Failure of n-uniqueness in amalgamation problems: a group theoretical approach

A joint work with P. Spiga

Elisabetta Pastori
University of Turin, Italy
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Introduction

Amalgamation properties concern the possibility of embedding increasingly complex systems of algebraically closed sets (boundedly closed sets, models,...) into a model of a theory.

We can also ask when the result is unique. This yields to properties called *n-existence* and *n-uniqueness* which we will define soon.

Amalgamation properties have nice uses! They have been used:

- To study the number of isomorphism classes of models in various cardinalities (Shelah).
- To generalize the "Group Configuration Theorem" to the context of simple theories (De Piro, Kim, Young).

More recently Hrushovski discovered surprising connections between amalgamation properties, definable groupoids and generalized imaginary sorts.

The category $\mathcal{C}_T$

Amalgamation properties can be formulated in categorical language. We focus on higher amalgamation over algebraically closed sets.

We work in a fixed stable or simple theory T. Let C be a saturated monster model of T and $\mathcal{C}_T$ be the category where:

- the objects Ob_T are all the algebraically closed subsets of C;
- the morphisms Mor_T are embeddings.

Functors from $\mathcal{P}(n)$

We think of the full power set $\mathcal{P}(n)$ on n elements, for $n \in \mathbb{N}$, as a category with inclusion maps as morphisms.

A functor $A : \mathcal{P}(n) \to \mathcal{C}_T$ is given by

- For every $u \subseteq n$, an algebraically closed set $A(u)$;
- For every pair (u, v) such that $u \subseteq v \subseteq n$ we have an embedding $A_{u,v} : A(u) \to A(v)$.

Functoriality says: whenever $u \subseteq v \subseteq w \subseteq n$,
$A_{v,w} \circ A_{u,v} = A_{u,w}$

n-Amalgamation Problems

Notation: $\mathcal{P}^-(n)$ =set of proper subsets of n

Definition

An *n-amalgamation problem* over acl($\emptyset$) is a functor

$$A : \mathcal{P}^-(n) \to \mathcal{C}_T$$

where

- $A(\emptyset)$ =acl($\emptyset$);
- for any $s \in \mathcal{P}^-(n)$, $A(s) = \text{acl}(A_{i,s}(A(i)) : i \in s)$;
- for any $s \in \mathcal{P}^-(n)$ the set $\{A_{i,s}(A(i)) : i \in s\}$ is independent over acl($\emptyset$).

Solutions of n-Amalgamation Problems

Definition

A *solution* of an n-amalgamation problem A over $\mathrm{acl}(\emptyset)$ is an extension of A to a functor

$$\bar{A} : \mathcal{P}(n) \to \mathcal{C}_T$$

on the full power set $\mathcal{P}(n)$ satisfying the same conditions (so including the case $s = \{1, \ldots, n\}$).

So, in order to determine a solution of A we need to find $\bar{A}(n)$ and n embeddings

$$\bar{A}_{n\setminus\{i\},n} : A(n \setminus \{i\}) \to \bar{A}(n)$$

compatible with A, that is,

$$\bar{A}_{n\setminus\{i\},n} \circ A_{s,n\setminus\{i\}} = \bar{A}_{n\setminus\{j\},n} \circ A_{s,n\setminus\{j\}}, \ \forall i,j \in n, \ \forall s \subseteq n \setminus \{i,j\}.$$

n-Existence and n-Uniqueness

Definition

T has *n-existence* over $\mathrm{acl}(\emptyset)$ if every n-amalgamation problem over $\mathrm{acl}(\emptyset)$ has at least one solution.

Definition

T has *n-uniqueness* over $\mathrm{acl}(\emptyset)$ if any n-amalgamation problem over $\mathrm{acl}(\emptyset)$ has at most one solution.

2- and 3-existence

2-existence says: any 2-amalgamation problem $A : \{\emptyset, \{1\}, \{2\}\} \to \mathcal{C}_T$ has an extension $\bar{A}$ to $\{1, 2\}$ such that $\bar{A}_{\{1\},\{1,2\}}(A(\{1\}))$ and $\bar{A}_{\{2\},\{1,2\}}(A(\{2\}))$ are independent over $\bar{A}_{\{\emptyset\},\{1,2\}}(A(\{\emptyset\}))$.

