out that $n > 2$. We prove that $(\dagger)$ holds for $k + 1$ intersections. Clearly, $V_{2,k+1} \subseteq V_{1,k+1}$. Let g be in $V_{1,k+1}$. We need to show that $g \in V_{2,k+1}$. By induction hypothesis (on the sets $A_1, \dots, A_k$), we have

$$V_{1,k+1} = \left(\left(\bigcap_{i=1}^{k} V_{A_i} \right) + \ker \beta^*_{n,n-1} \right) \cap (V_{A_{k+1}} + \ker \beta^*_{n,n-1}). \tag{6}$$

By Equation (6) and Proposition 2.2, we have

$$g = g_1 + \beta^*_{n-1,n-2} h_1 = g_2 + \beta^*_{n-1,n-2} h_2, \tag{7}$$

where $g_1 \in \cap_{i=1}^{k} V_{A_i}$, $g_2 \in V_{A_{k+1}}$ and $h_1, h_2 \in \mathbb{F}_2^{[\Omega]^{n-2}}$. We claim that (up to replacing h_1 by $h_1 + l$, where $l \in \ker \beta^*_{n-1,n-2}$), we may assume that $h_1 - h_2 \in \cap_{i=1}^{k} V_{A_i \cap A_{k+1}}$.

Let w be an $(n-1)$-subset of Ω contained in $A_i \cap A_{k+1}$ for some $i = 1, \dots, k$. Since $g_1 \in V_{A_i}$ and $g_2 \in V_{A_{k+1}}$, we see that $g_1(w) = g_2(w) = 0$. So, from Equation (7) we obtain

$$g(w) = (\beta^*_{n-1,n-2} h_1)(w) = (\beta^*_{n-1,n-2} h_2)(w),$$

that is, $(\beta^*_{n-1,n-2}(h_1 - h_2))(w) = 0$. As w is an arbitrary $(n-1)$-subset of $A_i \cap A_{k+1}$, Lemma 3.5 yields $h_1 - h_2 \in V_{A_i \cap A_{k+1}} + \ker \beta^*_{n-1,n-2}$. As i is an arbitrary element in $\{1, \dots, k\}$, we get

$$h_1 - h_2 \in \bigcap_{i=1}^{k} (V_{A_i \cap A_{k+1}} + \ker \beta^*_{n-1,n-2}).$$

Since $k+1 < n$, we have $k < n-1$ and so we may now apply our inductive hypothesis on the sets $A_1 \cap A_{k+1}, \ldots, A_k \cap A_{k+1}$. We have

$$h_1 - h_2 \in \left(\bigcap_{i=1}^{k} V_{A_i \cap A_{k+1}} \right) + \ker \beta^*_{n-1,n-2}. \tag{8}$$

From Equation (8), we get $h_1 - h_2 = h + l$, where $h \in \cap_{i=1}^k V_{A_i \cap A_{k+1}}$ and $l \in \ker \beta^*_{n-1,n-2}$. Set $h_1' = h_1 + l$. We have

$$h_1' - h_2 = h_1 + l - h_2 = h \in \cap_{i=1}^k V_{A_i \cap A_{k+1}}$$

and our claim is proved.

Let t be the element of $\mathbb{F}_2^{[\Omega]^{n-2}}$ defined by

$$t(w) = \begin{cases} h_1(w) & \text{if } w \subseteq A_i \text{ for some } i = 1, \ldots, k, \\ h_2(w) & \text{if } w \subseteq A_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the function t is well-defined. Indeed, recall that $n > 2$ and note that if w is an $(n-2)$-subset of Ω with $w \subseteq A_i \cap A_{k+1}$ (for some $i = 1, \ldots, k$), then $h_1(w) = h_2(w)$ as $h_1 - h_2 \in V_{A_i \cap A_{k+1}}$.

We claim that $g + \beta^*_{n-1,n-2} t \in \cap_{i=1}^{k+1} V_{A_i}$. We have to show that $g + \beta^*_{n-1,n-2} t$ vanishes in $[A_i]^{n-1}$, for $i = 1, \ldots, k+1$. Let w be an $(n-1)$-subset of Ω with $w \subseteq A_i$, for some $i = 1, \ldots, k+1$. If $i \leq k$, then we have

$$(g + \beta^*_{n-1,n-2} t)(w) = (g_1(w) + \beta^*_{n-1,n-2} h_1(w)) + \beta_{n-1,n-2} h_1(w) = 0,$$

where in the first equality we used Equation (7) and the fact that t and h_1 coincide in $[A_i]^{n-2}$, and in the second equality we used that $g_1 \in V_{A_i}$. Similarly, if $i = k+1$, then

$$(g + \beta^*_{n-1,n-2} t)(w) = (g_2(w) + \beta^*_{n-1,n-2} h_2(w)) + \beta_{n-1,n-2} h_2(w) = 0,$$

where in the first equality we used Equation (7) and the fact that t and h_2 coincide in $[A_{k+1}]^{n-2}$, and in the second equality we used that $g_2 \in V_{A_{k+1}}$.

