EXTREMAL CONVEX SETS FOR SYLVESTER-BUSEMANN TYPE FUNCTIONALS

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In memory of Carlo Pucci, remembering our multiple discussions about mathematics as well as life.

Abstract. The Sylvester $(d+2)$-points problem deals with the probability $S(K)$ that $d+2$ random points taken from a convex compact subset K of $\mathbb{R}^d$ are not the vertices of any convex polytope and asks for which sets $S(K)$ is minimal or maximal. While it is known that ellipsoids are the only minimizers of $S(K)$, the problem of the maximum is still open, unless $d=2$.

In this paper we study generalizations of $S(K)$, which include the Busemann functional – appearing in the formula for the volume of a convex set in terms of the areas of its central sections – and a functional introduced by Bourgain, Meyer, Milman and Pajor in connection with the local theory of Banach spaces.

We show that also for these more general functionals ellipsoids are the only minimizers and, in the two-dimensional case, triangles (or parallelograms, in the symmetric case) are maximizers.

1. Introduction.

Let $\mathcal{K}^d$ denote the class of all convex bodies in $\mathbb{R}^d$, i.e. of all d-dimensional compact convex set. For $K \in \mathcal{K}^d$, the functional

$$S(K;d) = \frac{1}{[V(K)]^{d+2}} \int_K \cdots \int_K [x_0, x_1, \dots, x_d]\, dx_0 dx_1 \dots dx_d \,, \tag{1.1}$$

where $V(K)$ denotes the volume of K and $[x_0, x_1, \dots, x_d]$ the volume of the convex hull of $x_0, x_1, \dots, x_d$, is the normalized mean expected value of the volume of the simplex whose vertices are randomly, independently and uniformly chosen from K.

This functional is well known in the literature because, if multiplied by $d+2$, it is exactly the probability that one of $d+2$ random points from K falls in the convex hull of the remainders and then answers to the relevant question posed by Sylvester in 1864 [Sy]. For the historical development of this problem, see Pfiefer [Pf].

Nowadays, by Sylvester's problem it is meant the following one: Which are the convex bodies giving the maximum or the minimum of $S(K;d)$?

Classical compactness arguments, based on the fact that the functional $S(K;d)$ is affinely invariant, imply the existence of maximizers and minimizers.

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In 1917 Blaschke [Bl] solved the problem in the case $d = 2$. He showed that ellipses are the only minimizers and that triangles are the only maximizers of $S(K; 2)$.

For $d > 2$, Groemer [G1] proved that ellipsoids are the only minimizers of $S(K; d)$. The problem of finding maximizers of $S(K; d)$ is still open and it is conjectured that simplices are the only solutions.

A functional of the same type as the Sylvester one appears in the Busemann formula which expresses the volume of a convex body in terms of the areas of its central sections. Assume that K contains the origin in its interior and keep one of the random points in (1.1) still at the origin. Thus, we have the functional

$$B(K; d) = \frac{1}{[V(K)]^{d+1}} \int_K \dots \int_K [0, x_1, \dots, x_d] \, dx_1 \dots dx_d \,. \tag{1.2}$$

The Busemann intersection formula (see [Bu]) states that

$$V(K)^{d-1} = \frac{d!}{2} \int_{S^{d-1}} V(K \cap u^\perp)^{d+1} B(K \cap u^\perp; d-1) \, du \,,$$

where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$, $u^\perp$ is the hyperplane through the origin orthogonal to u and the integration is performed with respect to the Hausdorff $(d-1)$-dimensional measure.

