Almost-free finite covers

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Abstract

Let W be a first-order structure and ρ be an $\mathrm{Aut}(W)$-congruence on W. In this paper we define the *almost-free* finite covers of W with respect to ρ, and we show how to construct them. These are a generalization of free finite covers.

A consequence of a result of [5] is that any finite cover of W with binding groups all equal to a simple non-abelian permutation group is almost-free with respect to some ρ on W. Our main result gives a description (up to isomorphism) in terms of the $\mathrm{Aut}(W)$-congruences on W of the kernels of principal finite covers of W with binding groups equal at any point to a simple non-abelian regular permutation group G. Then we analyze almost-free finite covers of $\Omega^{(n)}$, the set of ordered n-tuples of distinct elements from a countable set Ω, regarded as a structure with $\mathrm{Aut}(\Omega^{(n)}) = \mathrm{Sym}(\Omega)$ and we show a result about biinterpretability.

The material here presented addresses a problem which arises in the context of classification of totally categorical structures.

1 Introduction

Given a countable set W, consider the natural action of the symmetric group $\mathrm{Sym}(W)$ on W. This action yields a topology on $\mathrm{Sym}(W)$ in which pointwise stabilizers of finite sets give a base of open neighborhoods of the identity. Let Υ be a closed subgroup of $\mathrm{Sym}(W)$ that acts transitively on W and G a finite group acting on a finite set Δ. Consider the projection $\pi : \Delta \times W \to W$ given by $\pi(\delta, w) = w$. We denote by G^W the set of all functions from W to G. Let $\mathcal{F}$ be the set of closed subgroups of $\mathrm{Sym}(\Delta \times W)$ which satisfy three conditions: (1) Every $F \in \mathcal{F}$ preserves the partition of $\Delta \times W$ given by the fibres of π, (2) From (1) we have that every $F \in \mathcal{F}$ gives an induced map $\mu_F : F \to \mathrm{Sym}(W)$. We require that, for all $F \in \mathcal{F}$, $\mu_F(F) = \Upsilon$, (3) The permutation groups induced respectively by F and $\ker \mu_F$ on $\pi^{-1}(w)$, for all $w \in W$, are both equal to G.

Let $\mathcal{K} = \{\ker \mu_F,\ F \in \mathcal{F}\}$. In this paper we will deal with the following

Problem: Given G and Υ, find a description of the elements belonging to $\mathcal{K}$.

This problem, which is here formulated in terms of infinite permutation groups, is motivated by questions arising in model theory concerning finite covers (see [6]).

Definition 1 *Let C and W be two first-order structures. A finite to-one surjection $\pi : C \to W$ is a* finite cover *if its fibres form an $Aut(C)$-invariant partition of C, and the induced map $\mu : Aut(C) \to \mathrm{Sym}(W)$, defined by $\mu(g)(w) = \pi(g\pi^{-1}(w))$, for all $g \in Aut(W)$ and for all $w \in W$, has image $Aut(W)$.*
We shall refer to the kernel of μ as the kernel *of the finite cover π. If $\pi : C \to W$ is a finite cover, the* fibre group *$F(w)$ at $w \in W$ is the permutation group induced by $Aut(C)$ on $\pi^{-1}(w)$. The* binding group *$B(w)$ at $w \in W$ is the permutation group induced by the kernel on $\pi^{-1}(w)$.*

Using the terminology of finite covers, the problem above can be stated in the following equivalent version: given a finite group G and a first-order structure W with automorphism group Υ, describe the kernels of the finite covers of W with $F(w) = B(w) = G$ at any point, which have $\Delta \times W$ as domain of the covering structures and such that the finite-to-one surjections π are the projection on the second coordinate.

A more detailed commentary on finite covers and this problem is given in the last section. However, we avoid the model-theoretic methods using rather infinite permutation group techniques.

In [2] Ahlbrandt and Ziegler described the subgroups $K \in \mathcal{K}$, when G is an abelian permutation group. In this case G^W, the group of functions from W to G, is an Υ-module with $f^{\upsilon}(w) = f(\upsilon^{-1}w)$, where $\upsilon \in \Upsilon$ and $f \in G^W$ and the kernels in $\mathcal{K}$ are profinite Υ-modules. They proved that $\mathcal{K}$ is exactly the set of closed Υ-submodules of G^W .

In this paper, we deal with the case when G is a simple *non abelian* regular permutation group. Under this hypothesis our main result, which is stated and proved in Section 3), gives a description of the elements of $\mathcal{K}$ in terms of the Υ-congruences on W. A key ingredient in the proof is a result of Evans and Hrushovski ([5], Lemma 5.7).

Previous results are the following. In [10], Ziegler described the groups $K \in \mathcal{K}$ in the case when W is a countable set Ω and $\Upsilon = \mathrm{Sym}(\Omega)$ (the disintegrated case), for any group G. Increasing the complexity of the set W, it seems not possible to give a general description of the groups $K \in \mathcal{K}$ not depending on the group G. For example, if W is the set of n-subsets from a countable set Ω, $\Upsilon = \mathrm{Sym}(\Omega)$ and G is a cyclic group of order a prime p, then the groups $K \in \mathcal{K}$ are an intersection of kernels of certain

Υ-homomorphisms, as is described in [7]. While if G is a simple non abelian group, then $\mathcal{K} = \{G, G^W\}$ (see corollary 6).

In Section 4 we analyze the special case in which given a countable set Ω, W is defined as the subset of the n-fold cartesian product $\Omega^{(n)}$ whose elements are n-tuples with pairwise distinct entries. Defining Υ as Sym(Ω), in Proposition 15, 18 and 19 we give an explicit description of the equivalence classes of the Sym(Ω)-congruences on $\Omega^{(n)}$. In these Propositions we see that the blocks for Sym(Ω) in $\Omega^{(n)}$ can be either of finite or of infinite cardinalities. Proposition 21 shows that if $\pi : C \to \Omega^{(n)}$ is a cover of $\Omega^{(n)}$ with Aut(C) in $\mathcal{F}$ and G equal to a simple-non abelian finite group such that the kernel of π determines a Sym(Ω)-congruence on $\Omega^{(n)}$ (in the sense of Lemma 5) with classes of finite cardinality, then, for every $m \in \mathbb{N}$ greater then n, there exists a finite cover $\pi' : C' \to \Omega^{(m)}$ bi-interpretable with π with binding groups and fibre groups both equal to G at any point and kernel that determines a Sym(Ω)-congruence on $\Omega^{(m)}$ with classes of infinite cardinality.

In section 5.3 we define the almost-free finite covers. A posteriori we see that the results of sections 3 and 4 concern examples of almost-free finite covers with binding groups equal to the fibre groups at any point. Let W be a transitive structure, ρ be an Aut(W)-congruence on W and $[w_0]$ be a congruence class.

An almost-free finite cover π of W w.r.t ρ is a finite cover whose permutation group induced by its kernel on the union of the fibres of π over $[w_0]$ is isomorphic to the binding group at w_0, while the permutation group induced on the fibres over two elements not in the same congruence class is the direct product of the two respective binding groups. This definition generalizes the definition of free finite cover. More in detail a free finite cover of W is an almost- free finite cover of W with respect to the equality. In Theorem 24 we show how to construct an almost-free finite cover. The proof uses Lemma 2.1.2 of [6].

2 General results

Definition 2 *A* pregeometry *on a set X is a relation between elements $x \in X$ and finite subsets $X_0 \subset X$, called* dependence*, which satisfies:*

- *Reflexivity : x is dependent on $\{x\}$;*
- *Extension: x depends on X_0 and $X_0 \subseteq X_1$ implies x depends on X_1;*
- *Transitivity: x is dependent on X_0 and every $y \in X_0$ is dependent on X_1 implies x is dependent on X_1;*
- *Symmetry: x is dependent on $X_0 \cup \{y\}$ but not on X_0, implies y is dependent on $X_0 \cup \{x\}$.*

Remark 3 *A classical example of a pregeometry is a vector space with linear dependency.*

If Ω is any set then there is a natural topology on $\mathrm{Sym}(\Omega)$ which makes it into a topological group. The open sets are unions of cosets of pointwise stabilizers of finite subsets of Ω. We then make any permutation group P on Ω into a topological group by giving it the relative topology. If Ω is countable the topology is metrisable.

From now on W stands for a countable set, Υ for a closed subgroup of $\mathrm{Sym}(W)$ that acts transitively on W and G for a finite group acting on a finite set Δ. Consider the projection $\pi : \Delta \times W \to W$ given by $\pi(\delta, w) = w$. We denote by G^W the group of functions from W to G. Let $\mathcal{F}$ be the set of closed subgroups of $\mathrm{Sym}(\Delta \times W)$ which satisfy the following conditions:
(i) Every $F \in \mathcal{F}$ preserves the partition of $\Delta \times W$ given by the fibres of π.
(ii) Given $F \in \mathcal{F}$, let $\mu_F : F \to \mathrm{Sym}(W)$ be the naturally induced map given by point (i). We require that, for all $F \in \mathcal{F}$, $\mu_F(F) = \Upsilon$.
(iii) The permutation groups induced respectively by F and $\ker \mu_F$ on $\pi^{-1}(w)$, for all $w \in W$, are both equal to G.

It is easy to see that, with the above topology, G^W is a compact subgroup of $\mathrm{Sym}(\Delta \times W)$ and the $\ker \mu_F$ are closed subgroups of G^W and that μ_F are continuous and open maps (Lemma 1.4.2, [6]). We introduce now a notion of isomorphism among the elements of $\mathcal{F}$. We say that F_1 and F_2 are *isomorphic* if there exists a bijection $\phi : \Delta \times W \to \Delta \times W$ such that $\phi(\pi^{-1}(w)) = \pi^{-1}(w)$, for all $w \in W$ and such that the induced map $f_\phi : \mathrm{Sym}(\Delta \times W) \to \mathrm{Sym}(\Delta \times W)$ sends F_1 to F_2. Let $\mathcal{K} = \{\ker \mu_F, F \in \mathcal{F}\}$. We now introduce an equivalence relation $\mathcal{R}$ on $\mathcal{K}$: we shall say that $K_1 \, \mathcal{R} \, K_2$ if and only if there exists F_1, $F_2 \in \mathcal{F}$ such that $K_1 = \ker \mu_{F_1}$, $K_2 = \ker \mu_{F_2}$ and F_1 is isomorphic to F_2. We shall denote the $\mathcal{R}$-equivalence class of an arbitrary $K \in \mathcal{K}$ by $[K]$. (We shall say that K_1 is *isomorphic* to K_2 if they are equivalent.)

