On finite covers, groupoids and finite internal covers

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Abstract

We present a brief survey on finite covers in model theory. In particular, we focus on those covers whose automorphism group over the base structure is either abelian or finite. In the last section we show some recent results due to Hrushovski ([15]) concerning groupoids and finite internal covers.

Definition 0.1 *If C and W are first-order structures we say that $\pi : C \to W$ is a* finite cover *of W if*

1. *π is a surjection with fibres of finite cardinality;*
2. *the fibres of π are the equivalence classes of an $\emptyset$-definable equivalence relation on C;*
3. *for any $n \in \mathbb{N}$ all subsets of W^n which are $\emptyset$-definable in the 2-sorted structure (C, W, π) are already $\emptyset$-definable in the structure W.*

The general problem is, for a given W, to describe all possible C. A prominent place in model theory where this problem arises is in the study of totally categorical structures.

In 1904 Oswald Veblen stole a term from Kant and described a theory as *categorical* if it has just one model up to isomorphism (i.e. it has a model and all its models are isomorphic to each other). The bad news is that there are no categorical first-order theories with infinite models. This follows at once from the upward Loewenheim-Skolem theorem. Actually if T is a first-order theory with infinite models, then the strongest kind of categoricity we can hope for T is that, for certain infinite cardinals κ, T has exactly one model of cardinality κ up to isomorphism. This property is called κ-*categoricity.* If the theory is κ-categorical for all infinite cardinals κ then it is said to be *totally categorical.* A structure is said to be *κ-categorical* or *totally categorical* if its theory is so. For example, the set of rational numbers, seen as a countable dense ordered set without endpoints and the countable random graph are $\aleph_0$-categorical structures, while the theory of infinite dimensional vector spaces over a fixed finite field is an example of totally categorical theory.

The idea is that if a first-order theory T forces its models (of a fixed cardinality) to be all similar to each other, this can only be because the models of T have few irregularities and asymmetries. So there should be a good structural description of these models. Considerable work has been done in classifying totally categorical structures. Anyway, they remain quite mysterious objects. It is known that (up to interdefinability) there are only countably many countable totally categorical structures (see [16]). However, there is not a sufficiently satisfactory explicit description of them. We know from the Zilber's Ladder Theorem ([25]) that any totally categorical structure can be built out from some very well-understood structures (pure sets, projective or affine spaces over infinite dimensional vector spaces over finite fields...) by taking a sequence of (*finite, affine*) covers. In these notes we give a brief survey on finite covers. In particular we focus on those with abelian kernel (Section 3) and on those with finite kernel (Section 4). For the latter kind of covers there is a satisfactory description due to D. Evans (see Theorem 4.7 in Section 4 and [8]) by a system of certain continuous homomorphisms denoted in as *conjugate system of homomorphisms.*

Internality, and finite internal covers arise as well in the study of totally categorical theories. In particular, the notion of internality was discovered by Zilber as a tool to study the structure of strongly minimal theories ([26]) which are crucial in the understanding of $\aleph_1$-categorical theories. Later, Poizat realised in [22] that internality can be used to treat the Galois theory of differential equations. In more recent times, Hrushovski in [15] describes finite internal covers using definable groupoids. We give an account of this here in Section 5. We note that finite covers with finite kernel are an example of finite internal covers. Then it is natural to ask if and how the description of finite covers with finite kernel due to Evans is connected with the description of finite internal covers by definable groupoids given by Hrushovski. This is an open question we pose at the end of Section 5.

0.1 An introduction to model theory

We assume standard notions from logic to be known. We briefly recall that a theory is a collection of statements (axioms) written in a fixed formal language, and a model of the theory T is a structure consisting of an interpretation of the symbols in the language of T , in which all the axioms of T hold. For example, the theory ACF of algebraically closed fields can be written in a language with signature $(0, 1, +, -, \cdot)$ and a model of this theory is an algebraically closed field. A formula is written in the same formal language, but has free variables, into which elements of the model can be plugged. Any such formula $\phi(x_1, \dots x_n)$ (where $x_1, \dots, x_n$ are all the free variables of ϕ) determines a subset $\phi(M)$ of M^n, for any model M, namely, the set of all tuples $\bar{a} \in M^n$ for which $\phi(\bar{a})$ holds (in symbols $M \models \phi(\bar{a})$). A subset A of M^n is called *definable* if there exist $b_1, \dots, b_m \in M$ and a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ such that

$$A = \{\bar{a} \in M^n : M \models \phi(\bar{a}, \bar{b})\}.$$

If the parameters $\bar{b}$ can be taken from the subset $X \subseteq M$, then A is said to be X-definable. The union of the finite X-definable subsets of M is called the *algebraic closure* of X, denoted by $\mathrm{acl}_M(X)$, while the union of the X-definable singleton subsets of M is the *definable closure* of X, denoted by $\mathrm{dcl}_M(X)$. If M is the model of some theory, the set of all statements (in the underlying language) that are true in M is a theory $T(M)$, and M is a model of $T(M)$. The set of all the permutations of M which preserve the constants, the functions and relations of M is a group, called the *automorphism group* of M. It is denoted by $\mathrm{Aut}(M)$.

Let L and $L^- \subset L$ be two signatures and M be an L-structure. Then, we can turn M into an L^--structure by simply forgetting the symbols of L which are not in L^-. The resulting structure M^- is called the *reduct* of M to L^-. In particular, $\mathrm{Aut}(M) \leq \mathrm{Aut}(M^-)$. In the other way around if C is the reduct of M to L^-, then M is said an *expansion* of C to L.

We allow our structures to be *multi-sorted*. The universe of a multi-sorted structure is made of disjoint sets. The *sorts* are part of the signature and they play the role of names for the disjoint sets which constitute the universe of the structure. The signature of a multi-sorted structure also says on which sort the constants, the functions and relations are defined. For example a vector space $\mathcal{V}$ can be regarded as 2-sorted structure with sorts V and S. The universe is made of the two disjoint sets $\mathcal{V}_V = V$, the set of vectors, and $\mathcal{V}_S = S$, the scalars. In addition to the sorts, the signature consists of the obvious constants and functions for V (i.e. the standard signature for a group) and for S (i.e. the standard signature for a field). Moreover we have the function $\times$ which is interpreted in $\mathcal{V}$ as the scalar multiplication $\times^{\mathcal{V}} : \mathcal{V}_S \times \mathcal{V}_V \to \mathcal{V}_V$.