In [Bu] Busemann proved also that $B(K; d)$ attains its minimum if and only if K is an origin symmetric ellipsoid. Such a result is known as *Busemann's random simplex inequality.*

During the last century, different generalizations and variants of the Sylvester functional (1.1) were proposed. The most natural one sounds the extension proposed by Groemer, where the number of random points is arbitrary and the volume of the resultant random polytope is raised to the p-th power:

$$S(K; m; p) = \frac{1}{[V(K)]^{m+p+1}} \int_K \dots \int_K [x_0, x_1, \dots, x_m]^p \, dx_0 dx_1 \dots dx_m \,. \tag{1.3}$$

Groemer [G2] proved that ellipsoids are still the only minimizers of $S(K; m; p)$ for every $m \geq d$ and $p \geq 1$. Schöpf [So] obtained the same result for every $p > 0$ and $m = d$. In the planar case, Dalla and Larman [DL] proved that triangles are maximizers of $S(K; m; 1)$ and Giannopoulos [Gi] showed that they are the only maximizers.

Analogously, one can consider the following extension of Busemann's functional (1.2):

$$B(K; m; p) = \frac{1}{[V(K)]^{m+p}} \int_K \dots \int_K [0, x_1, \dots, x_m]^p \, dx_1 \dots dx_m \,. \tag{1.4}$$

Looking at the formula (1.1), we notice that the volume of the random simplex $[x_0, x_1, \dots, x_d]$ equals to

$$\frac{1}{d!} V \left(\sum_{i=1}^{d} \overline{x_i x_0} \right) ,$$

where $\overline{x_i x_0}$ denotes the segment joining x_i to x_0 and the sum is made according to the Minkowski addition of subsets of $\mathbb{R}^d$:

$$A + B = \{x + y : x \in A, y \in B\} .$$

This fact suggests the following extension of functional (1.4):

$$I(K; m; p) = \frac{1}{V(K)^{m+p}} \int_K \dots \int_K V\left(\sum_{j=1}^{m} \overline{0x_j}\right)^p dx_1 \dots dx_m \, , \tag{1.5}$$

which was introduced by Bourgain, Meyer, Milman and Pajor [B-P] in connection with some comparisons between norms in the local theory of Banach spaces. They considered the following more general version

$$I(K_1, K_2, \dots, K_m; p) = \frac{1}{(V(K_1) \dots V(K_m))^{\frac{m+p}{m}}} \int_{K_1} \dots \int_{K_m} V\left(\sum_{j=1}^{m} \overline{0x_j}\right)^p dx_1 \dots dx_m \, , \tag{1.6}$$

where $K_1, K_2, \dots, K_m \in \mathcal{K}^d$, and proved that, for $p > 0$,

$$I(K_1, \dots, K_m; p) \geq I(B_1, B_2, \dots, B_m; p) \, , \tag{1.7}$$

where B_i is the ball with the same volume as K_i centered at the origin.

Notice that the polytopes defined as Minkowski sums of segments, appearing in (1.5) and (1.6), are called *zonotopes* and play an important role in convex geometry (see [SW]).

In this paper we study the functional $I(K; m; p)$, for $p \geq 1$, and we prove that origin symmetric ellipsoids are the only minimizers (Theorem 7). Furthermore, we show that triangles are maximizers in the class of plane convex figures containing the origin and parallelograms in that of origin symmetric plane convex sets (Theorem 8). The main tool will be a general convexity property (Theorem 6) for the functional (1.6), which implies also inequality (1.7), for $p \geq 1$.

A further extension of the Sylvester-Busemann functional that we consider in the present paper can be defined by introducing the L^p*-centroid body* of a compact set of $\mathbb{R}^d$.

For $K \in \mathcal{K}^d$, the support function h_K is defined by

$$h_K(u) = \max_{z \in K} \langle z, u \rangle \, , \; u \in \mathbb{R}^d \, , \tag{1.8}$$

where $\langle \, , \rangle$ denotes the standard scalar product.

For each real number $p \geq 1$, the L^p-centroid body of K is the convex body $\Gamma_p K$ whose support function is

$$h_{\Gamma_p K}(u) = \left\{ \frac{1}{c_{d,p} V(K)} \int_K |\langle u, z \rangle|^p \, dz \right\}^{\frac{1}{p}} \, , \; u \in \mathbb{R}^d \, , \tag{1.9}$$

where

$$c_{d,p} = \frac{\kappa_{d+p}}{\kappa_2 \kappa_d \kappa_{p-1}}\,,$$

with

$$\kappa_r = \pi^{\frac{r}{2}} / \Gamma(1 + \frac{r}{2})\,.$$

Notice that κ_d is the volume of the unit ball B^d of $\mathbb{R}^d$ and the constant $c_{d,p}$ is such that $\Gamma_p B^d = B^d$, for every d and p.