Let $K \in \mathcal{K}$, $H \leq K$ and $w_1, \ldots, w_k \in W$. We define

$$H(w_1, \ldots, w_k) = \{f|_{\{w_1,\ldots,w_k\}} \mid f \in H\}$$

and, for simplicity, we shall refer to $H(w_1, \ldots, w_k)$ as H restricted to $w_1, \ldots, w_k$. If $w \in W$ and Y is a subset of $\Delta \times W$, we shall denote by $K(w/Y)$ the restriction to w of the pointwise stabilizer of Y in K. When using this restriction notation, we view $f \in K \leq G^W$ as a function on W.

The following definition and lemma justify the previous notations:

Definition 4 *Let $K \in \mathcal{K}$. Suppose $w_1, \ldots, w_k, w$ belong to W. We say that w depends on $w_1 \ldots, w_k$ and write $w \in \mathrm{cl}(w_1, \ldots, w_k)$, if*

$$K(w/\pi^{-1}(w_1), \ldots, \pi^{-1}(w_k)) = 1$$

We note that w depends on w_i if and only if $K(w, w_i) \cong G$.

Lemma 5 ([5], Lemma 5.7) *Let $K \in \mathcal{K}$ and $w_1, \ldots, w_k, w \in W$. Then, if G is a simple group, (W, cl) is a Υ-invariant pregeometry. If G is simple non-abelian, then (W, cl) reduces to an equivalence relation.*

The lemma states that, if G is simple non-abelian and w depends on $w_1, \ldots, w_k$, then there is an $i \in \{1, \ldots, k\}$ such that w depends on w_i and (W, cl) reduces to a Υ-congruence.

Corollary 6 *If Υ acts primitively on W and G is a simple non-abelian finite group, then $\mathcal{K} = \{G, G^W\}$.*

Here there is a result on topological groups that will be useful in the next section.

Proposition 7 *Let G be a topological group. Suppose G is metrisable. Let A be a compact subgroup of G and B a closed subgroup of G. Then AB and BA are closed sets.*

Proof. It is sufficient to show that AB is closed. Let $\{c_n\}_{n \in \mathbb{N}}$ be a sequence of elements of AB which converges to c. We have $c_n = a_n b_n$, where $a_n \in A$ and $b_n \in B$. Since A is compact, we can select from the sequence $\{a_n\}_{n \in \mathbb{N}}$ a subsequence $\{a_{n_k}\}$ which converges to an element $a \in A$. We conclude from the convergence of the sequences $\{c_{n_k}\}$ and $\{a_{n_k}\}$ that the sequence $\{b_{n_k}\}$ converges to the element $a^{-1}c$, which belongs to B, since B is closed. Hence $c = a(a^{-1}c) \in AB$ and the closure of the set AB is established. ■

3 Main Theorem

We will denote by $\mathcal{C}$ the set of all Υ-congruences on W.

Definition 8 *Let $\rho \in \mathcal{C}$. We define the subgroup of G^W*

$$K_\rho := \{f : W \to G : f \text{ constant on } Y,\ \forall\, Y \in W/\rho\}.$$

Theorem 9 *Suppose that G is a simple non-abelian finite permutation group acting regularly on Δ. Then there exists a bijection Ψ between $\mathcal{C}$ and $\mathcal{K}/\mathcal{R}$ given by $\Psi(\rho) = [K_\rho]$. The inverse mapping Φ of Ψ is given by $\Phi([K]) = \rho_K$, where ρ_K is defined by:*

$$w_i \rho_K w_j \Leftrightarrow K(w_i, w_j) \cong G.$$

Proof. We first show that Ψ maps $\mathcal{C}$ into $\mathcal{K}/R$.
Let $\rho \in \mathcal{C}$. Then K_ρ is a subgroup of G^W. First of all we embed K_ρ into $G^W \rtimes \Upsilon$ in the natural way:

$$\begin{array}{ccc} K_\rho & \hookrightarrow & G^W \rtimes \Upsilon \\ f & \mapsto & (f,1) \end{array}$$

and then we notice that K_ρ is normalized by Υ. Indeed, given $\sigma \in \Upsilon$, we have that

$$(\sigma(f),1)(\lambda,w) := (1,\sigma)\,(f,1)\,(1,\sigma^{-1})(\lambda,w) = (f(\sigma^{-1}w)\lambda,w).$$

Since $f \in K_\rho$, for every $w_i \in [w_j]_\rho$ in W we have $f(w_i) = f(w_j)$, but, since ρ is a Υ-congruence on W, we have $f(\sigma^{-1}w_i) = f(\sigma^{-1}w_j)$, for every $w_i \in [w_j]_\rho$ and so $(\sigma(f),1) \in K_\rho$.

Since K_ρ is normalized by Υ, we can consider the group:

$$H := K_\rho \rtimes \Upsilon.$$

This is a subgroup of $G^W \rtimes \Upsilon$ and if $\mu : GWr_W\Upsilon \mapsto \Upsilon$ is the map defined by $\mu(f,\gamma) = \gamma$, we then have that $\mu(H) = \Upsilon$ and $\ker\mu = K_\rho$. In order to prove that K_ρ is an element of $\mathcal{K}$ it is sufficient to show that H is a closed subgroup of $G^W \rtimes \Upsilon$. Indeed, $G^W \rtimes \Upsilon$ is closed in $\mathrm{Sym}(\Delta \times W)$.

The first step is to prove that K_ρ is closed. The finite group G has the discrete topology, while G^W has the product topology. An element $f \in G^W$ is a function from W to G. The w-projection map is the map $\pi_w : G^W \to G$ such that $\pi_w(f) = f(w)$. A basis for the product topology on G^W is the family of all finite intersections of $\pi_w^{-1}(U)$, where U is an open subset of G. In this topology the maps π_w are continuous. Hence, a member of this basis is of the form

$$\bigcap\{\pi_w^{-1}(U_w) \,:\, w \in F\}$$

where F is a finite subset of W.

Let $[w]_\rho$ be a ρ-class and g an element of the simple finite group G. By the continuity of π_w, $\pi_w^{-1}(g)$ is a closed subset of G^W. Let

$$M_{[w]_\rho}(g) := \bigcap_{v\in[w]_\rho} \pi_v^{-1}(g).$$

Then $M_{[w]_\rho}(g)$ is a closed set in G^W. We consider next

$$\bigcup_{g\in G} M_{[w]_\rho}(g)$$

and this is still a closed subset of G^W. Then, if Σ is the set of all the equivalence classes of ρ,

$$K_\rho = \bigcap_{[w]_\rho\in\Sigma}\bigcup_{g\in G} M_{[w]_\rho}(g)$$

and so K_ρ is closed in G^W.

Since K_ρ is a closed subgroup of the compact group G^W, K_ρ is compact. By Proposition 7, $H = K_\rho \rtimes \Upsilon$ is closed. Thus, we have shown that Ψ maps $\mathcal{C}$ to $\mathcal{K}/R$.

It is easy to see that the map Φ is well defined. Finally, Lemma 5 shows that $\Phi([K]) \in \mathcal{C}$.

In order to prove that Ψ is a bijection, we show that $\Phi \circ \Psi = \text{id}$ on $\mathcal{C}$. Let ρ be a Υ-congruence on W and let $\Phi([K_\rho]) = \bar{\rho}$. We want to prove that $\rho = \bar{\rho}$.

Let $w_i, w_j \in W$ such that $w_i \,\rho\, w_j$, then for every $f \in K_\rho$, f is constant on the equivalence class $[w_i]_\rho$, i.e. $f(w_i) = f(w_j)$. Hence, $K_\rho(w_i, w_j) \cong G$ and $[w_i]_\rho \subseteq [w_i]_{\bar{\rho}}$. Vice versa, let $w_i \in W$ and suppose there exists $w_j \in W$ such that $w_j \notin [w_i]_\rho$, but $w_j \in [w_i]_{\bar{\rho}}$. Since $w_j \notin [w_i]_\rho$, there exists an $f \in K_\rho$ such that $f(w_i) = g$ and $f(w_j) = 1$, where $g \in G$ and $g \neq 1$. Then $K_\rho(w_i, w_j) = G \times G$ and this yields a contradiction.

We shall finally prove that $\Psi \circ \Phi = \text{id}$.