Finally, as $\beta^*_{n-1,n-2} t \in \ker \beta^*_{n,n-1}$, we get that $g \in V_{2,k+1}$. □

Proposition 5.2 *The theory T_n has k-uniqueness for every $k \leq n$.*

Proof. Let k be an integer with $k \leq n$ and $a : P(k)^- \to \mathcal{C}_{T_n}$ be a k-amalgamation problem. We need to show that a has at most one solution up to isomorphism. Since every stable theory has 1- and 2-uniqueness, we may assume that $k \geq 3$. Set $\Gamma_1 = \mathrm{Aut}(a([k-1]) / \cup_{i=1}^{k-1} a([k] - i))$ and $\Gamma_2 = \mathrm{Aut}(a([k-1]) / \cup_{i=1}^{k-1} a([k-1] - i))$. By [9, Proposition 3.5], it is enough to prove that

$$\Gamma_1 = \Gamma_2, \tag{9}$$

i.e. $\overline{\Gamma_1}, \overline{\Gamma_2}$ give rise to the same action on $a([k-1])$ (see Definition 4.1).

By Remark 4.9, the algebraically closed sets of finite subsets of M_n are of the form $\mathrm{acl}(A)$, for some finite subset A of the sort Ω. By Corollary 4.6 the setwise stabilizer of $\mathrm{acl}(A)$ in $\mathrm{Aut}(M_n)$ is simply $(\mathrm{Sym}(\Omega \setminus A) \times \mathrm{Sym}(A)) \ltimes \mathrm{im}\, \beta^*_{n,n-1}$. Using Corollary 4.8, we get that the pointwise stabilizer of $\mathrm{acl}(A)$ in $\mathrm{Aut}(M_n)$ is $\mathrm{Sym}(\Omega \setminus A) \ltimes W_A$.

Let $a(i) = \mathrm{acl}(B_i)$, where B_i are finite subsets of M_n for $1 \leq i \leq k$. Set $A_i = \mathrm{supp}(B_i)$, for $1 \leq i \leq k$, and $A = \cup_{i=1}^{k-1} A_i$. Note that by definition of amalgamation problem and by Proposition 4.10, we have $a([k-1]) = \mathrm{acl}(A)$. Therefore, by the previous paragraph, as $k \geq 3$, we get that $\overline{\Gamma_1}$ is equal to

$$((\mathrm{Sym}(\Omega \setminus A) \times \mathrm{Sym}(A)) \ltimes \mathrm{im}\, \beta^*_{n,n-1}) \cap \bigcap_{i=1}^{k-1} (\mathrm{Sym}(\Omega \setminus ((A \cup A_k) \setminus A_i)) \ltimes W_{(A \cup A_k) \setminus A_i})$$

i.e.

$$\overline{\Gamma_1} = \mathrm{Sym}(\Omega \setminus (A \cup A_k)) \ltimes \bigcap_{i=1}^{k-1} W_{(A\cup A_k)\setminus A_i} \tag{10}$$

and $\overline{\Gamma_2}$ is equal to

$$((\mathrm{Sym}(\Omega \setminus A) \times \mathrm{Sym}(A)) \ltimes \operatorname{im}\beta^*_{n,n-1}) \cap \bigcap_{i=1}^{k-1} (\mathrm{Sym}(\Omega \setminus (A \setminus A_i)) \ltimes W_{A\setminus A_i})$$

i.e.

$$\overline{\Gamma_2} = \mathrm{Sym}(\Omega \setminus A) \ltimes \bigcap_{i=1}^{k-1} W_{A\setminus A_i}. \tag{11}$$

As $\mathrm{Sym}(\Omega\setminus(A\cup A_k))$ and $\mathrm{Sym}(\Omega\setminus A)$ act trivially on the elements of $\mathrm{acl}(A)$, by Equations (10) and (11), in order to prove that $\Gamma_1 = \Gamma_2$ it suffices to show that

$$W_1 = \bigcap_{i=1}^{k-1} W_{(A\cup A_k)\setminus A_i} \quad \text{and} \quad W_2 = \bigcap_{i=1}^{k-1} W_{A\setminus A_i}$$

induce the same action on $\mathrm{acl}(A)$. Also, W_1 and W_2 act trivially on the elements belonging to the sorts Ω and $[\Omega]^n$ of M_n. Thus, it suffices to study the action of W_1 and W_2 on the elements of $\mathrm{acl}(A)$ belonging to the sort $[\Omega]^n \times \mathbb{F}_2$, that is, on $[A]^n$. Clearly, $W_1 \subseteq W_2$. Therefore, it remains to show that for every element f of W_2 there exists an element $\overline{f}$ of W_1 such that f and $\overline{f}$ induce the same action on $[A]^n$.