3-existence says: suppose we are given a 3-amalgamation problem $A : \mathcal{P}^-(\{1, 2, 3\}) \to \mathcal{C}_T$. Think of this as giving the three sides of a triangle as $A(\{1, 2\})$, $A(\{1, 3\})$, $A(\{2, 3\})$.

$\Rightarrow$

A has an extension $\bar{A}$ to $\{1, 2, 3\}$ which independently amalgamates the three sides of the triangle.

n-Existence and n-Uniqueness: Examples

- 2-existence is equivalent to the existence of nonforking extensions

$\Rightarrow$ 2-existence is true in any simple theory.

- 2-uniqueness is true in any stable theory by stationary of strong types, but can fail for unstable simple theories:

The random graph

The theory of a random graph is simple so it has 2-existence. But if $A(i)$ is the vertex a_i (for $i = 1, 2$), then there are two solutions to the 2-amalgamation problem A: one with an edge between the points and one with no edge.

n-Existence and n-Uniqueness: Examples

- In simple theories k-uniqueness + k-existence for some $k \geq 2$ imply $k+1$-existence (Hrushovski).
- In particular in stable theories k-uniqueness for $2 \leq k \leq n$ implies $n+1$-existence

$\Rightarrow$ 3-existence is true in any stable theory.

- 3-existence holds in any simple theory with elimination of hyperimmaginaries (Kim and Pillay).

$(*)$ 3-uniqueness can fail for stable theories.

Fact

Among the example of type $(*)$, there is one due to Hrushovski, where the theory fails 3-uniqueness and 4-existence.

Failure of 4-existence: the tetrahedron-free hypergraph

Let T be the theory of the random tetrahedron-free hypergraph.

The language of T consists of a ternary relation R, which is "symmetric". Moreover, there is no set of four distinct points such that R holds on every three-element subset.

Fact

T is simple unstable and does not have 4-existence: if $A : \mathcal{P}^{-}(4) \to \mathcal{C}_T$ is a 4-amalgamation problem such that $A(i, j, k)$ is a set of three elements for which the R relation holds, then A has no solution.

P_n-structures

Let Ω be a countable set, $[\Omega]^n$ the set of n-subsets of Ω and $C = [\Omega]^n \times \mathbb{Z}/2\mathbb{Z}$. Also let $E \subseteq \Omega \times [\Omega]^n$ be the membership relation, and let P_n be the subset of C^{n+1} such that $((w_1, \delta_1), \ldots, (w_{n+1}, \delta_{n+1})) \in P_n$ if and only if there are distinct $c_1, \ldots, c_{n+1} \in \Omega$ such that $w_i = \{c_1, \ldots, c_{n+1}\} \setminus c_i$ and $\delta_1 + \cdots + \delta_{n+1} =_2 0$. Now let M_n be the structure with the 3-sorted universe $\Omega, [\Omega]^n, C$ and equipped with relations E, P_n and projection on the first coordinate $\pi : C \to [\Omega]^n$. Since M_n is a reduct of $(\Omega, \mathbb{Z}/2\mathbb{Z})^{\mathrm{eq}}$, we have that $\mathrm{Th}(M_n)$ is stable.

For $n = 2$ the P_2-structure described above is the example of Hrushovski of a stable structure witnessing failure of 3-uniqueness and 4-existence.

Aim of the talk

This talk: outline a proof of

Theorem

For any $n \geq 2$, $\mathrm{Th}(M_n)$ does not have $(n+1)$-uniqueness.

The proof uses mainly group-theoretical tools.

In our paper there is much more:

Theorem

For any $n \geq 2$, $\mathrm{Th}(M_n)$ has $(k+1)$-existence and k-uniqueness for any $k \leq n$, but $\mathrm{Th}(M_n)$ has neither $(n+2)$-existence nor $(n+1)$-uniqueness.

A criterion for n-uniqueness

Notation: Let M be a structure, $A, B \subset M$. Denote by $\mathrm{Aut}(A/B)$ the group of permutations of A which extend to automorphisms of M fixing pointwise B.

Proposition (Hrushovski)

A stable theory T has n-uniqueness if and only if for any independence-preserving functor $a : \mathcal{P}(n) \to \mathcal{C}_T$

$$\mathrm{Aut}(a(1,\ldots,n-1)/\bigcup_{i=1}^{n-1} a(1,\ldots,\hat{i},\ldots,n-1,n)) = \mathrm{Aut}(a(1,\ldots,n-1)/\bigcup_{i=1}^{n-1} a(1,\ldots,\hat{i},\ldots,n-1,)).$$

Hence, in order to use this criterion we need to understand $\mathrm{Aut}(M_n)$ and the algebraically closed sets in M_n^{eq}.