Up to constants, $\Gamma_1 K$ is known in the literature as the *centroid body* of K (see, for example, Gardner [Ga], Chapter 9, and Schneider [Sc], Section 7.4) and $\Gamma_2 K$ is the classical *Legendre ellipsoid* of K, which is related to the moments of inertia of K. The definition of L^p-centroid body for general p was introduced by Lutwak and Zhang in [LZ].

A basic result regarding L^p-centroid bodies is the *L^p-Busemann-Petty centroid inequality*

$$V(\Gamma_p K) \geq V(K)\,, \tag{1.10}$$

where $p \geq 1$ and equality holds if and only if K is an origin symmetric ellipsoid. For $p = 1$, (1.10) was first stated by Petty [P1] as a reformulation of Busemann's random simplex inequality (for more details see Lutwak [L]). For $p = 2$, inequality (1.10) was proved by Blaschke [B1] when $d = 3$ and by John [J1] in higher dimensions. For general p inequality (1.10) has been independently proved by Lutwak, Yang and Zhang [LYZ] and by the authors [CG1].

The volumes of $\Gamma_1 K$ and $\Gamma_2 K$ are related to the Busemann functional through the formulas

$$V(\Gamma_1 K) = \left(\frac{2}{c_{d,1}}\right)^d V(K) B(K; d; 1)\,,$$

$$V(\Gamma_2 K) = \kappa_d \left(\frac{d!}{c_{d,2}^d} V(K) B(K; d; 2)\right)^{1/2}$$

(see [P1], [B2]), while for $p \neq 1, 2$ an expression of the same kind for the volume of the L^p-centroid body is not available.

The volume of $\Gamma_p K$ strongly depends on the location of the body K. In [CG2], it is shown that $V(\Gamma_p(K - x))$ is a strictly convex function of x. In the same paper, the extrema of the functionals

$$\max_{x \in K} \frac{V(\Gamma_p(K - x))}{V(K)}\,,$$

$$\min_{x \in K} \frac{V(\Gamma_p(K - x))}{V(K)}\,,$$

have been investigated.

Here, the interest is focused on the functional

$$A(K; p; q) = \left[\frac{1}{V(K)^{q+1}} \int_K V(\Gamma_p(K - x))^q \, dx\right]^{1/q}\,, \tag{1.11}$$

where $p, q \geq 1$. Notice that (1.11) is related to the Sylvester functionals. Indeed,

$$A(K;1;1) = \frac{2^d}{c_{d,1}^d} S(K;d;1)\,,$$

$$A(K;2;2)^2 = \kappa_d^2 d!(d+2)^d S(K;d;2)\,.$$

We shall prove in Section 3 that in $\mathcal{K}^d$ ellipsoids are the only minimizers of $A(K;p;q)$ and that triangles (parallelograms, in the origin symmetric case) are maximizers, when $d = 2$.

2. Preliminaries.

The method we use here for dealing with the functionals (1.5), (1.6) and (1.11) is based on the idea of *shadow system* introduced by Rogers and Shephard (see [RS] and [Sh]). The same method was already used by Colesanti and the authors in [CCG], [CG1], [CG2].

Given a direction v and an arbitrary index set $\mathcal{I}$ such that, for every $i \in \mathcal{I}$, a_i is a point of a bounded subset A of $\mathbb{R}^d$, a shadow system along v is the family of convex bodies $K_t \subset \mathbb{R}^d$

$$K_t = conv\{a_i + \alpha(i)t\,v : i \in \mathcal{I}\}\,, \tag{2.1}$$

where α is a real bounded function on $\mathcal{I}$, and the parameter t runs in an interval of the real axis.