Let $K \in \mathcal{K}$, $\Phi([K]) = \rho_K$ and

$$\Psi(\Phi([K])) = [K_{\rho_K}].$$

We note that, for every equivalence class $[w]_{\rho_K}$, K restricted to $[w]_{\rho_K}$ is isomorphic to G. This implies that, for every $w \in W$, there exists an automorphism $\alpha_w \in \text{Aut}(G)$ with the following property: given any $f \in K$ and $[w]_{\rho_K}$, there exists $g \in G$ such that $f(\bar{w}) = \alpha_{\bar{w}}(g)$ for all $\bar{w} \in [w]_{\rho_K}$. We denote by $N_{\text{Sym}(\Delta)}(G)$ the normalizer of G in $\text{Sym}(\Delta)$. Since G acts regularly on Δ, for every $w \in W$ there exists n_w belonging to $N_{\text{Sym}(\Delta)}(G)$ such that $\alpha_w(g) = n_w^{-1} g n_w$, for $g \in G$. Consider the function $n : W \to N_{\text{Sym}(\Delta)}(G)$ given by $n(w) = n_w$. Let $F_{\rho_K} \in \mathcal{F}$ be a closed subgroup of $\text{Sym}(\Delta \times W)$ such that $K_{\rho_K} = F_{\rho_K} \cap G^W$. Since F_{ρ_K} is closed, $n^{-1} F_{\rho_K} n$ is closed. In fact, $n^{-1} F_{\rho_K} n \in \mathcal{F}$ and

$$K = n^{-1} K_{\rho_K} n = n^{-1} F_{\rho_K} n \cap G^W.$$

Since n is a bijection of $\Delta \times W$ which preserves the fibres of π, we have that $n^{-1} F_{\rho_K} n$ is isomorphic to F_{ρ_K} and then $[K] = [K_{\rho_K}]$. ∎

Remark 10 *It is clear from the previous proof that in every class $[K] \in \mathcal{K}/R$ there exists $\bar{K} \in [K]$ which is constant on the equivalence classes of $\Phi([K])$.*

4 Special case

Let H be a group acting on a set X, $a \in X$ and $\Delta \subseteq X$. We denote by $a^H = \{ha : h \in H\}$, by $H_{(\Delta)}$ the pointwise stabilizer of Δ in H and by

$H_{\{\Delta\}}$ the setwise stabilizer of Δ in H. We recall the following theorem, whose proof can be found in [4].

Theorem 11 ([4], Theorem 1.5A) *Let G be a group which acts transitively on a set Ω, and let $\alpha \in \Omega$. Let $\mathcal{D}$ be the set of blocks Δ for G containing α, let $\mathcal{H}$ denote the set of all subgroups H of G with $G_\alpha \leq H$. There is a bijection Ψ from $\mathcal{D}$ onto $\mathcal{H}$ given by $\Psi(\Delta) := G_{\{\Delta\}}$ whose inverse mapping Φ is given by $\Phi(H) := \alpha^H$. The mapping Ψ is order preserving in the sense that if $\Delta, \Theta \in \mathcal{D}$ then $\Delta \subseteq \Theta \Longleftrightarrow \Psi(\Delta) \leq \Psi(\Theta)$.*

From now on let W be $\Omega^{(n)}$, the set of ordered n-tuples of distinct elements of the countable set Ω. Let $\Upsilon = \mathrm{Sym}(\Omega)$ act on $\Omega^{(n)}$ in the natural way: if $\sigma \in \mathrm{Sym}(\Omega)$, then $\sigma(a_1, \dots, a_n) = (\sigma(a_1), \dots, \sigma(a_n))$. In the sequel we denote $\mathrm{Sym}(\Omega)$ by S when $\mathrm{Sym}(\Omega)$ acts on Ω. Let ρ be a Υ-congruence, and $\Delta \subseteq \Omega^{(n)}$ be the equivalence class of ρ containing the element $\alpha = (a_1, \dots, a_n)$. We will refer to Δ as a block of imprimitivity containing α.

Definition 12 *Let $\alpha = (a_1, \dots, a_n) \in \Omega^{(n)}$. We define*

$$\mathrm{supp}(\alpha) := \{a_1, \dots, a_n\}.$$

By Theorem 11, the subgroup $\Upsilon_{\{\Delta\}} = \{x \in \Upsilon | \quad x\Delta = \Delta\}$ contains the stabilizer $\Upsilon_\alpha = S_{(a_1, \dots, a_n)}$. A proof of the following lemma can be found in [4].

Lemma 13 ([4] Lemma 8.4B) *Let Σ_1 and Σ_2 be subsets of an arbitrary set Ω such that $|\Sigma_1 \cap \Sigma_2| = |\Sigma_1| \leq |\Sigma_2|$. Then*

$$\langle Sym(\Sigma_1), Sym(\Sigma_2) \rangle = Sym(\Sigma_1 \cup \Sigma_2),$$

(we identify $Sym(\Sigma)$ with the pointwise stabilizer of $\Omega \setminus \Sigma$).

Proposition 14 *Let $\alpha = (a_1, \dots, a_n) \in \Omega^{(n)}$. Let $\Delta \neq \Omega^{(n)}$ be a block containing α. Let $\{\Gamma_i\}_{i \in I}$ be the set of finite subsets of Ω such that*

$$\Upsilon_\alpha \leq S_{(\Gamma_i)} \leq \Upsilon_{\{\Delta\}}.$$

Let $\Gamma = \bigcap_{i \in I} \Gamma_i$. Then

$$\Upsilon_\alpha \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}.$$

Moreover Γ is finite and $\Gamma \subseteq \{a_1, \dots, a_n\}$.

Proof. We notice that the index set I is non-empty: for instance the set $\{a_1, \dots, a_n\}$ belongs to $\{\Gamma_i\}_{i \in I}$. Moreover, it is finite since every $\Gamma_i \subseteq \{a_1, \dots, a_n\}$. In order to prove that $\Upsilon_\alpha \leq S_{(\Gamma)}$ it is sufficient to notice that for every $i \in I$, $\Gamma \subseteq \Gamma_i$. Then $\Upsilon_\alpha \leq S_{(\Gamma_i)} \leq S_{(\Gamma)}$, for every $i \in I$.

We use Lemma 13 to prove the inclusion $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$. Let $\Sigma_i = \Omega \setminus \Gamma_i$, for $i \in I$. Then by Lemma 13 we have $\langle S_{(\Gamma_i)}, i \in I\rangle = S_{(\bigcap_{i\in I} \Gamma_i)}$ and so $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$.

Notice that Γ is the smallest subset of Ω such that $\Upsilon_\alpha \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$. We want to prove the set Γ has the smallest cardinality among the finite sets X of Ω such that $S_{(X)} \leq \Upsilon_{\{\Delta\}}$. Suppose not, then there exists a finite subset of Ω, say Σ, with $|\Sigma| \lneqq |\Gamma|$ and $S_{(\Sigma)} \leq \Upsilon_{\{\Delta\}}$. By Lemma 13, we have

$$\Upsilon_\alpha \leq S_{(\Gamma)} \leq S_{(\Gamma\cap\Sigma)} \leq \Upsilon_{\{\Delta\}}$$

and, since Γ is the smallest subset of Ω such that $\Upsilon_\alpha \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$, this yields a contradiction. Thus, the set Γ has the smallest cardinality among the finite subsets X of Ω such that $S_{(X)} \leq \Upsilon_{\{\Delta\}}$.

Let $x \in \Upsilon_{\{\Delta\}}$, then we have $S_{(x\Gamma)} = x^{-1} S_{(\Gamma)} x \leq \Upsilon_{\{\Delta\}}$, and so, applying again Lemma 13 we get that $\Upsilon_{\{\Delta\}} \geq \langle S_{(\Gamma)}, S_{(x\Gamma)}\rangle = S_{(\Gamma\cap x\Gamma)}$. Thus, for all $x \in \Upsilon_{\{\Delta\}}$, $\Gamma = x\Gamma$ by the minimality of Γ and $\Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}$. ■

As the following result shows, a ρ-class in $\Omega^{(n)}$ can be a finite subset or an infinite subset of $\Omega^{(n)}$.

Proposition 15 *Let $\Delta \neq \Omega^{(n)}$ be the equivalence class of a Υ-congruence ρ containing the element $(a_1, \ldots, a_n) \in \Omega^{(n)}$. Then*
a) *Δ is finite if and only if $S_{(a_1,\ldots,a_n)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{a_1,\ldots,a_n\}}$;*
b) *Δ is a countably infinite set if and only if $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}$, for some finite set $\Gamma \subsetneqq \{a_1, \ldots, a_n\}$.*

Proof.

a) Suppose Δ is a finite set in $\Omega^{(n)}$. If there doesn't exist any $\Gamma \subsetneqq \{a_1, \ldots, a_n\}$ such that $S_{(a_1,\ldots,a_n)} \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$, then by Proposition 14, since $S_{(a_1,\ldots,a_n)} \leq \Upsilon_{\{\Delta\}}$, we have $S_{(a_1,\ldots,a_n)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{a_1,\ldots,a_n\}}$.

Hence, suppose that there exists a finite set $\Gamma \subsetneqq \{a_1, \ldots, a_n\}$ such that $S_{(a_1,\ldots,a_n)} \lneqq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$. Let $x \in S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$, then $x\Delta = \Delta$. Take $a_i \in \{a_1, \ldots, a_n\} \setminus \Gamma$. Then pick $a \in \Omega$ such that $a \notin \mathrm{supp}(\delta)$, for every $\delta \in \Delta$. By k-transitivity of S, for any $k \in \mathbb{N}$, it is possible to choose an element x in $S_{(\Gamma)}$, such that $x(a_i) = a$. Then

$$x(a_1, \ldots, a_i, \ldots, a_n) = (x(a_1), \ldots, a, \ldots, x(a_n)) \in \Delta.$$

But this yields a contradiction, since $a \notin \mathrm{supp}(\delta)$, for every $\delta \in \Delta$.