Let f be in W_2. By Definition 3.6, we get that $f = \beta^*_{n,n-1}g$, for some $g \in \cap_{i=1}^{k-1}(V_{A\setminus A_i} + \ker\beta^*_{n,n-1})$. Lemma 5.1 (applied to $k-1, n$ and $(A \setminus A_1), \ldots, (A \setminus A_{k-1})$) yields

$$\bigcap_{i=1}^{k-1} \left(V_{A\setminus A_i} + \ker\beta^*_{n,n-1}\right) = \left(\bigcap_{i=1}^{k-1} V_{A\setminus A_i}\right) + \ker\beta^*_{n,n-1}.$$

Thence, up to replacing g by $g + l$ (for some $l \in \ker\beta^*_{n,n-1}$), we may assume that $g \in \cap_{i=1}^{k-1} V_{A\setminus A_i}$. Let $\overline{g}$ be the function in $\mathbb{F}_2^{[\Omega]^{n-1}}$ defined by

$$\overline{g}(w) = \begin{cases} g(w) & \text{if } w \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

Set $\overline{f} = \beta^*_{n,n-1}\overline{g}$. By construction, f and $\overline{f}$ coincide in $[A]^n$, that is, f and $\overline{f}$ induce the same action on $[A]^n$. Thus, it remains to prove that $\overline{f} \in W_1$, that is, $\overline{f}$ vanishes on every n-subset L of $(A \cap A_i) \setminus A_i$, for $i = 1, \ldots, k$. Let L be an n-subset of $(A \cup A_k) \setminus A_i$. We consider three cases $L \subseteq A$, $|L \cap A_k| \geq 2$ and $|L \cap A_k| = 1$.

If $L \subseteq A$, then $\overline{f}(L) = f(L) = 0$ (because f and $\overline{f}$ coincide on $[A]^n$).

If $|L \cap A_k| \geq 2$, then $(L \setminus \{x\}) \nsubseteq A$, for every x in L. By definition of $\overline{g}$, we have $\overline{g}(L \setminus \{x\}) = 0$ and $\overline{f}(L) = \sum_{x\in L} \overline{g}(L \setminus \{x\}) = 0$.

If $|L \cap A_k| = 1$ and $L \cap A_k = \{\overline{x}\}$, then (arguing as in the previous paragraph) $\overline{f}(L) = \sum_{x\in L} \overline{g}(L \setminus \{x\}) = g(L \setminus \{\overline{x}\})$. As $L \subseteq (A \cup A_k) \setminus A_i$, we have that $L \setminus \{\overline{x}\} \subseteq A \setminus A_i$. Since $g \in V_{A\setminus A_i}$, we get that $\overline{g}(L \setminus \{\overline{x}\}) = g(L \setminus \{\overline{x}\}) = 0$. □

J.Goodrick and A.Kolesnikov recently proved that if a complete stable theory T has k-uniqueness for every $2 \leq k \leq n$, then T has $n+1$-existence [7]. For completeness we report the proof of their result.

Theorem 5.3 *Let T be a complete stable theory. If T has k-uniqueness for every $2 \leq k \leq n$, then T has $n+1$-existence.*

P r o o f. Note that the existence and the uniqueness of nonforking extensions of types in a stable theory yields that any stable theory has both 2-existence and 2-uniqueness.

Since T is a complete stable theory, for every regular cardinal k, there exists a saturated model of cardinality k. In the sequel we shall consider the objects of $\mathcal{C}_T$ lying inside a very large saturated "monster model" $\mathfrak{C}$ of T.

Suppose a is an $(n+1)$-amalgamation problem. We have to prove that a has a solution a'. First, let B_0 and B_1 be sets of $\mathfrak{C}$ such that $\mathrm{tp}(B_0/a(\emptyset)) = \mathrm{tp}(a([n])/a(\emptyset))$, $\mathrm{tp}(B_1/a(\emptyset)) = \mathrm{tp}(a(\{n+1\})/a(\emptyset))$, and

$$B_0 \underset{a(\emptyset)}{\perp} B_1.$$

Let σ_0 and σ_1 be two automorphisms of $\mathfrak{C}$ fixing pointwise $a(\emptyset)$ and such that $B_0 = \sigma_0(a([n]))$, $B_1 = \sigma_1(a(\{n+1\}))$.

Define $a'([n+1])$ to be the algebraic closure of $B_0 \cup B_1$. To determine the solution a' of a, it remains to define the transition maps $a'_{s,[n+1]} : a'(s) \to a'([n+1])$, for all subsets s of $[n+1]$. The map $a'_{\emptyset,[n+1]}$ must be the identity on $a(\emptyset)$. For i in $[n]$, we let $a'_{\{i\},[n+1]} : a(\{i\}) \to a'([n+1])$ be the map $\sigma_0 \circ a_{\{i\},[n]}$, and we let $a'_{\{n+1\},[n+1]}$ be the map σ_1. Now, the following claim concludes the proof of the theorem.