In [RS] it is proved that *the volume $V(K_t)$ of a shadow system is a convex functions of t.*

Such a result was extended to other geometric quantities different than the volume. For example, all the *quermassintegrals* enjoy the same convexity property. We recall that the quermassintegrals of a convex body can be introduced through the coefficients of the polynomial expansion appearing in Steiner formula (see, e.g., [Ga], Ch. A.4, or [Sc], Ch. 4.2). The surface area and the mean width of a convex set are special instances of quermassintegrals.

A shadow system can be seen as a continuous transformation process of a given set. The most remarkable example is given by the Steiner symmetrization of a convex body K. In this case, the speed function $\alpha : K \to \mathbb{R}$ is constant on each chord of K parallel to the direction v of the movement and, at every t, the union of these chords is just K_t. A shadow system whose speed function has these properties is called a *parallel chord movement.* In such a movement the volume is a constant function of the parameter.

It is easy to check that, if the speed function α of a parallel chord movement is an affine function, that is, $\alpha : K \to \mathbb{R}$ and $\alpha(x) = \langle x, u\rangle + c$, for some vector $u \in v^\perp$ and real constant c, then K_t is an affine image of K, for every t.

A basic feature of shadow systems, that will be used later, regards the Minkowski addition and is expressed by the following lemma.

Lemma 1. *If $\{K_t : t \in [0,1]\}$ and $\{H_t : t \in [0,1]\}$ are shadow systems along the same direction v, then $\{K_t + H_t : t \in [0,1]\}$ is still a shadow system along v.*

Proof. Let $\{K_t : t \in [0,1]\}$ be defined by (2.1) and

$$H_t = conv\{c_j + \gamma(j)t\,v : j \in \mathcal{J}\}\,.$$

For every t, the Minkowski sum $K_t + H_t$ is the convex hull of its extreme points. Since every extreme point of $K_t + H_t$ is contained in the Minkowski sum of the sets of extreme points of K_t and H_t, we can write

$$\begin{aligned} K_t + H_t =& conv\{a_i + \alpha(i)t\, v + c_j + \gamma(j)t\, v : i \in \mathcal{I}, j \in \mathcal{J}\} \\ =& conv\{w_{i,j} + (\alpha(i) + \gamma(j))t\, v : (i,j) \in \mathcal{I} \times \mathcal{J}\}\,, \end{aligned}$$

where $w_{i,j} = a_i + c_j$. $\square$

In [CG1] it is proved the following result, which will be a basic tool in studying the functional $A(K;p;q)$.

Theorem 1. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then $\Gamma_p K_t$ is a shadow system along the same direction v. Furthermore, the volume of $\Gamma_p K_t$ is a strictly convex function of t unless the speed function of the movement $\{K_t : t \in [0,1]\}$ is linear.*

This theorem is also the main ingredient for the proof of the L^p-Busemann-Petty centroid inequality given in [CG1] and of its two-dimensional reverse forms (see [CG2]).

3. Extrema of $A(K;p;q)$.

In this section we consider the problem of finding convex sets at which $A(K;p;q)$ attains its minimum or maximum value.

An easy consequence of definitions (1.9) and (1.11) is the continuity of $A(K;p;q)$ with respect to the Hausdorff metric, for every $p, q \geq 1$. Let us check that $A(K;p;q)$ is affinely invariant. Given a linear map $L \in GL(d)$, (1.9) implies that

$$h_{\Gamma_p(LK)}(u) = h_{\Gamma_p K}(L^* u)\,, \text{ for every } u \in \mathbb{R}^d\,,$$

where L^* is the adjoint of L. On the other hand, (1.8) yields

$$h_{\Gamma_p K}(L^* u) = h_{L\Gamma_p K}(u)\,,$$

hence $\Gamma_p(LK) = L\Gamma_p K$. Therefore,

$$\begin{aligned} A(LK;p;q)^q =& \frac{1}{V(LK)^{q+1}} \int_{LK} V(\Gamma_p(LK - x))^q\, dx = \\ =& \frac{1}{|detL|^{q+1} V(K)^{q+1}} \int_K V(\Gamma_p(LK - Ly))^q |detL|\, dy = \\ =& A(K;p;q)^q\,. \end{aligned}$$

Thus, the invariance of $A(K;p;q)$ under translations and the above equality provide the invariance under affine transformations.