Topology

Let Ω be an infinite set. We consider $\mathrm{Sym}(\Omega)$ as a topological group giving it the tolpology where open sets are arbitrary unions of cosets of pointwise stabilizers of finite sets of Ω.

Important fact

Closed subgroups of $\mathrm{Sym}(\Omega)$ in this topology are precisely automorphism groups of first-order structures on Ω.

The Sym(Ω)-module $\mathbb{F}_2^{[\Omega]^n}$

Let $\mathbb{F}_2$ be the field with 2 elements and $\mathbb{F}_2^{[\Omega]^n}$ be the group of functions from $[\Omega]^n$ to $\mathbb{F}_2$. The natural action of $\mathrm{Sym}(\Omega)$ on $[\Omega]^n$ induces an action of $\mathrm{Sym}(\Omega)$ on $\mathbb{F}_2^{[\Omega]^n}$ given by

$$f^\sigma(w) = f(w^\sigma)$$

which makes $\mathbb{F}_2^{[\Omega]^n}$ into a $\mathrm{Sym}(\Omega)$-module.
There is a natural faithful action of $\mathbb{F}_2^{[\Omega]^n}$ on $[\Omega]^n \times \mathbb{F}_2$ given by

$$(w, \delta)^f = (w, f(w) + \delta).$$

Hence, $\mathbb{F}_2^{[\Omega]^n}$ with the relative topology becomes a topological $\mathrm{Sym}(\Omega)$-module and a profinite subgroup of $\mathrm{Sym}([\Omega]^n \times \mathbb{F}_2)$.

The α-maps

Definition

If $0 \leq j \leq n$, then the map $\alpha_{j,n} : \mathbb{F}_2^{[\Omega]^j} \to \mathbb{F}_2^{[\Omega]^n}$, given by

$$\alpha_{j,n}(f)(\omega) = \sum_{\omega' \in [\omega]^j} f(\omega') \qquad (\text{for } \omega \in [\Omega]^n)$$

is a continuous Sym(Ω)-homomorphism.

Proposition (Gray 1997)

Any closed Sym(Ω)-submodule of $\mathbb{F}_2^{[\Omega]^n}$ is a sum of images of α-maps.

Automorphism groups of the P_n-structures

Proposition (P.-Spiga)

Let M_n be the P_n-structure described above. Then

$$\mathrm{Aut}(M_n) = \operatorname{im} \alpha_{n-1,n} \rtimes \mathrm{Sym}(\Omega).$$

We sketch a proof.

- There is a natural action of $\mathrm{Sym}(\Omega)$ on M_n.
- The relations E, P_n and the partition given by the fibres of π are preserved by $\mathrm{Sym}(\Omega)$. Hence, $\mathrm{Sym}(\Omega) \leq \mathrm{Aut}(M_n)$.
- Let $\mu : \mathrm{Aut}(M_n) \to \mathrm{Sym}(\Omega)$ be the map given by restriction on the sort Ω of M_n.
- μ is surjective $\Rightarrow$ $\mathrm{Aut}(M_n)$ is a split extension of $\ker \mu$ by $\mathrm{Sym}(\Omega)$.

.....

- Every element of $\ker\mu$ preserves the fibres of π and fixes all the elements of $[\Omega]^n \Rightarrow \ker\mu$ is a closed Sym(Ω)-submodule of $\mathbb{F}_2^{[\Omega]^n}$.
- Let $((w_1,\delta_1),\ldots,(w_{n+1},\delta_{n+1}))$ be in P_n and f be in $\ker\mu$. Since $\ker\mu$ preserves P_n, we have

$$f(w_1)+\delta_1+\cdots+f(w_{n+1})+\delta_{n+1} =_2 0.$$

But $\delta_1+\cdots+\delta_{n+1} =_2 0 \Rightarrow$

$$\ker\mu = \{f \in \mathbb{F}_2^{[\Omega]^n} \mid \sum_{x\in[w]^n} f(x) =_2 0 \text{ for every } w \in [\Omega]^{n+1}\}$$

i.e. $\ker\mu = \ker\alpha_{n,n+1}$.

- $\ker\alpha_{n,n+1} = \operatorname{im}\alpha_{n-1,n}$.