Multi-sorted structures are used to deal with quotient constructions. Indeed, it is often convenient to enrich a structure with its equivalence relation classes and deal with them as elements of the structure. More formally, let M be an L-structure. We consider the set of sorts

$$S = \{S_E : E \text{ an } \emptyset\text{-definable equivalence relation on } M^n \text{ for some } n \in \mathbb{N}\}$$

and the many-sorted structure M^{eq} with sorts S, such that the sort S_E is interpreted in M^{eq} as M^n/E for E an $\emptyset$-definable equivalence relation on

M^n. Since $=$ is a definable equivalence relation on M, M can be identified with the sort $S_=$. All relations and functions of L are considered relations and functions on the sort M. Moreover for each $\emptyset$-definable equivalence relation E on M^n we have in M^{eq} an n-ary function $f_E : M^n \to S_E$ given by $f_E(\bar{x}) = \bar{x}/E$.

1 Notation and basic definitions about finite covers

As a first approach, it is natural to analyze the problem of describing finite covers of a given structure W, when W is countable and $\aleph_0$-categorical. In this case, we can apply the theorem of Ryll-Nardzewski, Svenonius and Engeler and obtain a reformulation of definitions in the group-theoretic language. Thus, most of the rest of this paper will be phrased in terms of permutation groups. For more details about finite covers we address the reader to [12].

Definition 1.1 *A* permutation structure $\langle W, G\rangle$ *consists of a set* W *and a closed subgroup* G *of* $\mathrm{Sym}(W)$*, the full symmetric group on* W*. We refer to* G *as the automorphism group of the permutation structure,* $G = \mathrm{Aut}(W)$*, and we simply denote by* W *the permutation structure.*

In Definition 1.1 *closed* is meant in the usual topology on $\mathrm{Sym}(W)$: when W is countable, it is the topology derived by the pointwise convergence. More generally it is the topology whose open sets are unions of cosets of pointwise stabilizers of finite subsets of W. In this way, $\mathrm{Sym}(W)$ is a topological group, and closed subgroups are exactly automorphism groups of relational structures on W (see [3]). Using the language of permutation structures, one can give an equivalent definition of finite covers.

Definition 1.2 *Suppose* C *and* W *are permutation structures. A map* $\pi : C \to W$ *is a finite cover of* W *if*

1. π *is surjective and each fibre* $C(w) := \pi^{-1}(w)$ *is finite, for any* $w \in W$*;*
2. *the set of fibres is an* $\mathrm{Aut}(C)$*-invariant partition of* C*;*
3. *the image of the induced map* $\mu : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$*, defined by* $\mu(g)(w) = \pi(g(C(w)))$*, for* $g \in \mathrm{Aut}(C)$ *and* $w \in W$*, is* $\mathrm{Aut}(W)$*.*

Note that μ is a continuous homomorphism. The *kernel of* π is by definition the kernel of μ. We say that the cover is *split* if the kernel of π admits a closed complement in $\mathrm{Aut}(C)$.

Notation: Let $\pi : C \to W$ be a finite cover of W. If A is a subset of C and B is a subset of W, then we denote by $\mathrm{Aut}(A/B)$ the subgroup of permutations of A which extend to elements of $\mathrm{Aut}(C)$ fixing pointwise every element of B in the induced action on W by μ. Using this notation the kernel of π can be denoted by $\mathrm{Aut}(C/W)$.

Example 1.3 Let W be any permutation structure. Let S be a set with two elements on which $\mathbb{Z}_2$ (the cyclic group of order two) acts non-trivially. Define $C = S \times W$ and let $\mathrm{Aut}(C)$ be the full wreath product $\mathbb{Z}_2 \mathrm{Wr}\, \mathrm{Aut}(W)$. The projection map $\pi : C \to W$ is a finite cover. The kernel of the cover is $\mathbb{Z}_2^W$, the group of functions from W to $\mathbb{Z}_2$. The cover is split.

Example 1.4 Let Ω be any infinite set. Let W and C be respectively the set of unordered and ordered pairs of distinct elements of Ω, considered as permutation structures with automorphism group $\mathrm{Sym}(\Omega)$. The two-to-one map $\pi : C \to W$ given by $\pi((a, b)) = \{a, b\}$ is a finite cover with trivial kernel.

Example 1.5 Let C be the set on non-zero vectors of an infinite dimensional vector space V over a finite field, and $\mathrm{Aut}(C) = GL(V)$ be the group of its linear transformations. Let W be the set of one-dimensional subspaces of V and $\mathrm{Aut}(W) = PGL(V)$, the permutations induced by $GL(V)$. The map $\pi : C \to W$ given by $\pi(c) = \langle c \rangle$ is a finite cover. It has finite kernel consisting of the scalar transformations (so it is isomorphic to the multiplicative group of the finite field). The cover is non-split since $GL(V)$ has no proper subgroups of finite index.

2 The cover problem

As indicated in § 1, one of the principal question in this subject is:

The Cover Problem. Given a permutation structure W, describe all the possible finite covers of W.

In the examples we have in mind, W is usually transitive (i.e. $\mathrm{Aut}(W)$ acts transitively on W), so in any finite cover $\pi : C \to W$, for all $w \in W$ the *fibre groups* $\mathrm{Aut}(C(w)/w)$, i.e. the groups of permutations induced by $\mathrm{Aut}(C)$ on the fibres, are all isomorphic to some finite group F, as well as the *binding groups* $\mathrm{Aut}(C(w)/W)$, i.e. the groups of permutations induced by the kernel on the fibres, are all isomorphic to some finite group B. For any $w \in W$, $\mathrm{Aut}(C(w)/W) \trianglelefteq \mathrm{Aut}(C(w)/w)$; moreover, for any $w \in W$, there is a homomorphism $\chi_w : \mathrm{Aut}(W/w) \to \mathrm{Aut}(C(w)/w)/\, \mathrm{Aut}(C(w)/W)$, defined by $\chi_w(g) = (h|C(w))\, \mathrm{Aut}(C(w)/W)$, where $g \in \mathrm{Aut}(W/w)$ and $h \in \mathrm{Aut}(C)$ is a permutation which extends g. This homomorphism is well defined, continuous and surjective ([12], Lemma 2.1.1). We shall refer to it as the *canonical homomorphism* of the cover.

Thus, we can attempt to describe the transitive finite covers of W with given binding group B, fibre group F and canonical homomorphism χ (which are called the *data* of a finite cover). Following the approach in [1] and [2], we can further subdivide the cover problem above as:

Part A Describe the possible kernels K.

Part B Determine the possible group extensions of K by $\mathrm{Aut}(W)$ which can arise as $\mathrm{Aut}(C)$.