Theorem 2. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement, then $A(K_t;p;q)$ is a convex function of t, for every $p, q \geq 1$. Moreover, it is strictly convex unless the speed function is an affine function.*

Proof. Let v be the direction and α be the speed function of the movement. Then

$$\begin{aligned} A(K_t;p;q) =& \left[\frac{1}{V(K_t)^{q+1}} \int_{K_t} V(\Gamma_p(K_t - x))^q\, dx\right]^{1/q} \\ =& \left[\frac{1}{V(K_0)^{q+1}} \int_{K_0} V(\Gamma_p(K_t - x - \alpha(x)tv))^q\, dx\right]^{1/q}\,, \end{aligned} \tag{3.1}$$

where we used the fact that the Jacobian of the map $x \to x + \alpha(x)t\,v$ equals 1.

Formula (3.1) can be rewritten as

$$A(K_t; p; q) = \|s(t, \cdot)\|_q \,,$$

where the norm is taken in $L^q(K_0)$ and

$$s(t,x) = \frac{V(\Gamma_p(K_t - x - \alpha(x)tv))}{V(K_0)^{1+\frac{1}{q}}} \,.$$

For every $x \in K_0$, the family $\{K_t - x - \alpha(x)tv : t \in [0,1]\}$ is a parallel chord movement with speed function $\alpha(\cdot) - \alpha(x)$. Therefore, by Theorem 1, $s(t,x)$ is a convex function of t.

Thus, if we fix λ, t_1, $t_2 \in [0,1]$, then, by Minkowski's inequality for L^q-norms, we have

$$\begin{aligned} A(K_{(1-\lambda)t_1+\lambda t_2}; p; q) &= \|s((1-\lambda)t_1 + \lambda t_2, \cdot)\|_q \\ \leq \|(1-\lambda)s(t_1,\cdot) + \lambda s(t_2,\cdot)\|_q &\leq (1-\lambda)\|s(t_1,\cdot)\|_q + \lambda\|s(t_2,\cdot)\|_q \\ &= (1-\lambda)A(K_{t_1}; p; q) + \lambda A(K_{t_2}; p; q) \,. \end{aligned} \tag{3.2}$$

Theorem 1 says that the first inequality of (3.2) is just an equality if and only if $\alpha(z) - \alpha(x) = \langle z - x, u\rangle$, for some $u \in v^\perp$, that means α is an affine function. In this case, $s(t,x)$ is constant with respect to t and then equality holds everywhere in (3.2). $\square$

Theorem 3. *For every $p, q \geq 1$, the minimum of $A(K; p; q)$ in the class of all convex bodies is attained if and only if K is an ellipsoid.*

Proof. Let $K \in \mathcal{K}^d$ and fix a direction v. If we denote with $K|v^\perp$ the orthogonal projection of K onto $v^\perp$, then K can be represented by

$$K = \{x + yv \in \mathbb{R}^d : x \in K|v^\perp, y \in \mathbb{R}, f_v(x) \leq y \leq g_v(x)\} \,,$$

where f_v and $-g_v$ are convex functions on $K|v^\perp$.

The Steiner process of symmetrization of K with respect to $v^\perp$ can be described by a parallel chord movement as follows. In (2.1), take $\alpha(x) = -(f_v(x|v^\perp) + g_v(x|v^\perp))$ and $t \in [0,1]$. For $t = 0$ we obtain K, for $t = 1$ the reflection K^v of K in the hyperplane $v^\perp$, and, for $t = \frac{1}{2}$, the Steiner symmetral of K with respect to $v^\perp$.