Algebraic closure in M^{eq}

Algebraic closure in M^{eq}

Let M be a countable $\aleph_0$-categorical structure and $A \subset_{\text{fin}} M$. The elements of $\text{acl}(A)$ corresponds (up to interdefinability) to closed subgroups of finite index of $\text{Aut}(M/A)$.

Proof of $\Rightarrow$

Let $c \in \text{acl}(A)$. Then $\text{Aut}(M/A, c)$ is a closed subgroup of finite index in $\text{Aut}(M/A)$.

Algebraic closure in M^{eq}

Proof of $\Leftarrow$

Let G be a closed subgroup of f. i. in $\mathrm{Aut}(M/A)$. By general topological arguments, G is open in $\mathrm{Aut}(M)$ $\Rightarrow$ there exists a tuple $\bar{b}$ such that $\mathrm{Aut}(M/\bar{b}) \leq G \leq \mathrm{Aut}(M/A)$. Consider $\{(\bar{b}, \bar{b}^{\gamma}) : \gamma \in G\}$. By $\aleph_0$-categoricity these pairs lie in a finite number of orbits under $\mathrm{Aut}(M)$: choose $(\bar{b}, \bar{b}^{\gamma_1}), \ldots, (\bar{b}, \bar{b}^{\gamma_m})$ as rapresentatives. Let

$$\mathcal{R} := \{(\bar{b}^{\alpha}, \bar{b}^{\gamma_i \alpha}) : i \leq m, \alpha \in \mathrm{Aut}(M)\}.$$

Then, $\mathcal{R}$ is a $\emptyset$-definable equivalence relation in M. Let $c = [\bar{b}]_{\mathcal{R}}$. Then, $\mathrm{Aut}(M/c) = \bigcup_{i \leq m} \gamma_i \,\mathrm{Aut}(M/\bar{b}) = G$ and $c \in \mathrm{acl}(A)$.

Proposition (P.-Spiga)

Let $A \subset_{\text{fin}} M_n$. Then $\text{Aut}(M_n/A)$ has a unique minimal closed subgroup of finite index.

Sketch of proof

For simplicity take $A \subset \Omega$. The proof is made of several lemmas.

1) Define

$$V_A = \{f \in \mathbb{F}_2^{[\Omega]^{n-1}} \mid f(w) = 0 \text{ for every } w \in [A]^{n-1}\}.$$

The module V_A has finite index in $\mathbb{F}_2^{[\Omega]^{n-1}}$. Also, if V is a closed $\text{Sym}(\Omega \setminus A)$-submodule of $\mathbb{F}_2^{[\Omega]^{n-1}}$ of finite index, then $V_A \subseteq V$.

Continuing

2) $\alpha_{n-1,n}(V_A) = \{g \in \operatorname{im} \alpha_{n,n-1} \mid g(w) = 0 \text{ for every } w \in [A]^n\}$
3) $\alpha_{n-1,n}(V_A)$ is the unique minimal closed $\mathrm{Sym}(\Omega \setminus A)$-submodule of $\operatorname{im} \alpha_{n-1,n}$ of finite index.
4) Now consider $\bar{A} := A \cup [A]^n \cup ([A]^n \times \mathbb{F}_2)$. Then ,

$$\mathrm{Aut}(M_n/\bar{A}) = \alpha_{n-1,n}(V_A) \rtimes \mathrm{Sym}(\Omega \setminus A)$$

is the unique minimal closed subgroup of finite index of $\mathrm{Aut}(M_n/A)$.

Algebraic closure in M_n

Definition

Let $A = A_1 \cup A_2 \cup A_3 \subset_{\text{fin}} M_n$, where $A_1 \subset \Omega$, $A_2 \subset [\Omega]^n$ and $A_3 \subset [\Omega]^n \times \mathbb{F}_2$. Consider $\tilde{A}_2 \subseteq \Omega$ the union of the elements in A_2 and $\tilde{A}_3 \subseteq \Omega$ the union of the elements in $\pi(A_3)$. We define

$$\mathrm{supp}(A) := A_1 \cup \tilde{A}_2 \cup \tilde{A}_3 \subset \Omega.$$

EX: if $A = \{1, 2, \{1, 4\}, (\{3, 7\}, 1)\}$, $\mathrm{supp}(A) = \{1, 2, 3, 4, 7\}$.