In the other direction, if $S_{(a_1,\ldots,a_n)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{a_1,\ldots,a_n\}}$, then

$$\Delta = (a_1, \ldots, a_n)^{\Upsilon_{\{\Delta\}}} \subseteq (a_1, \ldots, a_n)^{S_{\{a_1,\ldots,a_n\}}}$$

and $|(a_1, \ldots, a_n)^{S_{\{a_1,\ldots,a_n\}}}|$ is finite.

b) We now assume Δ is a countably infinite set. Suppose there does not exist any finite set $\Gamma_0 \subsetneqq \{a_1, \ldots, a_n\}$ such that $S_{(\Gamma_0)} \leq \Upsilon_{\{\Delta\}}$. By

Theorem 11 we have that $S_{(a_1,\ldots,a_n)} \leq \Upsilon_{\{\Delta\}}$. Since for every finite set $\Gamma_0 \subsetneqq \{a_1,\ldots,a_n\}$ we have $S_{(\Gamma_0)} \nleq \Upsilon_{\{\Delta\}}$, then $\{a_1,\ldots,a_n\}$ is the smallest subset of Ω such that $S_{(a_1,\ldots,a_n)} \leq \Upsilon_{\{\Delta\}}$ and so, by Proposition 14, $\Upsilon_{\{\Delta\}} \leq S_{\{a_1,\ldots,a_n\}}$. Take an element $(b_1,\ldots,b_n)$ of Δ, such that $\{b_1,\ldots,b_n\} \neq \{a_1,\ldots,a_n\}$; as Δ is infinite, this element exists. By the n-transitivity of S, there exists an element $x \in S$ such that $x(a_1) = b_1, \ldots, x(a_n) = b_n$. Then $x(a_1,\ldots,a_n) \in \Delta$ and so we have an element $x \in \Upsilon_{\{\Delta\}}$ but not in $S_{\{a_1,\ldots,a_n\}}$. This yields a contradiction. So there exists at least a set $\Gamma_0 \subsetneqq \{a_1,\ldots,a_n\}$ such that $S_{(\Gamma_0)} \leq \Upsilon_{\{\Delta\}}$. Take Γ to be the intersection of all such sets. By Proposition 14 we have that $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}$.

Conversely suppose $\Gamma \subsetneqq \{a_1,\ldots,a_n\}$, and $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}$. Then $(a_1,\ldots,a_n)^{S_{(\Gamma)}} \subseteq \Delta$, and since $(a_1,\ldots,a_n)^{S_{(\Gamma)}}$ is infinite, then Δ is infinite. ■

Remark 16 *If $|\Gamma| = n$, $n \geq 1$, then $S_{(\Gamma)} \trianglelefteq S_{\{\Gamma\}}$ and $S_{\{\Gamma\}}/S_{(\Gamma)} \cong Sym_n$ the symmetric group on n points. Given an element $\alpha = (a_1,\ldots,a_n) \in \Omega^{(n)}$ and a finite block Δ containing it, we have that $H = \Upsilon_{\{\Delta\}}$ satisfies the following inclusions: $S_{(\Gamma)} \leq H \leq S_{\{\Gamma\}} \leq S$, where $\Gamma = \{a_1,\ldots,a_n\}$. Then $H/S_{(\Gamma)}$ is isomorphic to a subgroup of Sym_n. There exists a bijection Θ between the subgroups of* Sym_n *and the subgroups of $S_{\{\Gamma\}}$ which contain $S_{(\Gamma)}$.*

Put

$$\mathcal{K}_F = \{K \in \mathcal{K} |\, \rho_K \text{ has finite equivalence classes}\}.$$

Proposition 17 *Let $\mathcal{L}$ be the set of subgroups of Sym_n. Then there exists a bijection*

$$\zeta : \mathcal{K}_F/\mathcal{R} \to \mathcal{L}.$$

Proof. By Theorem 9, it is sufficient to find a bijection between the set of finite blocks containing an element $\alpha = (a_1,\ldots,a_n)$ and $\mathcal{L}$. Let Δ be a finite block in $\Omega^{(n)}$ containing α. We have that

$$\Upsilon_\alpha = S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}} \leq S$$

where Γ =supp(α). Then by Remark 16, $\Upsilon_{\{\Delta\}}$ is the image by Θ of a subgroup of Sym_n. If $\Delta_1 \neq \Delta_2$ then $\Upsilon_{\{\Delta_1\}} \neq \Upsilon_{\{\Delta_2\}}$. By Remark 16, it follows that the map ζ is injective. In the other direction, let $H \in \mathcal{L}$. By the remark 16, $\Theta(H)$ is a subgroup L of $S_{\{\Gamma\}}$ which contains $\Upsilon_\alpha = S_{(\Gamma)}$. Then, by Theorem 11, we have a finite block α^L containing α. ■

Proposition 18 *Let $\alpha = (a_1,\ldots,a_n) \in \Omega^{(n)}$ and let $\mathcal{D}_F^\alpha$ be the set of the finite blocks in $\Omega^{(n)}$ containing α. Then the elements of $\mathcal{D}_F^\alpha$ are exactly the sets α^H, where H is a subgroup of* $\mathrm{Sym}\{a_1,\ldots,a_n\}$.

Proof. Let $\Delta \in \mathcal{D}_F^\alpha$. Let H' be the subgroup of $\mathrm{Sym}(\Omega)$ such that $\alpha^{H'} = \Delta$. Then

$$\Upsilon_\alpha \leq H' \leq S_{\{\Gamma\}},$$

where $\Gamma = \operatorname{supp}(\alpha)$. Since $S_{\{\Gamma\}}/S_{(\Gamma)} \cong Sym_n$ we have that $H' = H \times \operatorname{Sym}(\Omega \setminus \Gamma)$, where H is a subgroup of $\operatorname{Sym}\{a_1, \ldots, a_n\}$. Then $\Delta = \alpha^H$. Conversely, taken a subgroup $H \leq \operatorname{Sym}\{a_1, \ldots, a_n\}$, $\alpha^H = \alpha^{H \times \operatorname{Sym}(\Omega \setminus \Gamma)}$. By Theorem 11 α^H is a block in $\Omega^{(n)}$. ■

The same argument works for the following:

Proposition 19 *Let $\alpha = (a_1, \ldots, a_n) \in \Omega^{(n)}$ and let $\mathcal{D}_I^\alpha$ be the set of non-trivial infinite blocks in $\Omega^{(n)}$ containing α. Then the elements of $\mathcal{D}_I^\alpha$ are exactly the sets $\alpha^{L \times \operatorname{Sym}(\Omega \setminus \Xi)}$, where $\Xi \subsetneq \{\alpha_1, \ldots, \alpha_n\}$ and L is a subgroup of* $\operatorname{Sym}(\Xi)$.

Let us mention a little remark about Proposition 18. Let $\alpha = (a_1, \ldots, a_n)$. Denote $\operatorname{Sym}\{a_1, \ldots, a_n\}$ by Sym_n. Consider the set

$$\alpha^{\operatorname{Sym}_n} = \{\sigma(a_1, \ldots, a_n), \sigma \in \operatorname{Sym}_n\}.$$

Let $[K] \in \mathcal{K}_F/\mathcal{R}$, and $\bar{K} \in [K]$ be the largest subgroup of G^W which is constant on the equivalence classes of $\Phi(K)$. By Proposition 18 there exists a subgroup T of Sym_n such that K restricted to $\Delta = \alpha^T$ is constant on it. The block system which includes Δ is $\{g\Delta : g \in \operatorname{Sym}(\Omega)\}$. We look at the restriction of $\bar{K}$ to the set $\alpha^{\operatorname{Sym}_n}$. This is the subgroup of $G^{\alpha^{\operatorname{Sym}_n}}$ of the function from $\alpha^{\operatorname{Sym}_n}$ to G constant on the subsets $bT(\alpha)$, where bT are the left cosets of T in Sym_n. We notice that the cardinalities of the finite blocks in $\Omega^{(n)}$ are exactly the cardinalities of the subgroups of Sym_n.

5 Commentary

5.1 Finite Covers

As is well known, a subgroup of $\operatorname{Sym}(W)$ is closed if and only if it is the group of automorphisms of some first-order structure with domain W (see for instance Proposition (2.6) in [3]). Thus we state the following definition.

A *permutation structure* is a pair $\langle W, G \rangle$, where W is a non-empty set (the *domain*), and G is a closed subgroup of $\operatorname{Sym}(W)$. We refer to G as the automorphism group of W. If A and B are subsets of W (or more generally of some set on which $\operatorname{Aut}(W)$ acts), we shall refer to $\operatorname{Aut}(A/B)$ as the group of permutations of A which extend to elements of $\operatorname{Aut}(W)$ fixing every element of B and to $\operatorname{Aut}(A/\{B\})$ as the group of permutations of A which extend to elements of $\operatorname{Aut}(W)$ stabilizing setwise the set B.

Permutation structures are obtained by taking automorphism groups of first-order structures and we often regard a first-order structure as a permutation structure without explicitly saying so. When $\pi : C \to W$ is a finite cover (Definition 1), we frequently use the notation $C(w)$ to denote the fibre $\pi^{-1}(w)$ above w in the cover $\pi : C \to W$.

We recall that the *fibre group* $F(w)$ of π on $C(w)$ is $\mathrm{Aut}(C(w)/w)$, while the *binding group* $B(w)$ of π on $C(w)$ is $\mathrm{Aut}(C(w)/W)$. It follows that the binding group is a normal subgroup of the fibre group. If $\mathrm{Aut}(W)$ acts transitively on W, then all the fibre groups are isomorphic as permutation groups, as are the binding groups. There is a continuous epimorphism $\chi_w : \mathrm{Aut}(W/w) \to F(w)/B(w)$ called the *canonical epimorphism* (Lemma 2.1.1 [6]). Thus if $\mathrm{Aut}(W/w)$ has no proper open subgroup of finite index, then $F(w) = B(w)$.

Let $\pi_1 : C_1 \to W$ and $\pi_2 : C_2 \to W$ be two finite covers of W. Then π_1 is said to be *isomorphic over* W to π_2 if there exists a bijection $\alpha : C_1 \to C_2$ with $\alpha(\pi_1^{-1}(w)) = \pi_2(w)$ for all $w \in W$, such that the induced map $f_\alpha : \mathrm{Sym}(C_1) \to \mathrm{Sym}(C_2)$ satisfies $f_\alpha(\mathrm{Aut}(C_1)) = \mathrm{Aut}(C_2)$.

The *Cover Problem* is, given W and data $(F(w), B(w), \chi_w)$, to determine (up to isomorphism) the possible finite covers with these data.

If C and C' are permutation structures with the same domain and $\pi : C \to W$, $\pi' : C' \to W$ are finite covers with $\pi(c) = \pi'(c)$ for all $c \in C = C'$, we say that π' is a *covering expansion* of π if $\mathrm{Aut}(C') \leq \mathrm{Aut}(C)$.