We observe that, if $\pi : C \to W$ is a finite cover, then the kernel of π is a subgroup of the cartesian product of all its binding groups. The next constructive lemma (cf. Lemma 2.1.2 in [12]) describes how to construct a finite cover with kernel as big as possible; such covers are completely determined by the fibre and the binding groups together with the canonical homomorphism.

Proposition 2.1 *Let W be a transitive permutation structure and F a permutation group on a finite set X. Let $w_0 \in W$, suppose B is a normal subgroup of F and $\chi : \mathrm{Aut}(W/w_0) \to F/B$ a continuous epimorphism. Then there exists a finite cover $\sigma : M \to W$ with all fibre groups and binding groups equal to F and B, respectively, and kernel $\prod_{w \in W} B$, such that the canonical homomorphism χ_{w_0} is equal to χ. With these properties, σ is uniquely determined (up to isomorphism over W).*

Finite covers with kernel equal to the cartesian products of all its binding groups are called *free* finite covers. Such covers play an important role in the attempt of describing all finite covers with given data. In fact, every finite cover $\pi : C \to W$ is an expansion of a free finite cover with the same fibre groups, binding groups and canonical homomorphisms as π. The proof of this is quite simple. Let $B(w)$ be the binding groups of π for each $w \in W$. Then, it is easy to see that $\prod_{w \in W} B(w)\,\mathrm{Aut}(C)$ is a subgroup of $\mathrm{Sym}(C)$. Moreover, by a general fact about topological groups, compactness of $\prod_{w \in W} B(w)$ and closeness of $\mathrm{Aut}(C)$ imply that $\prod_{w \in W} B(w)\,\mathrm{Aut}(C)$ is a closed subgroup of $\mathrm{Sym}(C)$. Hence, $\prod_{w \in W} B(w)\,\mathrm{Aut}(C)$ is the automorphism group of a structure C_0 with universe C. In particular, C is an expansion of C_0. The map $\pi_0 : C_0 \to W$, defined in the same way as π, is a finite cover with kernel $\prod_{w \in W} B(w)$. Hence, π_0 is a free finite cover with binding groups equal to those of π. Also, for any $w \in W$ we have that the fibre group of π_0 at w is

$$\mathrm{Aut}(\pi_0^{-1}(w)/w) = B(w)\,\mathrm{Aut}(\pi^{-1}(w)/w) = \mathrm{Aut}(\pi^{-1}(w)/w)$$

so, π_0 and π have the same fibre groups. The definition of canonical homomorphisms shows that they also coincide in π and π_0.

It follows that, given an $\aleph_0$-categorical structure W, initial data $\mathcal{D} = (B(w), F(w), \chi_w)_{w \in W}$ and a free finite cover $\pi_0 : C_0 \to W$ determined by $\mathcal{D}$, the Cover Problem reduces to find the closed subgroups Γ of $\mathrm{Aut}(C_0)$ such that $\mu(\Gamma) = \mathrm{Aut}(W)$.

One of the most relevant cases is when the binding groups are abelian groups. In this case, as we shall later remind, Part A translates into a permutation module problem. The nicest answer to Part B is that all finite covers of W split. this means that, the automorphism group of any finite cover $\pi : C \to W$ of W is a semidirect product of a profinite subgroup (the kernel of π) and a closed subgroup isomorphic to $\mathrm{Aut}(W)$. Model-theoretically this is equivalent to say that any finite cover of W is a reduct of a finite cover with trivial kernel. There is no reason *a priori* to expect that this situation might arise. However, the following result shows that it is quite usual.

Theorem 2.2 *Suppose W is one of the following countable $\aleph_0$-categorical structures:*

1. *a pure set;*
2. *the rational numbers, as an ordered set;*
3. *any primitive homogeneous graph;*
4. *any primitive homogeneous directed graph not isomorphic to the countable universal homogeneous local order, myopic local order or the dense local partial order.*

   Then any finite cover of W splits.

Part (1) was proved independently by M. Ziegler [24], A.A. Ivanov [18] and W. Hodges, I. Hodkinson (unpublished), whereas was proved by A. A. Ivanov ([19]). Part (3) and (4) can be found in [7]. Homogeneity in (4) is meant in the sense of Fraisse, i.e. any isomorphism between finite sub-(directed) graphs extends to an automorphism of the whole graph. On the other hand, primitivity means that there are no non-trivial 0-definable equivalence relations. The proof of the above theorem uses the classification of the countable homogeneous graphs made by A. Lachlan and R. Woodrow in [21] and the classification of countable homogeneous directed graphs made by G. Cherlin [4].

We note that if π is a transitive free finite cover and if the fibre group splits over the binding group, then also the cover π splits (see Lemma 2.1.4 in [12]). As we will recall later on, most of the work around the cover problem has been done in the case when the fibre group splits over the binding group. In the following, we report a countable family of non-split free finite covers:

Example 2.3 Let Ω be a pure set, k a positive integer and $W_k = [\Omega]^k$ be the set of k-subsets of Ω. Regard W_k as a permutation structure with $\mathrm{Aut}(W_k) = \mathrm{Sym}(\Omega)$. Note that if $w \in W_k$, then the stabilizer in $\mathrm{Aut}(W_k)$ of w is isomorphic to $\mathrm{Sym}(w) \times \mathrm{Sym}(\Omega \setminus w)$, so in particular has a proper closed normal subgroup of finite index. Then there is a homomorphism $\chi_w : \mathrm{Aut}(W_k/w) \to \mathbb{Z}_2$ given by taking $\chi_w(g)$ to be the sign of g restricted to w. Let $\pi_k : C_k \to W_k$ be the free finite cover of W_k with fibre groups $\mathbb{Z}_4$ acting regularly, binding group $\mathbb{Z}_2$, and canonical homomorphisms χ_w.

The above free finite covers are not split, essentially because $\mathbb{Z}_4$ does not split over $\mathbb{Z}_2$.

In the next sections we describe some results which deal with the following special cases of the Cover Problem:

Part C Describe finite covers of W with **abelian** kernel.

Part D Describe finite covers W with **finite** kernel.