CLAIM: For every proper non-empty subset s of $[n+1]$, there is a way to define the transition maps $a'_{s,[n+1]}$, which is consistent with a and the definition of $a'_{\{i\},[n+1]}$ given above, and such that

$$a'_{s,[n+1]}(a(s)) = \mathrm{acl}\left(\bigcup_{i\in s} a(\{i\})\right).$$

We argue by induction on the size k of the set s. If $k=1$, then there is nothing to prove. Suppose we have defined $a'_{s,[n+1]}$ as in the claim, for all $s \subseteq [n+1]$ such that $|s| < k$. Let s be a subset of $[n+1]$ such that $|s| = k$. The family of sets $\{a(t) \mid t \subsetneq s\}$ forms a k-amalgamation problem with the same transition maps as a. Call a^1 this amalgamation problem. By the induction hypothesis, the family of sets $\{a'_{t,[n+1]}(a(t)) \mid t \subsetneq s\}$ forms another k-amalgamation problem with the transition maps given by set inclusions. Call a^2 this amalgamation problem. Notice that a^1 and a^2 are isomorphic, and that both have independent solutions. Namely, a^1 can be completed to $a(s)$ using the transition maps in a, and a^2 has a natural solution $(a^2)'$ such that

$$(a^2)'(s) = \mathrm{acl}\left(\bigcup_{i\in s} a(\{i\})\right),$$

where the transition maps are again given by set inclusions. So, by the k-uniqueness property, there is an isomorphism of these solutions, which yields the desired transition map $a'_{s,[n+1]}$ from $a(s)$ to $\mathrm{acl}(\bigcup_{i\in s} a(\{i\}))$.

□

Now we are ready to prove that T_n has k-existence for every $k \leq n+1$.

Proposition 5.4 *The theory T_n has k-existence for every $k \leq n+1$.*

P r o o f. By definition, $T_n = \mathrm{Th}(M_n)$ is complete. Since T_n is a stable theory, the proof of this proposition follows at once from Proposition 5.2 and Theorem 5.3. □

Next, we show that T_n does not have $n+1$-uniqueness.

Proposition 5.5 *The theory T_n does not have $n+1$-uniqueness.*

Proof. Recall that by construction $n \geq 2$. Let $a : P(n+1)^- \to \mathcal{C}_{T_n}$ be the $(n+1)$-amalgamation problem defined on the objects by $a(s) = \mathrm{acl}(s)$ and where the morphisms are inclusions. In order to prove this proposition we show the following equations:

$$|\operatorname{Aut}(\mathrm{acl}([n])/\cup_{i=1}^{n}\mathrm{acl}([n+1]-i))| = 1, \tag{12}$$
$$|\operatorname{Aut}(\mathrm{acl}([n])/\cup_{i=1}^{n}\mathrm{acl}([n]-i))| = 2. \tag{13}$$

In fact, by [9, Proposition 3.5], Equations (12), (13) yield that a has more than one solution up to isomorphism, i.e. T_n does not have $n+1$-uniqueness.

We start by proving Equation (12). Since $[n], [n+1]-i$ have size n, Proposition 4.4 yields $\mathrm{acl}_{M_n}([n]) = [n]\cup\{[n]\}\cup\{([n],0),([n],1)\}$ and $\mathrm{acl}_{M_n}([n+1]-i) = ([n+1]-i)\cup\{[n+1]-i\}\cup\{([n+1]-i,0),([n+1]-i,1)\}$.

By the description given in the previous paragraph, every permutation in $\mathrm{Sym}(\Omega)$ fixing pointwise the elements in $\cup_{i=1}^{n}\mathrm{acl}([n+1]-i)$ also fixes pointwise every element in $\mathrm{acl}([n])$. Therefore, it suffices to consider the elements in $\operatorname{im}\beta^*_{n,n-1}$. Let f be in $\operatorname{im}\beta^*_{n,n-1}$ and suppose that f fixes every element in $\cup_{i=1}^{n}\mathrm{acl}([n+1]-i)$, i.e. $f([n+1]-i) = 0$, for $1 \leq i \leq n$. Let $g \in \mathbb{F}_2^{[\Omega]^{n-1}}$ such that $f = \beta^*_{n,n-1}g$. We have

$$0 = \sum_{i=1}^{n} f([n+1]-i) = \sum_{i=1}^{n}\sum_{j\neq i}^{n+1} g([n+1]\setminus\{i,j\}). \tag{14}$$

Now, for $j \neq n+1$, the summand $g([n+1]\setminus\{i,j\})$ appears twice in Equation (14) and therefore over $\mathbb{F}_2$ their sum is zero. Hence

$$0 = \sum_{i=1}^{n} f([n+1]-i) = \sum_{i=1}^{n} g([n]-i) = (\beta^*_{n,n-1}g)([n]) = f([n]).$$

This yields that f fixes $([n],0),([n],1)$. Hence Equation (12) follows.