Suppose that C and W are two permutation structures and $\pi : C \to W$ is a finite cover. The cover is *free* if

$$\mathrm{Aut}(C/W) = \prod_{w \in W} \mathrm{Aut}(C(w)/W),$$

that is, the kernel is the full direct product of the binding groups.

The existence of a free finite cover with prescribed data depends on the existence of a certain continuous epimorphism.

Indeed, let W be a transitive permutation structure and $w_0 \in W$. Given a permutation group F on a finite set X, a normal subgroup B of F and a continuous epimorphism

$$\chi : \mathrm{Aut}(W/w_0) \to F/B,$$

then there exists a free finite cover $\sigma : M \to W$ with fibre and binding groups at w_0 equal to F and B, and such that the canonical epimorphism χ_{w_0} is equal to χ . With these properties σ is determined uniquely (see [6], Lemma 2.1.2).

A *principal* cover $\pi : C \to W$ is a free finite cover where the fibre and binding groups at each point are equal. Free covers are useful in describing finite covers with given data because every finite cover $\pi : C \to W$ is an expansion of a free finite cover with the same fibre groups, binding groups and canonical homomorphisms as in π (see [6], Lemma 2.1.3).

Let's go back to Section 2. Using the language of finite covers, $\mathcal{F}$ is the set of the expansions of the principal finite covers of $\langle W, \Upsilon \rangle$, with all fibre groups and binding groups equal to a given group G.

In the case when G is a simple non-abelian regular group, our main theorem shows that the Υ-congruences on W describe (up to isomorphisms over W) the kernels of expansions belonging to $\mathcal{F}$.

5.2 Bi-interpretability

Definition 20 *Two permutation structures are bi-interpretable if their automorphism groups are isomorphic as topological groups.*

For a model-theoretic interpretation, if the permutation structures arise from $\aleph_0$-categorical structures, see Ahlbrandt and Ziegler ([1]). Usually classification of structures is up to bi-interpretability.

Let $n \in \mathbb{N}$. Consider $\Omega^{(n)}$ as a first-order structure with automorphism group equal to $\mathrm{Sym}(\Omega)$.

Proposition 21 *Let $M_1 := \Delta \times \Omega^{(n)}$ and $\pi_1 : M_1 \to \Omega^{(n)}$ be a finite cover of $\Omega^{(n)}$ with all binding groups and fibre groups equal to a simple non-abelian finite group G acting on Δ. Let K_1 be the kernel of π_1.*

Suppose that the congruence classes which K_1 determine have finite cardinality. Then, $\forall m > n$ there exists a permutation structure $M_2 := \Delta \times \Omega^{(m)}$ and a finite cover $\pi_2 : M_2 \to \Omega^{(m)}$ with all fibre groups and binding groups equal to G such that M_1 is bi-interpretable with M_2 and the kernel K_2 of π_2 determines a $\mathrm{Sym}(\Omega)$*-congruence with equivalence classes of infinite cardinality.*

Proof. By the notation $M_1(\alpha)$, we mean the copy of Δ over the element $\alpha \in \Omega^{(n)}$. The kernel K_1, by Lemma 5, determines a $\mathrm{Sym}(\Omega)$-congruence ρ which, by hypothesis, has equivalence classes of finite cardinality. Let m be a positive integer greater than n and M_2 be the set

$M_2 := \{(\delta, w) : w = (\alpha, c_1, .., c_{m-n}) \in \Omega^{(m)}$ where $\alpha \in \Omega^{(n)}$ and $\delta \in M_1(\alpha)\}$

Obviously $M_2 = \Delta \times \Omega^{(m)}$. Let $\mu_1 : \mathrm{Aut}(M_1) \to \mathrm{Sym}(\Omega)$ be the map induced by π_1 and Λ be the subgroup of $\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega)$

$$\Lambda = \{(\sigma, g) : g = \mu_1(\sigma)\}.$$

Our claim is to show that $\langle M_2, \Lambda \rangle$ is a permutation structure and that $\pi_2 : M_2 \to \Omega^{(m)}$ given by $\pi_2(w, m) = w$ is a finite cover of $\Omega^{(m)}$ with $F(w) = B(w) = G$ and kernel K_2 which determines a $\mathrm{Sym}(\Omega)$-congruence with equivalence classes of infinite cardinality.

It is easy to check that Λ is a permutation group on M_2 which preserves the partition of M_2 given by the fibres of π_2.

We equip $\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega)$ with the product topology. This topology coincides with the topology of the pointwise convergence induced by $\mathrm{Sym}(M_1 \times \Omega^{(m)})$ on $\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega)$. The map Φ given by

$$\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega) \stackrel{p_2}{\to} \mathrm{Sym}(\Omega)$$

and the map Ψ given by

$$\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega) \stackrel{p_1}{\to} \mathrm{Aut}(M_1) \stackrel{\mu_1}{\to} \mathrm{Sym}(\Omega)$$

where p_1 and p_2 are the projections on the first and second component, respectively, are continuous. The permutation group Λ is equal to the difference kernel

$$Z = \{(\sigma, g) \in \mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega) : \Psi(\sigma, g) = \Phi(\sigma, g)\}$$

which, by Proposition 3 page 30 of [8], is closed in $\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega)$. Moreover, $\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega)$ is closed in $\mathrm{Sym}(M_1 \times \Omega^{(m)})$ and then $\langle M_2, \Lambda \rangle$ is a permutation structure. The usual map induced by π_2

$$\mu_2 : \Lambda \to \mathrm{Sym}(\Omega^{(m)})$$

has image $\mathrm{Sym}(\Omega)$. The kernel of μ_2, which we denote by K_2, is

$$K_2 = \{(\sigma, id) \in \Lambda : \sigma \in K_1\}.$$

Then $K_1 \cong K_2$. Let $(\delta, w) = (\delta, \alpha, c_1, \ldots, c_{m-n}) \in M_2$ where $\alpha \in \Omega^{(n)}$ and $c_1, \ldots, c_{m-n} \in \Omega \setminus \mathrm{supp}(\alpha)$ and are all distinct. Let (σ, id) be an element in K_2. If we restrict it to the fibre over w, we see that it is the same as restricting σ to the fibre over α. Hence the binding group over w, $B_2(w)$, is clearly isomorphic to G. The same holds for the fibre group: let $w = (\alpha, c_1, \ldots, c_{m-n})$, then $F_2(w)$ is the restriction of the group

$$\mathrm{Aut}(M_2/w) = \{(\sigma, g) \in \Lambda : g \in \mathrm{Sym}(\Omega)_{((\alpha, c_1, \ldots, c_{m-n})}\}$$

to the fibre over w. Since $g \in \mathrm{Sym}(\Omega)_{((\alpha, c_1, \ldots, c_{m-n})}\}$ then $g \in \mathrm{Sym}(\Omega)_{(\alpha)}$. Hence $\sigma \in \mathrm{Aut}(M_1/\alpha)$ and so $F_2(w)$ is isomorphic to G.

Moreover, if we consider two points of $\Omega^{(m)}$, say $w = (\alpha, c_1, \ldots, c_{m-n})$ and $w' = (\alpha', c'_1, \ldots, c'_{m-n})$, with $\alpha \rho \alpha'$, we have that $K_2(w, w') \cong G$. Vice versa if $K_2(w, w') \cong G$, it means that $K_1(\alpha, \alpha') \cong G$. Then the $\mathrm{Sym}(\Omega)$-congruence, ρ', that K_2 determines is given by $w \rho' w'$ if and only if $\alpha \rho \alpha'$. In the equivalence class of $w = (\alpha, c_1, \ldots, c_{m-n})$ for instance there are all the elements of the form $(\alpha, c'_1, \ldots, c'_{m-n})$, with $c_1, \ldots, c_{m-n} \in \Omega \setminus \mathrm{supp}(\alpha)$ and pairwise distinct. Then the equivalence classes of ρ' are of infinite cardinality.

Next we check the bi-interpretability. We consider the map

$$\begin{array}{rccc} \beta: & \Lambda & \to & \mathrm{Aut}(M_1) \\ & (\sigma, g) & \mapsto & \sigma \end{array}$$

The kernel of β is $\ker\beta = \{(id, g) \in \Lambda : g = \mu_1(id)\}$. Then β is injective. It is also surjective since, given $\sigma \in \mathrm{Aut}(M_1)$, $(\sigma, \mu_1(\sigma)) \in \Lambda$. Clearly the inverse map is given by $\beta^{-1}(\sigma) = (\sigma, \mu_1(\sigma))$.

It is a topological isomorphism. Indeed, take a basic open neighbourhood of the identity in $\mathrm{Aut}(M_1)$, say $\mathrm{Aut}(M_1)_{(\Gamma)}$, where $\Gamma = \{\delta_i\}_{i\in I}$ is a finite set of M_1. Each $\delta_i \in M_1(\alpha_i)$. Then

$$\beta^{-1}(\mathrm{Aut}(M_1)_{(\Gamma)}) = \{(\sigma, \mu_1(\sigma)) : \sigma \in \mathrm{Aut}(M_1)_{(\Gamma)}\}.$$

For each α_i, we choose $c_1^i, \ldots, c_{m-n}^i \in \Omega$ such that $w_i = (\alpha_i, c_1^i, \ldots, c_{m-n}^i)$ is an extension of α_i to an element of $\Omega^{(m)}$. The map

$$\begin{array}{rccc} \beta^{-1}: & \mathrm{Aut}(M_1) & \to & \mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega) \\ & \sigma & \mapsto & (\sigma, \mu_1(\sigma)) \end{array}$$

is continuous. The image of β^{-1} is Λ and as Λ has the topology induced by $\mathrm{Aut}(M_1) \times \mathrm{Sym}(\Omega)$, then $\beta^{-1} : \mathrm{Aut}(M_1) \to \Lambda$ is continuous. Hence, we have proved the bi-interpretability. ■

5.3 Almost-free finite covers

Let W be a transitive structure, ρ an $\mathrm{Aut}(W)$-congruence on W and $\pi : C \to W$ a finite cover. Given a ρ-equivalence class $[w]$, we shall denote by $C([w]) = \bigcup_{w_i\in[w]} C(w_i)$, by $F^\pi([w])$ the permutation group induced by $\mathrm{Aut}(C/\{[w]\})$ on $C([w])$, and by $B^\pi([w])$ the permutation group induced by the kernel of π on $C([w])$. Note that $B^\pi([w]) \trianglelefteq F^\pi([w])$.