3 Finite covers with abelian kernel

There are various results which emphisise the importance of reducing the analysis of the cover problem in the case of abelian kernels (for example, see [9], [8]). In this case we are in the following situation:

Lemma 3.1 *Suppose that $\pi : C \to W$ is a finite cover with abelian kernel K. Then, K is a topological* $\mathrm{Aut}(W)$*-module.*

Proof. Consider the action $\psi : \mathrm{Aut}(C) \times K \to K$ by conjugation. Since K is abelian, K is in the kernel of the action. So, we have an induced action of the quotient group $\mathrm{Aut}(C)/K \cong \mathrm{Aut}(W)$ on K. Consider $\mathrm{Aut}(W) \times K$ with the product topology. The $\mathrm{Aut}(W)$-action $\phi : \mathrm{Aut}(W) \times K \to K$ is continuous. Indeed, the action by conjugation ψ is continuous and if U is an open set in K then $\phi^{-1}(U) = (\mu \times id)\psi^{-1}(U)$. We claim that μ is continuous from which it follows that K is a a topological $\mathrm{Aut}(W)$-module. A typical basic open set of $\mathrm{Aut}(W)$ is the pointwise stabilizer of a finite set $F \subseteq W$, $\mathrm{Aut}(W/F)$. The preimage of this by μ is $\mathrm{Aut}(C/F)$ which contains $\mathrm{Aut}(C/\pi^{-1}(F))$. From the fact that $\pi^{-1}(F)$ is a finite set it follows that $\mathrm{Aut}(C/F)$ is an open subgroup of $\mathrm{Aut}(C)$. ∎

Notation: In the sequel, $\pi_0 : C_0 \to W$ will denote a free finite cover with abelian kernel K_0.

As said above, if fibre and binding groups are equal, then π_0 splits. Let T be a closed complement to K_0 in $\mathrm{Aut}(C_0)$. Note that any subgroup of K_0 which is normalized by T is actually normal in $\mathrm{Aut}(C_0)$ and vice versa, i.e. subgroups of K_0 normalized by T are exactly the $\mathrm{Aut}(W)$-submodules of K_0 (the action of $\mathrm{Aut}(W)$ on K_0 is the one described in Lemma 3.1). Hence KT is a closed subgroup of $\mathrm{Aut}(C_0)$ (closeness follows from a general fact about topological groups). Then, KT can be thought as the automorphism group of a split covering expansion of π_0 with kernel K. This gives part of the following which is a result of Ahlbrandt and Ziegler [2].

Theorem 3.2 *Let $\pi_0 : C_0 \to W$ be a free finite cover with fibre groups equal to the binding groups at each point and abelian kernel K_0. Regard K_0 as a topological* $\mathrm{Aut}(W)$*-module. Then K is the kernel of some covering expansion of π_0 if and only if K is a closed* $\mathrm{Aut}(W)$*-submodule of K_0.*

When the fibre groups and the binding groups are all cyclic groups of order p, K_0 can be identified with the $\mathrm{Aut}(W)$-module $\mathbb{F}_p^W$ of functions from W into $\mathbb{F}_p$, the field of integers modulo p (the $\mathrm{Aut}(W)$-action on $\mathbb{F}_p^W$ is given by $gf(w) = f(g^{-1}w)$, for $f \in \mathbb{F}_p^W$, $g \in \mathrm{Aut}(W)$ and $w \in W$). So we are interested in the closed $\mathrm{Aut}(W)$-invariant subspaces of $\mathbb{F}_p^W$. For example, in [1] the authors describe completely the kernels of covering expansions of π_0 in the case when W is as in Example 1.5, for the field $\mathbb{F}_2$, with binding and fibre groups equal to $\mathbb{F}_2$.

Sometimes the closed $\mathbb{F}_p \mathrm{Aut}(W)$-invariant subspaces of $\mathbb{F}_p^W$ can be more easily described making use of a simple instance of the Pontrjagin duality (see [23]).

Indeed, there is a natural pairing $\mathbb{F}_p^W \times \mathbb{F}_p W \to \mathbb{F}_p$ given by $(f, \sum_w a_w w) \mapsto \sum_w a_w f(w)$. By a standard application of Pontryagin duality, the closed $\mathbb{F}_p \mathrm{Aut}(W)$-submodules of $\mathbb{F}_p^W$ are of the form X^0 for $\mathbb{F}_p \mathrm{Aut}(W)$-submodules X of $\mathbb{F}_p W$, where X^0 denotes the annihilator of X with respect to this pairing. Moreover if $X \leq Y$ are $\mathbb{F}_p \mathrm{Aut}(W)$-submodules of $\mathbb{F}_p W$ then $Y^0 \leq X^0$ and X^0/Y^0 is isomorphic to the Pontrjagin dual S^* of $S = Y/X$.

In [14] D. Gray analyzed the case when $W = [\Omega]^n$ is the set of n-subsets from a countable set Ω and $\mathrm{Aut}(W) = \mathrm{Sym}(\Omega)$ naturally acts on $[\Omega]^n$. He showed that the closed $\mathrm{Sym}(\Omega)$-submodules of $\mathbb{F}_p W$ are intersection of kernels of certain $\mathrm{Sym}(\Omega)$-homomorphisms. The proofs use representation theory of finite symmetric groups, as developed in the book of G. D. James [17]. Dualising Gray's results we have a complete description of possible kernels of a finite cover of $[\Omega]^n$ with fibre and binding groups of order p.

3.1 Continuous cohomology groups and finite covers

When fibre and binding groups are different, Theorem 3.2 cannot apply. Hence, one needs a criterion to distinguish among closed $\mathrm{Aut}(W)$-

submodules of K_0 which are actually kernels of covering expansions of π_0. In [13] such a criterion is given in terms of continuous cohomology groups.

Any finite cover gives rise to a short exact sequence.

$$1 \to K \to \mathrm{Aut}(C) \to \mathrm{Aut}(W) \to 1. \tag{1}$$

All the groups involved in (1) are topological groups, in particular K is profinite. So the appropriate cohomological context is the continuous one as well as the appropriate category for this cohomological machinary is the one of permutation groups regarded as topological groups with continuous homomorphism. Therefore, these are all Hausdorff topological groups: a base of open neighbourhoods of the identity consists of open subgroups. We denote such category by $\mathcal{PG}$. The following definitions are taken from [10].

Let G be a closed permutation group, endowed with the topology induced by the full symmetric group, and K be a continuous profinite G-module. Denote by $C_c^n(G, K)$ the additive group of continuous functions $\varphi : G^n \to K$. The usual coboundary operator δ^n sends $C_c^n(G, K)$ to $C_c^{n+1}(G, K)$, so that $(C_c^n(G, K); \delta^n)_{n\in\mathbb{N}}$ is a cochain complex. The homology of this complex, $H_c^*(G, K)$, is the *continuous cohomology* of G with coefficients in K.