The affine invariant functional $A(K; p; q)$ attains the same value at K and K^v. Therefore, Theorem 2 implies that the value of $A(K; p; q)$ is not increased if we pass from K to its Steiner symmetral. Moreover, $A(K; p; q)$ strictly decreases unless the speed of the movement is an affine function, that is, unless all the midpoints of the chords of K parallel to v lie on a hyperplane.

It is well known (see, e.g., [P2]) that ellipsoids are the only bodies enjoying this property for every direction v.

To end the proof, we have only to notice that standard compactness arguments provide the existence of minimizers of $A(K; p; q)$ and that the affine invariance guarantees that $A(K; p; q)$ is constant on all the ellipsoids. $\square$

Let us consider now the problem of finding maximizers of the functional $A(K; p; q)$. As already noticed, such a problem is still open even for the original Sylvester functional, unless $d = 2$. We also solve the problem for $A(K; p; q)$ in the two-dimensional case.

Theorem 4. *For every $p, q \geq 1$ and $d = 2$, the maximum of $A(K; p; q)$ in the class of all convex bodies is attained if K is a triangle.*

Proof. Let P be a polygon with n vertices, where $n > 3$. Call v_1, v_2, v_3 three consecutive vertices of P and u the direction of $v_3 - v_1$. The shadow system $\{P_t : t \in [t_0, t_1]\}$, with $t_0 < 0 < t_1$, along u, with speed 1 at v_2 and 0 at all the other vertices, is a parallel chord movement if we choose $[t_0, t_1]$ as the largest interval such that the area of P_t is constant for all $t \in [t_0, t_1]$. We notice that P_{t_0} and P_{t_1} have $n - 1$ vertices and, by Theorem 2,

$$A(P; p; q) < max\{A(P_{t_0}; p; q), A(P_{t_1}; p; q)\}\,.$$

The inequality is strict because the speed of the movement cannot be affine. If $n > 4$, iterations of this procedure lead to the fact that triangles are the only maximizers of $A(K; p; q)$ in the class of polygons.

By the continuity of $A(K; p; q)$, an approximation argument ends the proof. $\square$

The above proof can be easily adapted for obtaining the following result.

Theorem 5. *For every $p, q \geq 1$ and $d = 2$, the maximum of $A(K; p; q)$ in the class of all centrally symmetric convex bodies is attained if K is a parallelogram.*

4. Extrema of $I(K; m; p)$.

In this section we deal with the functional $I(K; m; p)$, defined in (1.5).

The continuity of I in $\mathcal{K}^d$ with respect to the Hausdorff metric can be easily checked. Moreover, for every linear map $L \in GL(d)$, we have

$$\begin{aligned}
I(LK; m; p) =& \frac{1}{V(LK)^{m+p}} \int\limits_{LK} \cdots \int\limits_{LK} V\left(\sum_{i=1}^{m} \overline{0x_i}\right)^p dx_1 \ldots dx_m \\
=& \frac{1}{|detL|^{m+p} V(K)^{m+p}} \int\limits_{K} \cdots \int\limits_{K} V\left(\sum_{i=1}^{m} \overline{0L(y_i)}\right)^p |detL|^m dy_1 \ldots dy_m \\
=& I(K; m; p)\,,
\end{aligned}$$

where we used the fact that linear transforms commute with the Minkowski addition.

Standard arguments provide the existence of the minimum of $I(K; m; p)$. As far as the maximum is concerned, it is easy to see that $I(K; m; p)$ is not bounded in $\mathcal{K}^d$. Nevertheless, if we restrict ourselves to the convex bodies containing the origin, then the maximum exists. This fact, owing to the continuity and the linear invariance of the functional, is a consequence of John's theorem ([J2], Theorem 3), which ensures that every convex body contains an ellipsoid E and is contained in a copy of E, rescaled by a factor d.

We give here a characterization of ellipsoids as the only minimizers of $I(K; m; p)$ in all dimensions, and, for $d = 2$, we show that triangles are maximizers.