Lemma 22 *Let W be a transitive structure, ρ an $\mathrm{Aut}(W)$-congruence on W and $\pi : C \to W$ be a finite cover. Then, for every ρ-class $[w]$ in W*

1. *there exists a finite-to-one surjection*

$$\pi_{[w]} : C([w]) \to [w]$$

such that its fibres form an $F^\pi([w])$-invariant partition of $C([w])$;

2. *there is a continuous epimorphism*

$$\chi^\pi_{[w]} : \mathrm{Aut}(W/\{[w]\}) \to F^\pi([w])/B^\pi([w]).$$

Proof. The first point is clear.

The second point requires a little proof. Let $g \in \mathrm{Aut}(W/\{[w]\})$. Then there exists $h \in \mathrm{Aut}(C/\{[w]\})$ which extends g. Let $\psi : \mathrm{Aut}(W/\{[w]\}) \to \mathrm{Aut}(C/\{[w]\})/\mathrm{Aut}(C/W)$ be the map defined by $\psi(g) = h\,\mathrm{Aut}(C/W)$. This map is well defined. Suppose that also $\bar{h}$ extends g. Then $h^{-1}\bar{h} \in \mathrm{Aut}(C/W)$ and so $h\,\mathrm{Aut}(C/W) = \bar{h}\,\mathrm{Aut}(C/W)$. Consider the restriction to the set of fibres over $\{[w]\}$. So we have a map $\xi_{[w]} : \mathrm{Aut}(C/\{[w]\})/\mathrm{Aut}(C/W) \to \mathrm{Sym}(C([w])/B^{\pi}([w])$, given by $\xi_{[w]}(h\,\mathrm{Aut}(C/W)) = h_{|C([w])}B^{\pi}([w])$, which is clearly onto $F^{\pi}([w])/B^{\pi}([w])$. Let $g \in \mathrm{Aut}(W/\{[w]\})$. We define $\chi^{\pi}_{[w]}(g) := \xi_{[w]}\psi(g)$. In order to prove that $\chi^{\pi}_{[w]}$ is continuous, we show that ψ and $\xi_{[w]}$ are continuous.

The restriction map $\xi_{[w]}$ is continuous by Lemma 1.4.1 of [6]. Consider $\mathrm{Sym}(C([w])$ with the topology of pointwise convergence and $\mathrm{Sym}(C([w])/B^{\pi}([w])$ with the quotient topology. Let $\mu_{|\,\mathrm{Aut}(C/\{[w]\})} : \mathrm{Aut}(C/\{[w]\}) \to \mathrm{Aut}(W/\{[w]\})$ be the map induced by μ. Since $[w]$ is a ρ-equivalence class $\mathrm{Aut}(C/\{[w]\})$ is an open subgroup of $\mathrm{Aut}(C)$. Indeed, let $c \in C([w])$. Take $h \in \mathrm{Aut}(C/c)$. Then $h(C([w])) = C([w])$. If $g = \mu(h)$, we have $g(w) = w$, and $[w]$ being an $\mathrm{Aut}(W)$-congruence class, this implies that $g([w]) = [w]$. Hence $\mathrm{Aut}(C/c) \subseteq \mathrm{Aut}(C/\{[w]\})$ which implies that $\mathrm{Aut}(C/\{[w]\})$ is an open subgroup of $\mathrm{Aut}(C)$. By the same reasoning we get that $\mathrm{Aut}(W/\{[w]\})$ is open in $\mathrm{Aut}(W)$. Now, since μ is open also $\mu_{|\,\mathrm{Aut}(C/\{[w]\})}$ will be open. Hence by Proposition 1, page 21 of [8], we have the continuity of ψ. ■

Definition 23 *Let W be a transitive structure and ρ an* $\mathrm{Aut}(W)$-*congruence on W. Let $\pi : C \to W$ be a finite cover of W, $w \in W$, with binding groups isomorphic to a group G and kernel K. We shall say that π is* **almost free** *with respect to ρ if*

1. *$K([w]) \cong G$ for each $[w] \in W/\rho$*
2. *$K(w_1, w_2) \cong G \times G$ for each $w_2 \notin [w_1]$.*

For a class of almost free finite covers take for example the set of finite covers of a transitive structure with all the binding groups and fibre groups isomorphic to a simple non-abelian group G.

Let $R := W/\rho$. Given a transitive structure W and an $\mathrm{Aut}(W)$-congruence ρ, naturally we have an induced map

$$M : \mathrm{Aut}(W) \to \mathrm{Sym}(R).$$

The map M is continuous, but the image of $\mathrm{Aut}(W)$ by M is not necessarily closed. The following counterexample is due to Peter Cameron (private communication).

Take the generic bipartite graph B, and consider the group G of automorphisms fixing the two bipartite blocks, acting on the set of edges of the graph. On the set of edges there are two equivalence relations, "same vertex in the first bipartite block", and "same vertex in the second bipartite block". Clearly G is precisely the group preserving these two equivalence relations, and so is closed. But the group induced on the set of equivalence classes of each relation is highly transitive and not the symmetric group, therefore not closed.

Let C and W be two structures. We shall call $\pi : C \to W$ a *cover* of W if π is a surjective map, the fibres of π form an $\mathrm{Aut}(C)$ -invariant partition of C and the induced map $\mu : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$ has image $\mathrm{Aut}(W)$. The difference with finite covers is that here we don't require the fibres of the cover to have finite cardinality. We shall extend the terminology of finite covers to covers in the obvious way: for example the fibre group at $w \in W$ of π will be $\mathrm{Aut}(C(w)/w)$, the binding group at w $\mathrm{Aut}(C(w)/W)$ and so on. In particular, a cover π is a free cover if the kernel of π is the full direct product of all its binding groups.

Theorem 24 *Let W be a countable $\aleph_0$-categorical transitive structure and ρ an $\mathrm{Aut}(W)$-congruence on W. We suppose that the following assumptions hold:*

1. *Let F be a closed permutation group on a set X. Fix $w_0 \in W$ and let $[w_0]$ be the ρ-equivalence class of w_0.*

   Suppose that there exists a finite -to-one surjection

   $$\sigma : X \to [w_0]$$

   such that the fibres form an F-invariant partition of X and that the induced map $T : F \to \mathrm{Sym}([w_0])$ has image $\mathrm{Aut}(W/\{[w_0]\}_{|\{[w_0]\}}$.

2. *Let B be the kernel of T and G be the permutation group induced by B on $\sigma^{-1}(w_0)$. Suppose that B is isomorphic to G and that the index of B in F is at most countable.*

3. *Assume that the map M is injective, open and with closed image.*

The map T induces a map $\chi : \mathrm{Aut}(W/\{[w_0]\}) \to F/B$ defined as $\chi(g) = hB$, where $h \in F$ and $T(h) = g_{|[w_0]}$.

Then there exists an almost free finite cover π_0 of W with respect to ρ with binding groups isomorphic to G, $F^{\pi_0}([w_0]) = F$, $B^{\pi_0}([w_0]) = B$ and map $\chi^{\pi_0}_{[w_0]}$ equal to χ. Moreover, if $\tilde{\pi}_0$ is an almost free finite cover with respect to ρ with $F^{\tilde{\pi}_0}([w_0])$ and $B^{\tilde{\pi}_0}([w_0])$ isomorphic as permutation groups to F and B respectively via a bijection γ such that $\gamma(\tilde{\pi}_0^{-1}(w)) = \pi_0^{-1}(w)$

for every $w \in [w_0]$*, and* $\chi^{\tilde{\pi}_0}_{[w_0]}$ *equal to* χ *(up to isomorphism), then* $\tilde{\pi}_0$ *is isomorphic over* W *to* π_0*.*

Proof. We give to R the first-order structure with automorphism group the image of M. Let $r_0 = [w_0]$. It is easy to see that the map χ and the map

$$M^{-1} : \mathrm{Aut}(R/r_0) \to \mathrm{Aut}(W/\{[w_0]\})$$

are continuous. If we compose the map M^{-1} with χ, we obtain a continuous map from $\mathrm{Aut}(R/r_0)$ to F/B, which we continue to denote by χ. The reason of using the same notation for the two χ is that they are essentially the same from a group theory point of view, since M^{-1} is an isomorphism. In the proof it will be clear from the context which one of the two maps we are referring to. The proof is developed in a series of steps.

Step a) We are going to build a free cover $\pi : S \to R$ with fibre group and binding group at r_0 respectively equal to F and B, with fibre $S(r_0)$ equal to X and canonical epimorphism at r_0 equal to χ. After having built the free cover π, we are going to prove that if ν is a free cover with fibre and binding group at r_0 isomorphic as permutation groups to F and B respectively, and χ_{r_0} equal to χ (up to isomorphism), then ν is isomorphic over R to π.

The arguments we shall use for the construction of π are essentially the same as those of the proof of Lemma 2.1.2 in [6]. Note that in Lemma 2.1.2 in [6] the authors deal with *finite* covers. Nevertheless, the extra hypothesis that the group F is closed in $\mathrm{Sym}(X)$ allows the arguments used there to work as well in our case although the cardinality of X may be infinite. So, even if it would be sufficient for Step a) to address the reader to [6], since we are going to use specific steps out of the proof of Lemma 2.1.2, we shall give the general lines of it for the use of the reader. For the details we refer to [6].