The most important fact is that, any $\mathcal{PG}$-extension of a profinite G-module K admits a continuous closed section (see[10]). For this reason, cohomology of low degree continuous cocycles on profinite G-modules retains their familiar applications: $H_c^1(G, K)$ classifies closed complements in the split extension and $H_c^2(G, K)$ classifies all $\mathcal{PG}$-extensions of K by G. More details about continuous chomology can be found in [10].

With this set-up, the criterion we mentioned above is the following:

Theorem 3.3 (Theorem 2.1 of [13]) *Suppose $\pi : C \to W$ is a finite cover with abelian kernel K and H a closed $\mathrm{Aut}(W)$-submodule of K. Let e_0 be the element in $H_c^2(\mathrm{Aut}(W), K)$, which gives rise to $\mathrm{Aut}(C)$ (as group extension of K by $\mathrm{Aut}(W)$).*

Let

$$0 \to H \xrightarrow{i} K \to \bar{H} \to 0$$

be the natural short exact sequence where i is the inclusion map. Consider

$$\cdots \to H_c^1(\mathrm{Aut}(W), \bar{H}) \to H_c^2(\mathrm{Aut}(W), H) \xrightarrow{i^*} H_c^2(\mathrm{Aut}(W), K),$$

where i^* is the induced map by i in cohomology. Then there exists a covering expansion of π with kernel H if and only if there exists an element $e \in H_c^2(\mathrm{Aut}(W), H)$ such that $i^*(e) = e_0$.

Continuation of Example 2.3 Using notation as in Example 2.3, we have an exact sequence:

$$0 \to \mathbb{Z}_2^{[\Omega]^k} \to \mathrm{Aut}(C_k) \to \mathrm{Sym}(\Omega) \to 1.$$

The possible kernels of covering expansions of π_k with same fibre and binding groups as π_k can be easily described using the method developed in [14]: they are sums of images of various of the following $\mathrm{Sym}(\Omega)$-homomorphisms: $\alpha_{j,k} : \mathbb{F}^{[\Omega]^j} \to \mathbb{F}^{[\Omega]^k}$ where $\alpha_{j,k}(f)(w) = \sum_{v\in[w]^j} f(v)$, for $f \in \mathbb{F}^{[\Omega]^j}$ and $w \in [\Omega]^k$.

Consider the case $k = 2$. Since the exact sequence above is not split, the 2-cocycle class $e_0 \in H^2_c(\mathrm{Sym}(\Omega), \mathbb{Z}_2^{[\Omega]^2})$, which gives rise to $\mathrm{Aut}(C_2)$ as a group extension of $\mathbb{Z}_2^{[\Omega]^2}$ by $\mathrm{Sym}(\Omega)$, is non-zero. If K is a closed $\mathrm{Sym}(\Omega)$-submodule of $\mathbb{Z}_2^{[\Omega]^2}$, then it is shown in [13] that $H^2_c(\mathrm{Sym}(\Omega), K) = 0$. These calculations are obtained applying the Shapiro's Lemma and other standard cohomological techniques adapted to the continuous case. Hence, by Theorem 3.3 we have that the free cover π_2 does not admit any proper covering expansion.

For $k > 2$ there exist proper covering expansions of π_k. The following theorem describes the minimal ones, i.e. finite covers $\pi : C \to W_k$ such that $\mathrm{Aut}(C) \leq \mathrm{Aut}(C_k)$ and if $G \leq \mathrm{Aut}(C)$ then $\mu(G) \neq \mathrm{Aut}(W_k)$, where μ is the induced map defined in Definition 1.2.

Theorem 3.4 (Theorem 4.8 [13]) *For $2 < k$ there exist continuous homomorphisms $\gamma_{2,k} : \mathrm{Aut}(C_2) \to \mathrm{Aut}(C_k)$ which extend the natural $\mathrm{Sym}(\Omega)$-homomorphisms $\alpha_{2,k} : \mathbb{Z}_2^{[\Omega]^2} \to \mathbb{Z}_2^{[\Omega]^k}$. The $\mathrm{Aut}(C_k)$-conjugates of $\gamma_{2,k}(\mathrm{Aut}(C_2))$ are the minimal covering expansions of π_k.*

A direct consequence of this result is the solution of the Cover Problem for the permutation structures W_k with data given as in Example 2.3. Indeed, it is easy to see that every automorphism group of a finite cover π with kernel K is of the form $K\Gamma$ where Γ is the automorphism group of some minimal covering expansion of π (minimal covering expansions always exists, by a result of Cossey, Kegel and Kovacs; see [6]). So, once we know all the minimal covering expansions of a free finite cover π_0 of W and all the possible kernels, we are able to describe all the finite covers of W with the same data as π_0.

4 Finite covers with finite kernel

In this section we give a brief survey on finite covers with finite kernel. The situation here is rather well understood. We start presenting the comprehensive work of Evans on finite covers with finite kernels ([8]).

Definition 4.1 *The permutation structure W is said to be* irreducible *if $\mathrm{Aut}(W)$ has no proper closed subgroups of finite index.*

We note that if $\pi : C \to W$ is an irreducible finite cover with finite kernel $\mathrm{Aut}(C/W)$, then $\mathrm{Aut}(C/W)$ is central in $\mathrm{Aut}(C)$. Indeed, as $\mathrm{Aut}(C/W)$ is a finite normal subgroup of $\mathrm{Aut}(C)$, its centrilizer in $\mathrm{Aut}(C)$ is a closed subgroup of finite index in $\mathrm{Aut}(C)$. Then, from irreducibility of $\mathrm{Aut}(C)$ it follows that $\mathrm{Aut}(C/W)$ is central in $\mathrm{Aut}(C)$.

The notion of irreducibility of Definition 4.1 has a model- theoretic meaning in terms of the algebraic and definable closure. Indeed, an easy conseguence of the definition of algebraic closure is that if $x \in \mathrm{acl}_M(A)$ for M an L-structure and $A \subseteq M$ then the orbit of x under the action of $\mathrm{Aut}(M/A)$ is of finite cardinality. Respectively, if $x \in \mathrm{dcl}_M(A)$, then x is fixed by all the elements of $\mathrm{Aut}(M/A)$. The algebraic closure in M^{eq} is described by the following result:

Proposition 4.2 *Let M be a countable $\aleph_0$-categorical structure and A be a finite subset of M. The elements of $\mathrm{acl}_{M^{eq}}(A)$ corresponds to closed subgroups of finite index of $\mathrm{Aut}(M/A)$.*

Proof Let $c \in \mathrm{acl}_{M^{eq}}(A)$. Then $\mathrm{Aut}(M/A, c)$ is a closed subgroup of finite index in $\mathrm{Aut}(M/A)$.