First, we prove a convexity result for the general functional $I(K_1, \ldots, K_m; p)$ introduced by (1.6).

Theorem 6. *Let $\{K_{i,t} : t \in [0, 1]\} \subset \mathcal{K}^d$, $i = 1, 2, \ldots, m$, with $m \geq d$, be parallel chord movements along the same direction $v \in \mathbb{R}^d$. Then, for $p \geq 1$, the functional $I(K_{1,t}, \ldots, K_{m,t}; p)$ is a convex function of t.*

Proof. Let $\alpha_i : K_i \to \mathbb{R}$ be the speed function of the parallel chord movement $\{K_{i,t} : t \in [0,1]\}$, $i = 1, 2, \ldots, m$. Then, starting from definition (1.6), a change of variables gives

$$I(K_{1,t}, K_{2,t}, \ldots, K_{m,t}; p) =$$
$$\frac{1}{(V(K_{1,0}) \ldots V(K_{m,0}))^{\frac{m+p}{m}}} \int_{K_{1,0}} \ldots \int_{K_{m,0}} V\left(\sum_{i=1}^{m} \overline{0(x_i + \alpha(x_i)t\,v)}\right)^p dx_1 \ldots dx_m\,,$$

as the Jacobian of the map $x_i \to x_i + \alpha(x_i)t\,v$ is 1.

Now,

$$V\left(\sum_{i=1}^{m} \overline{0(x_i + \alpha(x_i)t\,v)}\right)$$

is a convex function of t. Indeed, each term of the Minkowski sum is a shadow system and Lemma 1 implies that the sum is a family of polytopes which is still a shadow system. The Rogers-Shephard theorem, quoted in Section 2, provides the convexity in t of the volume, which is kept by its p-th power, since $p \geq 1$.

The statement follows from the fact that integrals of convex functions are still convex. $\square$

A first consequence of Theorem 6, is the Bourgain-Meyer-Milman-Pajor inequality (1.7), for $p \geq 1$. Indeed, the functional $I(K_1, \ldots, K_m; p)$ does not increase when we apply Steiner symmetrizations to K_1, $\ldots$, K_m with respect to the same hyperplane $v^\perp$. On the other hand, it is well known (see, e.g. , Bonnesen and Fenchel [BF], Ch. 9) that iterated Steiner symmetrizations along a suitable sequence of directions, independent of i, reduce K_i to a ball of the same volume, for every i.

Theorem 7. *If $K \in \mathcal{K}^d$, $m \geq d$ and $p \geq 1$, then*

$$I(K; m; p) \geq I(B^d; m; p)\,,$$

where B^d denotes the unit d-ball. Equality holds if and only if K is an origin symmetric ellipsoid.

Proof. We have only to prove the "only if" part of the statement. To do this, let us fix a direction v and consider the parallel chord movement $\{K_t : t \in [0,1]\}$ corresponding to the Steiner process of symmetrization of K with respect to $v^\perp$ (see the proof of Theorem 3). We show that $I(K_{\frac{1}{2}}; m; p) < I(K_0; m; p) = I(K_1; m; p)$, unless the speed of the movement is a linear function in K. This fact will imply that, for a minimizer, all the midpoints of the chords parallel to v lie on a hyperplane through the origin. The same argument used in the proof of Theorem 3 leads to the conclusion.

If $I(K_{\frac{1}{2}}; m; p) = I(K_0; m; p) = I(K_1; m; p)$, then, by Theorem 6, $I(K_t; m; p)$ is independent of t and so $V\left(\sum_{i=1}^{m} \overline{0(x_i + t\alpha(x_i)v)}\right)$ is constant in t, for almost every m-ples of points from K. By continuity, "almost" can be deleted.