First the following cover is constructed. Let $\chi : \mathrm{Aut}(R/r_0) \to F/B$ and C be the set of left cosets of $\ker \chi$ in $\mathrm{Aut}(R)$. Consider the map $\theta : C \to R$ given by $\theta(g \ker \chi) = gr_0$. The permutation group $\mathrm{Aut}(R)$ induces a group of permutations on C. The induced group is a closed subgroup of $\mathrm{Sym}(C)$ and so we can consider C as a relational structure with automorphism group isomorphic to $\mathrm{Aut}(R)$. Then the map θ is a cover with trivial kernel.

Let $Y = \theta^{-1}(r_0) \sqcup X$ be the disjoint union of $\theta^{-1}(r_0)$ and X. The group F acts on Y: the action of $h \in F$ on $m \in \theta^{-1}(r_0)$ is $h(m) = (\chi^{-1}(hB))(m)$. Put on Y the relational structure given by F. For every $r \in R$ choose $g_r \in \mathrm{Aut}(R)$ such that $g_r r = r_0$ (with $g_{r_0} = id$). Then $g_r(\theta^{-1}(r)) = \theta^{-1}(r_0)$ and it induces an embedding $\eta_r : \theta^{-1}(r) \to Y$.

The next step is the following: we build a cover $\pi' : S' \to R$, where the domain of S' is made of the disjoint union of R, C and $R \times Y$ and π' is the identity on R, acts as θ on C, and as the projection to the first coordinate on $R \times Y$. We also have an injection $\tau : C \to R \times Y$ given by $\tau(c) = (r, \eta_r(c))$, whenever $\theta(c) = r$. Moreover, the structure of S' is made up of the original structure on R and C and for each n-ary relation R on Y we have an n-ary relation R' on $R \times Y$ given by

$$R'((r_1, y_1), \cdots, (r_n, y_n)) \text{ iff } r_i = r_j, \text{ for all } i, j \text{ and } R(y_1, \cdots, y_n).$$

Now we see how to extend an automorphism of R to a permutation of S' which preserves the above structure.

If $g \in \mathrm{Aut}(R)$, then g determines an automorphism of C. Also, if $gr_1 = r_2$, then via τ there is a bijection $\tau g \tau^{-1}$ from $\{r_1\} \times \theta^{-1}(r_0)$ to $\{r_2\} \times \theta^{-1}(r_0)$. In fact, let $h \in \mathrm{Aut}(R/r_0)$ and $h \ker \chi \in \theta^{-1}(r_0)$, since $\tau^{-1}(r_1, h \ker \chi) = g_{r_1}^{-1} h \ker \chi$, we have that $\tau g \tau^{-1}(r_1, h \ker \chi) = (r_2, g_{r_2} g g_{r_1}^{-1} h \ker \chi)$.

Since $g_{r_2} g g_{r_1}^{-1} \in \mathrm{Aut}(R/r_0)$, if we choose a representative z in the class $\chi(g_{r_2} g g_{r_1}^{-1})$ then $z(h \ker \chi) = g_{r_2} g g_{r_1}^{-1} h \ker \chi$ and this extends to a permutation $\beta(r, g)$ of Y. If we also denote by $\beta(r, g)$ the induced map from $r \times Y$ to $gr \times Y$, then $\omega(g) = g \cup \bigcup_{r \in R} \beta(r, g)$ is a permutation of S' which preserves the structure we put on S' and extends g.

Let π be the restriction of π' to $S = R \times X$ considered as permutation structure with $\mathrm{Aut}(S')$ acting. Then $\pi : S \to R$ is a free cover of R with kernel isomorphic to G^R.

Now the uniqueness, the last step. Let $\nu : N \to R$ be a cover with fibre group and binding group at r_0 isomorphic as permutation groups to F and B respectively. Let $\gamma : N(r_0) \to X$ be the bijection which gives rise to the isomorphism (we call it $\tilde{\gamma}$) as permutation groups which sends the fibre group at r_0 of ν to F and the binding group at r_0 of ν to B. Then $\chi_{r_0} = \tilde{\gamma} \circ \chi$. For each $r \in R$, g_r can be extended to an automorphism $\hat{g}_r \in \mathrm{Aut}(N)$. We define the map $\beta : N \to R \times X$ in the following way: if $n \in \nu^{-1}(r)$, define $\beta(n) := (r, \gamma(\hat{g}_r(n))) \in R \times X$. As is shown on Lemma 2.1.2 in [6], Step 5, this is a bijection which gives rise to an isomorphism of covers.

Step b). Let $g_r \in \mathrm{Aut}(R)$ be the permutations used above for constructing the free cover S. Then we construct a finite cover of W in the following way. Consider the set

$$C_0 := \{(w, k) : w \in r \text{ and } k \in \sigma^{-1}(M^{-1}(g_r)(w))\}.$$

Let $\pi_0 : C_0 \to W$ be the map given by $\pi_0(w, k) = w$. Since $\sigma : X \to [w_0]$ is a finite-to-one surjection, we have that π_0 is a finite-to-one surjection as well.

Let $\alpha : R \times X \to C_0$ be the map defined in the following way: let $k \in X$, then there exists $w \in [w_0]$ such that $k \in \sigma^{-1}(w)$. We define

$$\alpha(r, k) := ((M^{-1}(g_r^{-1}))w, k).$$

Then α is a bijection. Indeed, it is surjective because, given $(w, k) \in C_0$ with $w \in r$ and $k \in \sigma^{-1}(M^{-1}(g_r)(w))$, we have that

$$(w, k) = (M^{-1}(g_r^{-1})(M^{-1}(g_r)(w)), k) = \alpha(r, k).$$

In order to prove that α is injective take $(r_1, k_1) \neq (r_2, k_2) \in R \times X$ with $k_1 \in \sigma^{-1}(w_1)$ and $k_2 \in \sigma^{-1}(w_2)$ and suppose that $((M^{-1}(g_{r_1}^{-1}))w_1, k_1) = ((M^{-1}(g_{r_2}^{-1}))w_2, k_2)$. Then k_1 must be equal to k_2 which implies that $w_1 = w_2$. If $r_1 \neq r_2$, then $M^{-1}(g_{r_1}^{-1})w_1$ and $M^{-1}(g_{r_2}^{-1})w_2$ belong to different equivalence class and so we have a contradiction. Let $f_\alpha : \mathrm{Sym}(S) \to \mathrm{Sym}(C_0)$ be the map induced by α. The image by f_α of $\mathrm{Aut}(S)$ is closed in $\mathrm{Sym}(C_0)$. We denote it by $\mathrm{Aut}(C_0)$.

Let $C_0(w)$ be the fibre over w of π_0. If $w \in r_1$ then $C_0(w) = \sigma^{-1}(M^{-1}(g_{r_1})(w))$. We have that $\alpha^{-1}C_0(w) = (r_1, \sigma^{-1}(M^{-1}(g_{r_1})(w))$. Take an element g of $\mathrm{Aut}(S)$. We are going to show that $f_\alpha(g) = \alpha g \alpha^{-1}$ preserves the partition of C_0 given by the fibres of π_0.

Let $\bar{g} \in \mathrm{Aut}(W)$ such that $M(\bar{g})$ is the induced permutation on R by g. If $M(\bar{g})r_1 = r_2$, there exists $f \in F$ such that

$$g(r_1, \sigma^{-1}(M^{-1}(g_{r_1})w)) = (r_2, f(\sigma^{-1}(M^{-1}(g_{r_1})w))) = (r_2, \sigma^{-1}(M^{-1}(g_{r_2})\bar{g}w))).$$

By the proof of Lemma 2.1.2 in [6], we see that the element f is a representative of the class $\chi(M^{-1}(g_{r_2})\bar{g}M^{-1}(g_{r_1}^{-1}))$. Hence $g(r_1, \sigma^{-1}(M^{-1}(g_{r_1})w)) = (r_2, \sigma^{-1}(M^{-1}(g_{r_2})\bar{g}w))$ and then

$$\alpha g \alpha^{-1} C_0(w) = C_0(\bar{g}w),$$

i.e. the fibres of π_0 form an $\mathrm{Aut}(C_0)$-invariant partition of C_0.

Let $\mu_0 : \mathrm{Aut}(C_0) \to \mathrm{Sym}(W)$ be the induced homomorphism. Take an element $g \in \mathrm{Aut}(W)$ and an extension $\tilde{g} \in \mathrm{Aut}(S)$ of $M(g)$. The argument above shows as well that $\mathrm{Im}\mu_0$ is equal to $\mathrm{Aut}(W)$. The kernel of μ_0 is $\alpha \ker \pi \alpha^{-1}$. It is isomorphic to G^R. Since $\ker \pi$ induces on $\sigma^{-1}(w)$ and on X a group isomorphic to G, then $\ker \pi_0$ induces on any fibre of π_0 and on $C_0([w_0])$ a group isomorphic to G as well. So condition 1) of Definition 23 is verified. Let now w_1 and w_2 be two elements of W belonging to two different ρ-equivalence classes, say r_1 and r_2. Since π is a free cover on R, $\ker \pi$ induces on $S(r_1) \cup S(r_2)$ a group isomorphic to $G \times G$. Then $\ker \pi_0$ induces on $C_0([w_1]) \cup C_0([w_2])$ a group isomorphic to $G \times G$, and on $C_0(w_1)$ and on $C_0(w_1)$ a group isomorphic to G. This implies that also condition 2) of Definition 23 is verified. So we have an almost free finite cover $\pi_0 : C_0 \to W$ as required.

Step c). Let $\nu_0 : N_0 \to W$ be a finite cover of W with binding groups isomorphic to a finite group G, with kernel isomorphic to G^R and with $B^{\nu_0}([w_0])$ and $F^{\nu_0}([w_0])$ isomorphic as permutation groups to B and F respectively via a bijection

$$\gamma : N_0([w_0]) \to X \tag{1}$$

such that $\gamma(\tilde{\pi}_0^{-1}(w)) = \pi_0^{-1}(w)$ for every $w \in [w_0]$. Suppose that $\chi^{\nu_0}_{[w_0]}$ is equal to χ.