Let G be a closed subgroup of finite index in $\mathrm{Aut}(M/A)$. By general topological arguments, G is open in $\mathrm{Aut}(M)$. Then, there exists a tuple $\bar{b}$ such that $\mathrm{Aut}(M/\bar{b}) \leq G \leq \mathrm{Aut}(M/A)$. Consider $\{(\bar{b}, \bar{b}^{\gamma}) : \gamma \in G\}$. By $\aleph_0$-categoricity these pairs lie in a finite number of orbits under $\mathrm{Aut}(M)$: choose $(\bar{b}, \bar{b}^{\gamma_1}), \ldots, (\bar{b}, \bar{b}^{\gamma_m})$ as rapresentatives. Let

$$\mathcal{R} := \{(\bar{b}^{\alpha}, \bar{b}^{\gamma_i \alpha}) : i \leq m, \alpha \in \mathrm{Aut}(M)\}.$$

Then, $\mathcal{R}$ is a $\emptyset$-definable equivalence relation in M. Let $c = [\bar{b}]_{\mathcal{R}}$. Then, $\mathrm{Aut}(M/c) = \bigcup_{i \leq m} \gamma_i \mathrm{Aut}(M/\bar{b}) = G$ and $c \in \mathrm{acl}_{M^{eq}}(A)$. ■

So, for a countable $\aleph_0$-categorical structure W, the notion of irreducibility it is equivalent to saying that $acl_{W^{\mathrm{eq}}}(\emptyset) = dcl_{W^{\mathrm{eq}}}(\emptyset)$.

In the following, by a *digraph* (L, R) we mean either a graph or a directed graph with vertex set L and edge set R.

Definition 4.3 *Suppose that (A, R) and (B, R') are digraphs.*

1. *A function $\sigma : A \to B$ is a* homomorphism *if $\{(\sigma a, \sigma a') \mid (a, a') \in R\} = R'$.*

2. *A homomorphism of digraphs $\sigma : A \to B$ is a* covering *if it is onto and it is locally an isomorphism of digraphs, that is, for all $a \in A$, the restriction of σ to the subdigraph $a^+ := \{a' \in A : (a, a') \in R\}$ ($a^- =: \{a' \in A : (a', a) \in R\}$, respectively) is an isomorphism with $(\sigma a)^+$ (with $(\sigma a)^-$, respectively).*

If (L, R) is a connected digraph one can construct a universal covering $\sigma : (U, R'') \to (L, R)$ in terms of homotopy classes of paths from some fixed basepoint in (L, R). This factors through any other connected covering and it is uniquely determined by this property. Any automorphism of (L, R) lifts to an automorphism of (U, R'') preserving the fibres of σ. The *deck transformations* Δ are the automorphisms of (U, R'') which stabilize each fibre of σ. Quotients by suitable normal subgroups of Δ give rise to finite covers of (L, R), which can be irreducible, transitive finite covers of L with finite kernels, according to the fact that (L, R) is sufficiently "nice".

Conversely, assuming irreducibility of certain stabilizers and the existence of a certain type of orbit on triples of vertices, any irreducible transitive finite cover with finite kernel arises from a digraph covering.

Definition 4.4 *Suppose W is a permutation structure. We say that* $\mathrm{Aut}(W)$*-orbits $P \subseteq W^3$ and $Q, R \subseteq W^2$ form a* graphic triple (P, Q, R) *if*

- *$(w, x, y) \in P$ implies $(w, x), (w, y) \in Q$ and $(x, y) \in R$;*
- *the digraph with edge set R is connected;*
- *one of the following conditions hold*
  1. *if $(x, y), (y, z), (x, z) \in R$ then there exists $w \in W$ with $(w, x, y), (w, y, z), (w, x, z) \in P$*
  2. *$P = \{(w, x, y)(w, x), (w, y), (x, y) \in R\}$.*

Theorem 4.5 ([8], Theorem 1.13) *Suppose W is a transitive irreducible permutation structure with a graphic triple (P, Q, R). Suppose further that if $(w, x, y) \in P$ then $\mathrm{Aut}(W/w), \mathrm{Aut}(W/w, x), \mathrm{Aut}(W/x, y)$ are irreducible.*

Let $\pi : C \to W$ be an irreducible finite cover with finite kernel. Then there is an $\mathrm{Aut}(C)$-invariant digraph relation R' on C such that $\pi : (C, R') \to (W, R)$ is a covering of digraphs.

The hypotheses of the theorem above are not satisfied from one of the most natural example of irreducible finite cover with finite kernel: the non-zero elements of a vector space over a finite field F covering its projective space (Example 1.5). Indeed, the set of triples of independent points in the projective space gives a graphic triple, but the stabilizer of two independent points x and y is not irreducible (there is a closed normal subgroup, the pointwise stabilizer of the subspace generated by x and y, such that the quotient group is isomorphic to the multiplicative group of the field F). To state the following theorem we need a definition.

Definition 4.6 *Let W be a permutation structure. A* strong type *on W is a function p which assigns to each finite subset X of W an* $\mathrm{Aut}(W/X)$*-orbit $p|X \subseteq W \setminus X$ such that*

- *if* $X \subseteq X'$ *then* $p|X' \subseteq p|X$*;*
- *if* $g \in \mathrm{Aut}(W)$ *then* $g(p|X) = p|(gX)$.

For example, if W is a projective space over a finite field we can take as $p|X$ the points of W independent from X. As another example, suppose $W = \mathbb{Q}$, considered as an ordered set; then we can take $p|X$ to be $\{a \in W : a < x \, \forall x \in X\}$.

If there is a strong type on W we obtain a graphic triple (P, Q, R) by taking

$$P = \{(w, x, y) \in W^3) \mid w, x \in p|\{y\}, w \in p|\{x, y\}\}$$

$$R = \{(x, y) \in W^2 : x \in p|\{y\}\}$$

$$Q = R$$

In this case the digraph (W, R) is simply connected and hence, the universal covering is an isomorphism. So if additionally the irreducibility conditions of Theorem 4.5 are satisfied, any irreducible finite cover with finite kernel of W is one-to-one. The following result describes what happens if there is not irreducibility of the two-point stabilizers.