We now prove Equation (13). Since $[n]-i$ has size $n-1$, Proposition 4.4 implies $\mathrm{acl}_{M_n}([n]-i) = [n]-i$. Therefore,

$$\cup_{i=1}^{n}\mathrm{acl}_{M_n}([n]-i) = \cup_{i=1}^{n}([n]-i) = [n].$$

Also, $\mathrm{acl}_{M_n}([n]) = [n]\cup\{[n]\}\cup\{([n],0),([n],1)\}$. Corollary 4.6 and Corollary 4.8 yield that every element of $\operatorname{Aut}(\mathrm{acl}([n])/\cup_{i=1}^{n}\mathrm{acl}([n]-i))$ fixes the elements belonging to the sorts Ω and $[\Omega]^n$ of $\mathrm{acl}_{M_n}([n])$. Hence, in order to prove Equation (13), it suffices to find an automorphism of $\mathrm{acl}_{M_n}([n])$ mapping $([n],0)$ into $([n],1)$. Let $g \in \mathbb{F}_2^{[\Omega]^{n-1}}$ with $g([n-1]) = 1$ and $g(w) = 0$ for $w \neq [n-1]$. Set $f = \beta^*_{n,n-1}g$ and note that $f([n]) = 1$. As $\operatorname{Aut}(M_n) = \operatorname{im}\beta^*_{n,n-1} \rtimes \mathrm{Sym}(\Omega)$, the map f is an automorphism of M_n. By construction f is an automorphism of $\mathrm{acl}_{M_n}([n])$ and $([n],0)^f = ([n], 0 + f([n])) = ([n],1)$. □

Finally, we show that T_n does not have $n+2$-existence.

Proposition 5.6 *The theory T_n does not have $n+2$-existence.*

Proof. We construct an $n+2$-amalgamation problem a over $\emptyset$ (that is, $a(\emptyset) = \emptyset$) for T_n with no solution.

Let g be the element of $\mathbb{F}_2^{[\Omega]^{n-1}}$ defined by

$$g(w) = \begin{cases} 1 & \text{if } w = [n-1], \\ 0 & \text{if } w \neq [n-1]. \end{cases}$$

Consider $f = \beta^*_{n,n-1}g$ and note that, as $\operatorname{Aut}(M_n) = \operatorname{im}\beta^*_{n,n-1} \rtimes \mathrm{Sym}(\Omega)$, the element f is an automorphism of M_n.

Let a be the functor $a : P(n+2)^- \to \mathcal{C}_{T_n}$ defined on the objects by $a(s) = \mathrm{acl}(s)$ and with morphisms defined by

$$a_{s,s'} = \begin{cases} f|_{a(s)} & \text{if } s = [n] \text{ and } s' = [n+1], \\ \text{inclusion} & \text{otherwise,} \end{cases} \tag{15}$$

where $f|_{a(s)}$ denotes the restriction of the automorphism f to $a(s)$. It is not obvious from Equation (15) that a is a functor. Therefore, in the following paragraph, we prove that a is well-defined, that is, $a_{s_2,s_3}\circ a_{s_1,s_2}=a_{s_1,s_3}$ for every s_1,s_2,s_3 in $P(n+2)^-$ with $s_1\subseteq s_2\subseteq s_3$.

If $s_2\neq[n+1]$ and $s_3\neq[n+1]$, then (by Equation (15)) the morphisms a_{s_1,s_2},a_{s_2,s_3} and a_{s_1,s_3} are inclusions and so clearly $a_{s_2,s_3}\circ a_{s_1,s_2}=a_{s_1,s_3}$. If $s_2=[n+1]$, then s_2 is a maximal element of the partially ordered set $P(n+2)^-$. Thence $s_3=s_2$ and, by Equation (15), a_{s_2,s_3} is the identity map. Thus $a_{s_2,s_3}\circ a_{s_1,s_2}=a_{s_1,s_3}$. In particular, from now on we may assume that $s_3=[n+1]$ and $s_2\neq[n+1]$. As $s_1\subseteq s_2$, if $s_2\neq[n]$, then $s_1\neq[n]$ and so, by Equation (15), the morphisms a_{s_1,s_2},a_{s_2,s_3} and a_{s_1,s_3} are inclusions and $a_{s_2,s_3}\circ a_{s_1,s_2}=a_{s_1,s_3}$. If $s_2=s_1=[n]$, then a_{s_1,s_2} is the identity map and $a_{s_2,s_3}\circ a_{s_1,s_2}=a_{s_1,s_3}$. The only case that remains to consider is $s_3=[n+1]$, $s_2=[n]$ and $s_1\neq[n]$. Thence a_{s_1,s_2} and a_{s_1,s_3} are inclusion maps and $a_{s_2,s_3}=f|_{a(s_2)}$. Since $s_1\subseteq s_2=[n]$ and $s_1\neq[n]$, we have $|s_1|<n-1$. Therefore, $a(s_1)=\mathrm{acl}(s_1)$ and by Proposition 4.4 $\mathrm{acl}_{M_n}(s_1)=s_1$ consists only of elements belonging to the sort Ω of M_n. As f acts trivially on the elements belonging to the sort Ω, by Proposition 4.5 we obtain $a_{s_2,s_3}\circ a_{s_1,s_2}=(f|_{a(s_2)})|_{a(s_1)}=f|_{a(s_1)}=a_{s_1,s_3}$. Finally, this proves that $a:P(n+2)^-\to\mathcal{C}_{T_n}$ is a functor.