Let us choose x_1, x_2, $\ldots$, x_{d-1} from K such that $det(v, x_1, \ldots, x_{d-1}) \neq 0$. We can assume that $\alpha(x_1) = \alpha(x_2) = \cdots = \alpha(x_{d-1}) = 0$, by adding possibly a suitable linear function. For every x_d lying on the hyperplane through 0, $x_1, \dots, x_{d-1}$, setting $x_{d+1} = x_{d+2} = \cdots = x_m = x_d$, we have that

$$\begin{aligned}
&V\left(\sum_{i=1}^{m} \overline{0(x_i + t\alpha(x_i)v)}\right) \\
=&V\left(\overline{0x_1} + \overline{0x_2} + \cdots + \overline{0x_{d-1}} + (m-d+1)\overline{0(x_d + t\alpha(x_d)v)}\right) \\
=&(m-d+1)|det(x_1, x_2, \dots, x_{d-1}, x_d + t\alpha(x_d)v)| \\
=&(m-d+1)t\alpha(x_d)|det(v, x_1, x_2, \dots, x_{d-1})|\,.
\end{aligned}$$

Since the last quantity has to be independent of t, we must have $\alpha(x_d) = 0$.

Thus, we have shown that, in every section of K through the origin, which is not parallel to v, the function α is linear, that is, there exists a function $w(u) : S^{d-1} \setminus v^{\perp} \to v^{\perp}$ such that

$$\alpha(x) = \langle x, w(u)\rangle\,, \text{ for every } x \in K \cap u^{\perp}\,.$$

By taking into account that $\{K_t : t \in [0,1]\}$ is a parallel chord movement, we deduce that α is linear in the union of all the chords of K, parallel to v, intersecting $K \cap u^{\perp}$.

This fact easily implies that, if the origin is contained in K or $d > 2$, then the function w is constant and, consequently, the function α is linear in K.

In case that $d = 2$ and $0 \notin K$, we deduce that w is constant on each connected component of $K \setminus \{rv : r \in \mathbb{R}\}$. If there are two connected components, then it is sufficient to consider the restriction of the movement to the interval $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$, with ε so small that $0 \in K_t$, for all t in such an interval. $\square$

Theorem 8. *For $d = 2$, $m \geq 2$ and $p \geq 1$, the maximum of $I(K; m; p)$ in the class of all convex bodies containing the origin is attained when K is a triangle with a vertex at the origin and the maximum of $I(K; m; p)$ in the class of all origin symmetric convex bodies is attained when K is parallelogram.*

Proof. Since translations are trivial cases of parallel chord movements, Theorem 6 implies that $I(K - x; m; p)$ is a convex function of x. Thus, if P is a polygon containing the origin, then we can assume that the origin is one of the vertices of P. Now, following the same method used in the proof of Theorem 4 (keeping fixed the vertex at the origin), we can reduce P to a triangle without decreasing the value of the functional. An approximation argument leads to the conclusion.

The second part of the theorem easily follows by adapting the proof for the not symmetric case. $\square$

As already claimed in Section 2, every quermassintegral of a shadow system is a convex function of the parameter. This means that the functionals

$$\left[\frac{1}{V(K)^{q+1-\frac{iq}{d}}} \int\limits_K W_i(\Gamma_p(K-x))^q\, dx\right]^{1/q},$$

$$\frac{1}{V(K)^{m+p-\frac{ip}{d}}} \int\limits_K \dots \int\limits_K W_i\left(\sum_{j=1}^{m} \overline{0x_j}\right)^p dx_0 \dots dx_m\,,$$

where W_i denotes the i-th quermassintegral, keep the same convexity property as $A(K;p;q)$ and $I(K;m;p)$. Notice that W_0 is the volume, W_1 and W_{d-1}, up to constants, the surface area and the mean width, respectively. The last two coincide with the perimeter when $d = 2$.

Hence, ellipsoids are minimizers of such new functionals and triangles (or parallelograms) are maximizers, for $d = 2$.

For the functional $S(K;m;p)$, defined in (1.3), this type of generalization has been recently considered by Hartzoulaki and Paouris in [HP], where a characterization of balls as minimizers is shown.

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