Theorem 4.7 ([8], Corollary 3.9) *Let W be a countable, transitive $\aleph_0$-categorical structure with a strong type p. Suppose* $\mathrm{Aut}(W)$ *and* $\mathrm{Aut}(W/w)$ *are irreducible (for $w \in W$). Let A be any finite abelian group. Then there is a one-to-one correspondence between*

1. *irreducible finite covers with kernel A*
2. *system of continuous surjective homomorphisms* $(\phi_{w,x} : (w, x) \in R)$

$$\phi_{w,x} : \mathrm{Aut}(W/w, x) \to A$$

satisfying

a) $\phi_{gw,gx}(ghg^{-1}) = \phi_{w,x}(h)$ *for* $g \in \mathrm{Aut}(W)$ *and* $h \in \mathrm{Aut}(W/w, x)$*;*

b) if $(w, x, y) \in P$ *and* $g \in \mathrm{Aut}(W/w, x, y)$ *then*

$$\phi_{w,y}(g) = \phi_{w,x}(g)\phi_{x,y}(g).$$

A family $(\phi_{w,x} : (w, x) \in R)$ as in Theorem 4.7 is called a *conjugate system of homomorphisms* for (W, R, A).

Some of the ideas of Theorem 4.7 have been generalized by Jeffrey Koshan not assuming that W has a strong type (see [20]).

5 Groupoids and internal covers

In this section we present part of the results of [15] about internal finite covers and groupoids. In particular we show that there is a bijective correspondence between finite internal covers and definable groupoids.

5.1 Definable groupoids

A *category* $\mathcal{C}$ is a 2-sorted structure with sorts O, M, where O names the objects and M the morphisms. We equip $\mathcal{C}$ with maps $i_0, i_1 : M \to O$, where if $m \in M$, $i_0(m)$ is the *source* object of m, while $i_1(m)$ is the *target* object of m, a partial composition map between morphisms $\circ : M \times M \to M$ satisfying the usual associative laws and an identity map $Id : O \to M$ such that $Id(x) : x \to x$ is the identity, i.e. for every morphism $f : x \to y$, $Id(y) \circ f = f = f \circ Id(x)$. The language of categories is then 2-sorted with relation symbols $i_0, i_1, Id, \circ$.

A *groupoid* $\mathcal{G}$ is a category $\mathcal{G} = (\mathrm{Ob}\,\mathcal{G}, \mathrm{Mor}\,\mathcal{G})$ where every morphism is invertible. Every groupoid defines an equivalent relation on $\mathrm{Ob}\,\mathcal{G}$, $\mathrm{Iso}_\mathcal{G}$ defined by: $\mathrm{Iso}_\mathcal{G}(x, y)$ if and only if there exists a morphism in $\mathrm{Mor}\,\mathcal{G}(x, y)$ from x to y. On the other hand, for every $a \in \mathrm{Ob}\,\mathcal{G}$ there is a group $G_a := \mathrm{Mor}\,\mathcal{G}(a, a)$. Clearly, if $\mathrm{Iso}_\mathcal{G}(a, b)$, then G_a is isomorphic to G_b. Thus, groupoids generalizes, at different extremes, both groups and equivalence relations: if $\mathrm{Ob}\,\mathcal{G} = \{a\}$, then the groupoid $\mathcal{G}$ reduces to the group G_a, while if $G_a = 1$ for every $a \in \mathrm{Ob}\,\mathcal{G}$, then $\mathcal{G}$ coincides with the equivalence relation $\mathrm{Iso}_\mathcal{G}$.

For the rest of the paper we suppose $\mathcal{G}$ has a single $\mathrm{Iso}_\mathcal{G}$-class. In this case $\mathcal{G}$ is said to be *connected*.

Let T be a complete theory with quantifier elimination and $\mathbb{U}$ a monster model of T. Given a groupoid $\mathcal{G}$, we shall say that $\mathcal{G}$ is a *definable groupoid* for T if $\mathrm{Ob}\,\mathcal{G}, \mathrm{Mor}\,\mathcal{G}$, the maps i_0, i_1 and the composition map are definable in $\mathbb{U}$. Let $\mathrm{Def}(\mathbb{U})$ be the category of parameter definable subsets of $\mathbb{U}$ where the morphisms are definable maps.

Definition 5.1 *A funtor $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ is definable if*

$$\{(a, b, c, d, e) : a, b \in \mathrm{Ob}\,\mathcal{G}, c \in \mathrm{Mor}\,\mathcal{G}(a, b), d \in F(a), e \in F(b), F(c)(d) = e\}$$

is definable.

If F is a *faithful* functor, refer to $(\mathcal{G}, F)$ as a *concrete definable groupoid.*

5.2 From definable groupoids to internal covers

In this paragraph we show how to associate to a definable groupoid a finite internal cover. We start giving the definition of finite internal cover.

Definition 5.2 ([15], Definition 2.2) *Let N be a structure, M the union of some of the sorts of N. The structure N is a* finite internal cover *of M if M is stably embedded in N and $\mathrm{Aut}(N/M)$ is finite. Equivalently, $N \subseteq \mathrm{dcl}(M, c)$ for some finite $c \in \mathrm{acl}(M)$. Similarly, we shall say that the theory of N is a finite internal cover of the theory of M.*

Saying that M is stably embedded in N is equivalent to the fact that the restriction map to the sorts of M $\mu : \mathrm{Aut}(N) \to \mathrm{Aut}(M)$ is surjective (see Appendix of [5]). Note that finite covers with finite kernel are finite internal covers.

We now show how to obtain, given a theory T, a finite internal cover of T from a definable groupoid for T.

Let T be a complete L-theory with quantifier elimination, $\mathbb{U}$ a monster model of T, $\mathcal{G}$ a definable concrete groupoid for T and $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ a faithful definable functor. Consider the following language L' expanding L: L' is the language of T expanded by extra sorts O, C, D, an extra relation symbol F', and the language of categories for a groupoid $\mathcal{G}' = (O, C)$. Let $T' \supset T$ be the L'-theory such that the axioms of T' coincides with those of T on the L-sorts, together with the statements that $\mathcal{G}' = (O, C)$ is a groupoid, the objects of $\mathcal{G}'$ are the objects of $\mathcal{G}$ plus a single extra object $*$ (i.e. $O = \mathrm{Ob}\,\mathcal{G} \cup \{*\}$), $F'(*) = D$ and $(\mathcal{G}', F')$ is a concrete groupoid extending $(\mathcal{G}, F)$, such that $\mathcal{G}$ is a *full* subgroupoid of $\mathcal{G}'$.