By Proposition 4.3, $a(\emptyset)=\mathrm{acl}(\emptyset)=\emptyset$. Therefore, the functor a is an $n+2$-amalgamation problem over $\emptyset$ for M_n.

We claim that a cannot be extended to $P(n+2)$. We argue by contradiction. Let $\overline{a}:P(n+2)\to\mathcal{C}_{T_n}$ be a solution of a. In particular, $\overline{a}$ is an extension of a to the whole of $P(n+2)$. Denote by x_i the morphisms $\overline{a}_{[n+2]-i,[n+2]}$, for $1\leq i\leq n+2$. So, by definition of morphism, x_i is the restriction to $\mathrm{acl}([n+2]-i)$ of an automorphism $f_i\sigma_i$ of M_n, where $f_i\in\mathrm{im}\,\beta^*_{n,n-1}$ and $\sigma_i\in\mathrm{Sym}(\Omega)$.

Since $\overline{a}$ is a functor and $\overline{a}$ extends a, we get

$$\begin{aligned}x_i\circ a_{[n+2]\setminus\{i,j\},[n+2]-i}&=\overline{a}_{[n+2]-i,[n+2]}\circ\overline{a}_{[n+2]\setminus\{i,j\},[n+2]-i}\\&=\overline{a}_{[n+2]-j,[n+2]}\circ\overline{a}_{[n+2]\setminus\{i,j\},[n+2]-j}\\&=x_j\circ a_{[n+2]\setminus\{i,j\},[n+2]-j}.\end{aligned}\tag{16}$$

Let i and j be in $[n+2]$ with $i\neq j$. Fix an enumeration of $\mathrm{acl}_{M_n}([n+2]\setminus\{i,j\})$ and denote it as $\overline{b_{ij}}=(b_{ij_1},\dots)$. Then, as it is shown in Proposition 4.5 $\overline{b_{ij}}\in\mathrm{acl}([n+2]\setminus\{i,j\})$ and, of course, also in $\mathrm{acl}([n+2]\setminus\{i\})$. By Proposition 4.4 the ordered pair $([n+2]\setminus\{i,j\},0)$ belongs to the sort $[\Omega]^n\times\mathbb{F}_2$ of M_n and lies in $\mathrm{acl}_{M_n}([n+2]\setminus\{i,j\})$. Set $b_{ij_1}=([n+2]\setminus\{i,j\},0)$. We have

$$\begin{aligned}x_i(\overline{b_{ij}})&=x_i(([n+2]\setminus\{i,j\},0),\dots)\\&=((([n+2]\setminus\{i,j\})^{\sigma_i},0+f_i([n+2]\setminus\{i,j\})),\dots)\\&=((([n+2]\setminus\{i,j\})^{\sigma_i},m_{ij}),\dots),\end{aligned}\tag{17}$$

where

$$m_{ij}=f_i([n+2]\setminus\{i,j\}).\tag{18}$$

Consider the matrix $M=(m_{ij})_{ij}$, with $m_{ii}=0$.

Let i and j be in $[n+2]$ with $i\neq j$ and $\{i,j\}\neq\{n+1,n+2\}$. By Equation (15) and by hypothesis on $\{i,j\}$, the morphism $a_{[n+2]\setminus\{i,j\},[n+2]-i}$ is an inclusion map and so it fixes $([n+2]\setminus\{i,j\},0)$. Therefore,

$$\begin{aligned}x_i\circ a_{[n+2]\setminus\{i,j\},[n+2]-i}(\overline{b_{ij}})&=x_i\circ a_{[n+2]\setminus\{i,j\},[n+2]-i}(([n+2]\setminus\{i,j\},0),\dots)\\&=x_i(([n+2]\setminus\{i,j\},0),\dots)\\&=((([n+2]\setminus\{i,j\})^{\sigma_i},m_{ij}),\dots),\end{aligned}$$

where in the last equality we used Equations (17) and (18). Similarly, replacing i with j, we obtain

$$\begin{aligned}x_i\circ a_{[n+2]\setminus\{i,j\},[n+2]-i}(\overline{b_{ij}})&=x_j\circ a_{[n+2]\setminus\{i,j\},[n+2]-j}(([n+2]\setminus\{i,j\},0),\dots)\\&=x_j(([n+2]\setminus\{i,j\},0),\dots)\\&=((([n+2]\setminus\{i,j\})^{\sigma_j},m_{ji}),\dots).\end{aligned}$$