Proposition (P.-Spiga)

Let $A \subset_{\text{fin}} M_n$. Then

$$\mathrm{acl}(A) \cap M_n = \mathrm{supp}(A) \cup [\mathrm{supp}(A)]^n \cup ([\mathrm{supp}(A)]^n \times \mathbb{F}_2)$$

Algebraic closure in M_n^{eq}

Proposition (P.-Spiga)

Let $A \subset_{\text{fin}} M_n$. Then,

$$\operatorname{dcl}(\operatorname{acl}(A) \cap M_n) = \operatorname{acl}(A)$$

Corollary (P.-Spiga)

Let $A, B \subset_{\text{fin}} M_n$. Then,

$$\operatorname{Aut}(\operatorname{acl}(A) \cap M_n / \operatorname{acl}(B) \cap M_n) = \operatorname{Aut}(\operatorname{acl}(A) / \operatorname{acl}(B)).$$

Failure of n-uniqueness

Proposition (P.-Spiga)

The theory $T = \mathrm{Th}(M_n)$ does not have $n+1$-uniqueness for every $n \geq 2$.

We sketch a proof for $n = 2$. (The proof in the general case is similar, just more complicated notation).
Think of Ω as $\mathbb{N}$. Let $a : \mathcal{P}(3) \to \mathcal{C}_T$ be the 3-independence preserving functor defined on the objects by
$a(s) = \mathrm{acl}(\{1, \ldots, s\})$. We show the following equations hold:

$$|\,\mathrm{Aut}(\mathrm{acl}(\{1,2\})/\,\mathrm{acl}(\{1,3\}), \mathrm{acl}(\{2,3\}))| = 1$$

and

$$|\,\mathrm{Aut}(\mathrm{acl}(\{1,2\})/\,\mathrm{acl}(1), \mathrm{acl}(2))| > 1.$$

Failure of n-uniqueness

By the results we have shown $\mathrm{acl}(\{i,j\}) \cap M_2 = \{i, j, \{i,j\}, \{(\{i,j\},0), (\{i,j\},1)\}$ and $\mathrm{Aut}(M_2/\,\mathrm{acl}(\{1,3\}), \mathrm{acl}(\{2,3\})) = K \rtimes \mathrm{Sym}(\Omega \setminus \{1,2,3\})$ where $K = \{f \in \operatorname{im}\alpha_{1,2} : f(\{1,3\}) = f(\{2,3\}) = 0\}$. By definition of $\operatorname{im}\alpha_{1,2}$ for every $f \in K$ there exists $g \in \mathbb{F}_2^{\Omega}$ such that $f(\{i,j\}) = g(i) + g(j)$.

$$\begin{cases} f(\{1,3\}) = g(1) + g(3) = 0 \pmod 2 \\ f(\{2,3\}) = g(2) + g(3) = 0 \pmod 2 \end{cases}$$

Failure of n-uniqueness

$\Rightarrow$ $g(1)+g(2)=f(\{1,2\})=0$ and f fixes also $\{(\{1,2\},0),(\{1,2\},1)\}$

$\Rightarrow$ $\mathrm{Aut}(\mathrm{acl}(\{1,2\})/\,\mathrm{acl}(\{1,3\}),\mathrm{acl}(\{2,3\}))$=id

Now we consider $\mathrm{Aut}(\mathrm{acl}(\{1,2\})/\,\mathrm{acl}(1),\mathrm{acl}(2))$.

- $\mathrm{acl}(i)\cap M_2=\{i\}$.

$\Rightarrow$ $\mathrm{Aut}(M_2/\,\mathrm{acl}(1),\mathrm{acl}(2))=\mathrm{im}\,\alpha_{1,2}\rtimes\mathrm{Sym}(\Omega\setminus\{1,2\})$.

- Let $g\in\mathbb{F}_2^{\Omega}$ such that $g(1)=1$ and $g(a)=0$ for all other $a\in\Omega$. Then $f=\alpha_{1,2}(g)\in\mathrm{im}\,\alpha_{1,2}\rtimes\mathrm{Sym}(\Omega\setminus\{1,2\})$ is an automorphism of $\mathrm{acl}(\{1,2\})$ such that $(\{1,2\},0)^f=(\{1,2\},0+f(\{1,2\}))=(\{1,2\},1)$.

$\Rightarrow$ $|\,\mathrm{Aut}(\mathrm{acl}(\{1,2\})/\,\mathrm{acl}(1),\mathrm{acl}(2))|>1$.

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