Let $\nu : N_0 \to R$ be given in the obvious way by $\nu(\delta) = r$ if $\delta \in N_0[w]$ and $[w] = r$. The fibres of ν form a partition of N_0 invariant under the action of $\mathrm{Aut}(N_0)$. Indeed, let $g \in \mathrm{Aut}(R)$, consider $M^{-1}(g)$ which extends to $\bar{g} \in \mathrm{Aut}(N_0)$. Then, if $\delta \in N_0([w])$ there exists $n \in [w]$ such that $\delta \in N_0(n)$ and $\bar{g}\delta \in N_0(M^{-1}(g)w) \subseteq N_0(g[w])$.

The fibre group at r_0 is equal to $F^{\nu_0}[w_0]$ and the binding group at r_0 is equal to $B^{\nu_0}([w_0])$. The map $\chi_{r_0} : \mathrm{Aut}(R/r_0) \to F^{\nu_0}([w_0])/B^{\nu_0}([w_0])$ is exactly the composition of $M^{-1} : \mathrm{Aut}(R/r_0) \to \mathrm{Aut}(W/\{[w_0]\})$ and $\chi^{\nu_0}_{[w_0]}$. Since the data of π and ν are the same up to isomorphism, by Lemma 2.1.2 in [6] ν and π are isomorphic over R via the bijection $\beta(\delta) = ([w], \gamma(\hat{g_r}(\delta)))$, if $\delta \in N_0([w])$ and $\hat{g_r} \in \mathrm{Aut}(N_0)$ is an extension of $M^{-1}g_r$.

Let $\delta \in N_0(w)$ (so $\hat{g_r}\delta \in N_0(M^{-1}(g_r)w)$). Since $\gamma(\tilde{\pi}_0^{-1}(w)) = \pi_0^{-1}(w)$ for every $w \in [w_0]$, we have that $\gamma(\hat{g_r}\delta) \in \sigma^{-1}(M^{-1}(g_r)w)$ and then

$$\alpha([w], \gamma(\hat{g_r}\delta)) = (M^{-1}(g_r^{-1}g_r)w, \gamma(\hat{g_r}\delta)) = (w, \gamma(\hat{g_r}\delta)).$$

Consider the bijection

$$\begin{array}{ccccc} N_0 & \overset{\beta}{\to} & S & \overset{\alpha}{\to} & C_0 \\ \delta & \mapsto & ([w], \gamma(\hat{g_r}\delta)) & \mapsto & (w, \gamma(\hat{g_r}\delta)) \end{array} \tag{2}$$

Then $\alpha\beta\,\mathrm{Aut}(N_0)\beta^{-1}\alpha^{-1} = \alpha\,\mathrm{Aut}(S)\alpha^{-1} = \mathrm{Aut}(C_0)$, i.e. $\mathrm{Aut}(N_0)$ and $\mathrm{Aut}(C_0)$ are isomorphic over W. ■

Corollary 25 *Let W be a transitive structure, $w_0 \in W$, and ρ be an $\mathrm{Aut}(W)$-congruence on W. Assume that the permutation group induced by $\mathrm{Aut}(W/\{[w_0]\})$ on $[w_0]$ is closed in $\mathrm{Sym}([w_0])$. Moreover suppose that the map M is injective, open and with closed image. Let G be a finite permutation group acting on a finite set Δ. Then there exists an almost-free finite cover of W w.r.t. ρ and with binding groups equal to G.*

Proof. We shall denote by A the permutation group induced by $\mathrm{Aut}(W/\{[w_0]\})$ on $[w_0]$. Consider the wreath product $G Wr_{[w_0]} A$ acting in the usual way on $[w_0] \times \Delta$. Let $\sigma : [w_0] \times \Delta \to [w_0]$ be given by $\sigma(w, \delta) = w$.

Denote by B the diagonal subgroup of $G^{[w_0]}$: it is normalized by A and so we can make the semidirect product $F := B \rtimes A$. This is closed by Proposition 7. Using the notation of Theorem 24 we have that χ is the homomorphism induced by restriction on $[w_0]$. The hypotheses of Theorem 24 are satisfied and so we have an almost free finite cover $\pi : W \times \Delta \to W$. ■

The following remark establish a link between section 3 and this section.

Remark 26 *Let W be a transitive structure which satisfies the hypotheses of Corollary 25. Let $\pi' : W \times \Delta \to W$ be the finite cover given by $\pi'(w, \delta) = w$ with $\mathrm{Aut}(W \times \Delta) = K_\rho \rtimes \mathrm{Aut}(W)$ (for K_ρ see the notation of Theorem 9) acting on $W \times \Delta$ in the usual way: $(f, \sigma)(w, \delta) = (\sigma w, f(w)\delta)$. Then the almost free finite cover which we constructed in the proof of Corollary 25 is exactly π'.*

Indeed, let F and B be as in the proof of Corollary 25. Since $F^{\pi'}([w_0]) = F$, $B^{\pi'}([w_0]) = B$ and $\chi^{\pi'}_{[w_0]} = \chi$ we have that the identity map can be taken as the bijection γ. If $g_r \in \mathrm{Aut}(R)$ for $r \in R$ are the permutations used in Step a) of the proof of Theorem 24, we take as $\hat{g_r} \in K_\rho \rtimes \mathrm{Aut}(W)$ the permutations (id, g_r). Since $\sigma^{-1}(w) = \Delta$ for every $w \in [w_0]$, we see that the structure C_0 is $W \times \Delta$ and then it follows immediately that the bijection $\alpha\beta : W \times \Delta \to C_0$ of Step c) is the identity map.

Corollary 27 *Let W be a transitive structure, $w_0 \in W$, and ρ be an $\mathrm{Aut}(W)$-congruence on W. Assume that the permutation group induced by $\mathrm{Aut}(W/\{[w_0]\})$ on $[w_0]$ is closed in $\mathrm{Sym}([w_0])$. Moreover suppose that the map M is injective, open and with closed image. Let G be a simple non-abelian finite permutation group acting on itself by conjugation. Then there exist at least two non-isomorphic almost free finite covers with respect to ρ, with isomorphic kernels.*

Proof. As above, we shall denote by A the permutation group induced by $\mathrm{Aut}(W/\{[w_0]\})$ on $[w_0]$. Take $\Delta = G$ and let $\pi : W \times G \to W$ be the almost free finite cover cover built in Corollary 25. Using the notation of Corollary 25 and the topological results in section 1.4 of [6] we have that the map $T : F \to A$ is continuous, maps closed subgroups to closed subgroups and is open. Then the isomorphism map $S : A \to F/B$ is a topological isomorphism.

Since $B^\pi([w_0]) = B \cong G$, by conjugation of G by elements of $F^\pi([w_0]) = F$ we get a map $\gamma : F^\pi([w_0])/G \to \mathrm{Out}(G)$. The image of γ is H/G, for some $H \leq \mathrm{Aut}(G)$. Composing S with γ, we have a map

$$\bar{S} : A \to H/G.$$

We see that γ is continuous. Indeed, the kernel of γ is $C_{F^\pi([w_0])}(G)G/G$, where $C_{F^\pi([w_0])}(G)$ is the centralizer of G in $F^\pi([w_0])$. The group G is finite and hence closed in $F^\pi([w_0])$. Its orbits on $[w_0] \times G$ are finite and so it is also compact. Moreover, $C_{F^\pi([w_0])}(G)$ is closed.

By Proposition 7 we have that $C_{F^\pi([w_0])}(G)G$ is closed in $F^\pi([w_0])$. Since it has finite index in $F^\pi([w_0])$, $C_{F^\pi([w_0])}(G)G$ is open in $F^\pi([w_0])$ and hence $C_{F^\pi([w_0])}(G)G/G$ is open in $F^\pi([w_0])/G$.

Let $P : H \to H/G$ be the quotient map and

$$F_1 := \{(\sigma, h) : \sigma \in A,\ h \in H \text{ and } P(h) = S(\sigma)\}$$

be the fibre product between A and H. This is a permutation group on $[w_0] \times G$ with action given by: $(\sigma, h)(w, g) = (\sigma w, h(g))$. By the same reasoning as in Proposition 21, we have that F_1 is closed in $\mathrm{Sym}([w_0] \times G)$.

The group $B_1 := \{(id, g) : id \in \mathrm{Sym}(\{[w_0]\}),\ g \in G\}$ is a normal subgroup of F_1. Let $\chi : Aut(W/\{[w_0]\}) \twoheadrightarrow F_1/B_1$ be the map given by

$$\chi(g) = (g_{|[w_0]}, h)B_1,$$

where h belongs to the coset $S(g_{|[w_0]})$. The map χ is well defined.

Let

$$\sigma : [w_0] \times G \to [w_0]$$

be the projection on the first component. The induced map $F_1 \to \mathrm{Sym}([w_0])$ has image A. Hence, by Theorem 24, we can build an almost-free finite cover π_1 w.r.t ρ with binding groups isomorphic to G. Note that the kernel is isomorphic to K_ρ. ■

5.4 Problems

We described in an explicit way the kernels of expansions of the free finite cover of $\langle \Omega^{(n)}, \mathrm{Sym}(\Omega)\rangle$, when the fibre groups and the binding groups are both equal to a simple non-abelian regular finite permutation group G.

1. What happens for finite covers where the base structure is a Grassmannian of a vector space over a finite field?

2. What happens for finite covers of $\Omega^{(n)}$ if the fibre groups and the binding groups are isomorphic to a simple abelian group? Here one would need to work with the closed $\mathrm{Sym}(\Omega)$-submodules of $\mathbb{F}_p^{\Omega^{(n)}}$. We recall that the case where the base permutation structure is $\langle [\Omega]^n, \mathrm{Sym}(\Omega)\rangle$, where $[\Omega]^n$ is the set of n-subsets from Ω with $\mathrm{Sym}(\Omega)$ acting on it in the obvious way, was solved by Gray ([7]).

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