Now, by Equation (16), we have

$$\begin{aligned} x_i \circ a_{[n+2]\setminus\{i,j\},[n+2]-i}(\overline{b_{ij}}) &= x_i \circ a_{[n+2]\setminus\{i,j\},[n+2]-i}(([n+2]\setminus\{i,j\},0),\dots) \\ &= x_j \circ a_{[n+2]\setminus\{i,j\},[n+2]-j}(([n+2]\setminus\{i,j\},0),\dots). \end{aligned}$$

In particular,

$$m_{ij} = m_{ji}, \qquad \text{for every } i,j \text{ with } \{i,j\} \neq \{n+1,n+2\}. \tag{19}$$

By Equation (15) the morphism $a_{[n+2]\setminus\{n+1,n+2\},[n+2]-(n+1)}$ is an inclusion map and so it fixes $([n+2]\setminus\{n+1,n+2\},0)$. Therefore,

$$\begin{aligned} & x_{n+1} \circ a_{[n+2]\setminus\{n+1,n+2\},[n+2]-(n+1)}(\bar{b}_{n+1,n+2}) \\ = \ & x_{n+1} \circ a_{[n+2]\setminus\{n+1,n+2\},[n+2]-(n+1)}(([n+2]\setminus\{n+1,n+2\},0),\dots) \\ = \ & x_{n+1}(([n+2]\setminus\{n+1,n+2\},0),\dots) \\ = \ & ((([n+2]\setminus\{n+1,n+2\})^{\sigma_{n+1}}, m_{(n+1)(n+2)}),\dots). \end{aligned}$$

By Equation (15) the morphism $f|_{a([n])} = a_{[n],[n+1]} = a_{[n+2]\setminus\{n+1,n+2\},[n+2]-(n+2)}$ maps $([n+2]\setminus\{n+1,n+2\},0)$ to $([n+2]\setminus\{n+1,n+2\},1)$. Therefore,

$$\begin{aligned} & x_{n+2} \circ a_{[n+2]\setminus\{n+1,n+2\},[n+2]-(n+2)}(\bar{b}_{n+1,n+2}) \\ = \ & x_{n+2} \circ a_{[n+2]\setminus\{n+1,n+2\},[n+2]-(n+2)}(([n+2]\setminus\{n+1,n+2\},0),\dots) \\ = \ & x_{n+2} \circ f|_{a([n])}(([n+2]\setminus\{n+1,n+2\},0),\dots) \\ = \ & x_{n+2}(([n+2]\setminus\{n+1,n+2\},1),\dots) \\ = \ & ((([n+2]\setminus\{n+1,n+2\})^{\sigma_{n+2}}, m_{(n+2)(n+1)}+1),\dots). \end{aligned}$$

By Equation (16) (applied to $i = n+1$ and $j = n+2$), we have

$$\begin{aligned} & (([n+2]\setminus\{n+1,n+2\})^{\sigma_{n+1}}, m_{(n+1)(n+2)}) \\ = \ & (([n+2]\setminus\{n+1,n+2\})^{\sigma_{n+2}}, m_{(n+2)(n+1)}+1) \end{aligned}$$

and

$$m_{(n+1)(n+2)} = m_{(n+2)(n+1)} + 1. \tag{20}$$

Now, we are ready to get a contradiction. We claim that each row of M adds up to zero. We have

$$\begin{aligned} \sum_{j=1}^{n+2} m_{ij} &= \sum_{j\in([n+2]-i)} m_{ij} = \sum_{j\in([n+2]-i)} f_i([n+2]\setminus\{i,j\}) \\ &= (\beta^*_{n+1,n} f_i)([n+2]-i) = 0, \end{aligned}$$

where in the first equality we used that $m_{ii} = 0$, in the second equality we used Equation (18) and in the last equality we used that $f_i \in \operatorname{im}\beta^*_{n,n-1} = \ker\beta^*_{n+1,n}$. In particular, the sum of all the entries of M is zero. Hence

$$0 = \sum_{ij} m_{ij} = \sum_{i<j}(m_{ij} + m_{ji}).$$

By Equation (19), $m_{ij} = m_{ji}$ if $\{i,j\} \neq \{n+1,n+2\}$. So, in the previous sum there is only one non-zero summand. Namely, $m_{(n+1)(n+2)} + m_{(n+2)(n+1)} = 0$. Now, Equation (20) yields

$$m_{(n+1)(n+2)} + m_{(n+2)(n+1)} = m_{(n+1)(n+2)} + m_{(n+1)(n+2)} + 1 = 1,$$

a contradiction. This contradiction finally proves that the extension $\overline{a}$ does not exist. □

Now, Theorem 1.2 follows at once from Proposition 5.2, 5.4, 5.5, 5.6. Finally, we point out that Proposition 5.5 also follows from Theorem 5.3 and Proposition 5.6.