Lemma 5.3 ([15], Lemma 2.5) *Any model M of T extends to a unique model M' of T' up to isomorphism over M. Moreover, for any $a \in \mathrm{Ob}\,\mathcal{G}(M)$ $\mathrm{Mor}\,\mathcal{G}(a, a) \cong \mathrm{Aut}(M'/M)$.*

Proof. We sketch how to extend M to M'. Fix an element $a \in \mathrm{Ob}\,\mathcal{G}(M)$. We construct (O, C, D, F') so that anything involving $*$ in $((O, C), F')$ is a copy of the corresponding thing in M replacing $*$ by a. More precisely, for any $b \in \mathrm{Ob}\,\mathcal{G}(M)$, let $\mathrm{Mor}\,\mathcal{G}'(*, b)$ be a copy of $\mathrm{Mor}\,\mathcal{G}(a, b)$, i.e. $\mathrm{Mor}\,\mathcal{G}'(*, b) = \{\tilde{c} : c \in \mathrm{Mor}\,\mathcal{G}(a, b)\}$. Then,

$$C = \mathrm{Mor}\,\mathcal{G} \cup \mathrm{Mor}\,\mathcal{G}(*, *) \cup \bigcup_{b \in \mathrm{Ob}\,\mathcal{G}(M)} (\mathrm{Mor}\,\mathcal{G}(*, b) \cup \mathrm{Mor}\,\mathcal{G}(b, *)),$$

$F'(*)$ is a copy of $F(a)$, i.e. $F'(*) = \{\tilde{d} : d \in F(a)\}$, and $D = F'(*)$ and

$$F' = F \cup \{(*, b, \tilde{c}, \tilde{d}, e) : b \in \mathrm{Ob}\,\mathcal{G}(M), c \in \mathrm{Mor}\,\mathcal{G}(a, b), d \in F(a), F(c)(d) = e\}.$$

∎

In Lemma 5.3 we consider the groupoid $\mathcal{G}$ having a unique $\mathrm{Iso}\,\mathcal{G}$-class. In the case of several isomorphism classes, a covering model M' can be obtained picking up a representative r_ν from each isomorphism class ν and adding to the objects of $\mathcal{G}$ a symbol $*_\nu$ for every ν, such that $F'(*_\nu)$ is a copy of $F(r_\nu)$.

Let us call $M_{(G,F)}$ the structure M' constructed above.

Theorem 5.4 ([15], Part of Theorem 2.8) *If* $\operatorname{Mor}\mathcal{G}(a,a)$ *is finite, then,* $M_{(\mathcal{G},F)}$ *is a finite internal cover of* M*.*

5.3 From internal finite covers to concrete groupoids

Surprisingly, a finite internal cover gives rise to a definable groupoids.

Theorem 5.5 ([15], Proposition 1.5) *Suppose* T' *is a finite internal cover of* T*, with an extra sort* S*. Then there exist connected definable concrete groupoids* $(\mathcal{G}, F)$ *in* T *and* $(\mathcal{G}', F')$ *in* T' *such that*

1. $\operatorname{Ob}(\mathcal{G}') = \operatorname{Ob}(\mathcal{G}) \cup \{*\}$*;*
2. $\mathcal{G}$ *is a full subgroupoid of* $\mathcal{G}'$*;*
3. F' *extends* F*;*
4. $F'(*) = S$*;*
5. $\operatorname{Mor}\mathcal{G}'(*,*) \cong \operatorname{Aut}(M'/M)$ *for any* $M' \models T'$ *(where* M *is restriction of* M' *to the* T*-sorts).*

Proof. We just describe how to construct the definable concrete groupoids $(\mathcal{G}, F)$ and $(\mathcal{G}', F')$. Assume that M' is a sufficiently saturated model of T' and that the restriction of M' to the T-sorts is M. By internality we have that M is stably embedded in M' and that $S \subseteq \operatorname{dcl}(M, c)$ for some finite tuple $c \in M'$. Hence, there exists a c-definable $D \subseteq M^n$, for some n, and a c-definable surjective function $g_c : D \to S$. By stable embeddedness, D and the equivalence relation given by g_c (i.e. $x \cong y$ if and only if $g_c(x) = g_c(y)$) are b-definable in M for some tuple b of parameters in M. Thus, we have a b-definable set $D_b \subseteq M^{\mathrm{eq}}$ and a c-definable bijection $f_c : S \to D_b$. Actually, f_c is given by g_c^{-1}. As $\operatorname{Aut}(M'/M)$ is finite we can assume $c \in \operatorname{acl}_{M'}(b)$ and $\operatorname{tp}^{M'}(c/b) \vdash \operatorname{tp}^{M'}(c/M)$. Now we let (b, c) vary in a definable set $B \times C$. So there exists a definable family of bijections

$$f_{c'} : S \to D_{b'}$$

such that if $(b', c'), (b', c'') \in B \times C$ then $\operatorname{tp}(c'/b') = \operatorname{tp}(c''/b')$. We are now ready to construct the concrete groupoids. Let $\operatorname{Ob}(\mathcal{G}) = B$, $\operatorname{Ob}(\mathcal{G}') = B \cup \{*\}$, $F'(*) = S$, $F(b') = D_b$, for $b' \in B$. Then, let

- $\operatorname{Mor}\mathcal{G}'(*, b') = \{c' : (b', c') \in B \times C\}$, for $b' \in B$;
- $F'(c') = f_c : S \to D_b'$ for $c' \in C$;
- $\operatorname{Mor}\mathcal{G}'(b', *) = \{\tilde{c}' : (b', c') \in B \times C\}$;
- $F'(\tilde{c}') = f_{c'}^{-1} : D_{b'} \to S$.

Now, it is natural to define, for $b', b'' \in B$ $\operatorname{Mor} \mathcal{G}(b', b'') = \{[c'', c'] = f_{c''} \circ f_{c'}^{-1} : D_{b'} \to D_{b''} | c'' \in \operatorname{Mor} \mathcal{G}'(*, b''), \tilde{c}' \in \operatorname{Mor} \mathcal{G}'(b', *)\}$ and $F([c'', c']) = f_{c''} \circ f_{c'}^{-1}$, while $\operatorname{Mor} \mathcal{G}'(*, *) = \{(c'', c') = f_{c''}^{-1} \circ f_{c'} : S \to S : c', c'' \in \operatorname{Mor} \mathcal{G}'(*, b'), b' \in B\}$. ■

In [15] many other results are proved. In particular, in Section 3 Hrushovski relates finite internal covers to some higher amalgamation properties. We don't treat them here. Under the light of the results we presented here, we pose the following question:

Question 5.6 *Is there any relation between the results of Section 4 and the results of Section 5? In particular, between Theorem 4.7 and Theorems 5.4 and 5.5?*

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