6 Extension of Example 1.1

In this section we remark that for every $n \geq 2$ the theories T_n are stable and that the family of examples $\{M_n\}_{n\geq 2}$ generalizes the example due to E.Hrushovski given in [3], see Example 1.1 in Section 1.

Definition 6.1 Let Ω be a countable set, and $C = [\Omega]^n \times \mathbb{Z}/2\mathbb{Z}$. Also let $E \subseteq \Omega \times [\Omega]^2$ be the membership relation, and let P be the subset of C^{n+1} such that $((w_1, \delta_1), \ldots, (w_{n+1}, \delta_{n+1})) \in P$ if and only if there are distinct $c_1, \ldots, c_{n+1} \in \Omega$ such that $w_i = \{c_1, \ldots, c_{n+1}\} \setminus c_i$ and $\delta_1 + \cdots + \delta_{n+1} = 0$. Now let $\overline{M}_n$ be the model with the 3-sorted universe $\Omega, [\Omega]^n, C$ and equipped with relations E, P and projection on the first coordinate $\pi : C \to [\Omega]^n$. Since $\overline{M}_n$ is a reduct of $(\Omega, \mathbb{Z}/2\mathbb{Z})^{\mathrm{eq}}$, we get that $\mathrm{Th}(\overline{M}_n)$ is stable.

Proposition 6.2 *Let $\overline{M}_n$ be the structures described in Definition 6.1. Then* $\mathrm{Aut}(\overline{M}_n) = \mathrm{im}\,\beta^*_{n,n-1} \rtimes \mathrm{Sym}(\Omega)$. *In particular, $\overline{M}_n$ and M_n are interdefinable.*

Proof. First we show that $\mathrm{Sym}(\Omega)$ is a subgroup of $\mathrm{Aut}(\overline{M}_n)$. Indeed, the group $\mathrm{Sym}(\Omega)$ acts with its natural action on the sorts Ω and $[\Omega]^n$ of $\overline{M}_n$. Also, if $g \in \mathrm{Sym}(\Omega)$ and $(\{a_1, \ldots, a_n\}, \delta) \in C$, then we set $(\{a_1, \ldots, a_n\}, \delta)^g = (\{a_1^g, \ldots, a_n^g\}, \delta)$. This defines an action of $\mathrm{Sym}(\Omega)$ on $\overline{M}_n$. It is straightforward to see that the relations E, P and the partition given by the fibers of π are preserved by $\mathrm{Sym}(\Omega)$. Hence, $\mathrm{Sym}(\Omega) \leq \mathrm{Aut}(\overline{M}_n)$.

Let $\mu : \mathrm{Aut}(\overline{M}_n) \to \mathrm{Sym}(\Omega)$ be the map given by restriction on the sort Ω of $\overline{M}_n$. Since μ is a surjective homomorphism, we have that $\mathrm{Aut}(\overline{M}_n)$ is a split extension of $\ker\mu$ by $\mathrm{Sym}(\Omega)$. Every element of $\ker\mu$ preserves the fibres of π and fixes all the elements of $[\Omega]^n$. So $\ker\mu$ is a closed $\mathrm{Sym}(\Omega)$-submodule of $\mathbb{F}_2^{[\Omega]^n}$.

Let $((w_1, \delta_1), \ldots, (w_{n+1}, \delta_{n+1}))$ be in P and f be in $\ker\mu$. Since $\ker\mu$ preserves P, we have

$$f(w_1) + \delta_1 + \cdots + f(w_{n+1}) + \delta_{n+1} = 0.$$

From the definition of P and $\beta^*_{n+1,n}$, we get

$$\ker\mu = \{f \in \mathbb{F}_2^{[\Omega]^n} \mid \sum_{x\in[w]^n} f(x) = 0 \text{ for every } w \in [\Omega]^{n+1}\} = \ker\beta^*_{n+1,n}.$$

By Proposition 2.2, we have that $\ker\beta^*_{n+1,n} = \mathrm{im}\,\beta^*_{n,n-1}$. Therefore $\mathrm{Aut}(\overline{M}_n) = \mathrm{Aut}(M_n)$ and $\overline{M}_n, M_n$ are interdefinable. □

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