ESTIMATES OF LOOMIS-WHITNEY TYPE FOR INTRINSIC VOLUMES

STEFANO CAMPI AND PAOLO GRONCHI

Abstract. The Loomis-Whitney inequality is a sharp estimate from above of the volume of a compact subset of $\mathbb{R}^n$ in terms of the product of the areas of its projections along the coordinate directions.

This paper deals with estimates from above of intrinsic volumes of a convex body in terms of sums of intrinsic volumes of finitely many orthogonal projections of the body itself.

We show that suitable polytopes maximize the surface area in the class of convex bodies whose projections along fixed directions have assigned surface area. A sharp estimate of the mean width of a convex body in terms of the mean widths of the coordinate projections is proved. An analogous estimate for intrinsic volumes of any order is conjectured and discussed. We prove that the conjecture holds true under the assumption that the coordinate projections satisfy an equilibrium condition and we show that such a condition is fulfilled in special classes of convex bodies.

1. Introduction

The estimation of the size of a 3d object from measurements related to 2d projections of the object itself is a problem arising in various applied contexts.

Examples of this type can be found in the microscopical study of biological tissues, when one is interested in evaluating number or volume or sizes of particular cells from samples which usually correspond to sections of the tissues. A mathematical approach, based mainly on stochastic methods, is supplied in this case by Stereology (see the book [33, Ch. 1] and the reviews [22] and [23]). As showed in [34], the data involved in the stereological procedures can be achieved through measures on projections. Note that the results in [34] apply to automatic processes in bio-agriculture (see also [35]) and that the objects under observations are here of larger size than cells. Estimations of the average size of convex particles in a 3d microstructure from projected images are given in [21].

Another relevant example is supplied in geochemistry by the study of fluid inclusions in minerals (see, for instance, [1] and [32, Ch. 3]), when one attempts to estimate volume-fractions of liquid and vapour phases from the two-dimensional projections of the inclusion.

Further examples leading to the same type of problem can be taken from astrophysics. This is the case, for instance, when the object to study is an asteroid and its movement allows to take different images, namely different projections of the celestial body. We refer to [15] and [29], where geometric and physical features of an asteroid are recovered from its lightcurve,

2010 *Mathematics Subject Classification.* 52A40.

i.e from the area of the projection as a function of time. See also [25] for an updated review on this subject.

Finally, in a wider setting, we can mention also problems of estimation of sizes in computing systems, when large high-dimensional data set have to be processed [24], or, more in general, problems from the emerging area of Compressed Sensing [16].

In this paper the objects we deal with are convex bodies in $\mathbb{R}^n$ and we are interested in estimating intrinsic volumes of a body in terms of intrinsic volumes of finitely many projections of the body itself. Our approach can be set in the framework of Geometric Tomography as well as in that of the Brunn-Minkowski theory for convex sets. The books by Gardner [19] and Schneider [30] are exhaustive treatises on these subjects and we shall refer to them for all the basic results and formulas quoted in what follows.

First, let us recall the notion of intrinsic volume.

Let K be a *convex body* in $\mathbb{R}^n$, i.e. a compact convex subset of $\mathbb{R}^n$. If B denotes the unit ball in $\mathbb{R}^n$ and t is a positive parameter, then the n-dimensional volume λ_n of the vector sum $K + tB$ can be expressed by the Steiner formula:

$$\lambda_n(K + tB) = \sum_{i=0}^{n} \kappa_{n-i} V_i(K) t^{n-i} ,$$

where κ_m is the volume of the unit ball in $\mathbb{R}^m$, $\kappa_0 = 1$, and the $V_i(K)$'s are just the *intrinsic volumes* of K. Clearly, $V_n(K) = \lambda_n(K)$ and, up to a constant, $V_{n-1}(K)$ and $V_1(K)$ are the surface area and the mean width of K, respectively. If the dimension of K satisfies $\dim K \le i$, then $V_i(K)$ coincides with the i-dimensional Lebesgue measure $\lambda_i(K)$ of K, and it turns out that intrinsic volumes do not depend on the dimension of the ambient space.

Note that, for $1 \le m \le n$, intrinsic volumes can be expressed in terms of *mixed volumes* of K and B by

$$V_m(K) = \frac{\binom{n}{m}}{\kappa_{n-m}} V(\underbrace{K, \dots, K}_{m}, \underbrace{B, \dots, B}_{n-m}) . \tag{1}$$

Alternatively, they can be expressed in terms of *mixed area measures* $S_m(K; \cdot)$ of K by

$$V_m(K) = \frac{\binom{n}{m}}{n\kappa_{n-m}} S_m(K; S^{n-1}) = \frac{1}{n} \int_{S^{n-1}} h_K(v)\, dS_{m-1}(K; v) ; \tag{2}$$

see [30, Ch. 5] or [19, App. A]. Note that $S_0(K; \cdot)$ coincides with the $(n-1)$-dimensional Hausdorff measure. Here, h_K is the *support function* of the convex body K, which is defined by

$$h_K(x) = \max\{\langle x, y \rangle : y \in K\},\ x \in \mathbb{R}^n , \tag{3}$$

where $\langle x, y \rangle$ denotes the scalar product in $\mathbb{R}^n$.

For every u in the unit sphere S^{n-1} of $\mathbb{R}^n$, if $u^\perp$ denotes the hyperplane through the origin perpendicular to u, $K|u^\perp$ denotes the orthogonal projection of K onto $u^\perp$, and $[-u, u]$ the segment joining $-u$ and u, then

$$V_m(K|u^\perp) = \frac{n\binom{n-1}{m}}{2\kappa_{n-m-1}} V(\underbrace{K,\dots,K}_{m},\underbrace{B,\dots,B}_{n-m-1},[-u,u])\,. \tag{4}$$

The projection body ΠK of the convex body K is the origin-symmetric convex body whose support function is the *brightness function* of K, that is, for $u \in S^{n-1}$,

$$h_{\Pi K}(u) = \lambda_{n-1}(K|u^\perp) = \frac{1}{2}\int_{S^{n-1}} |\langle u, v\rangle|\, dS_{n-1}(K;v)\,, \tag{5}$$

and, for $1 \le m < n-1$, the mth projection body $\Pi_m K$ of K is the origin-symmetric convex body whose support function is the mth *girth function* of K, that is, for $u \in S^{n-1}$,

$$h_{\Pi_m K}(u) = \frac{\kappa_{n-m-1}}{\binom{n-1}{m}} V_m(K|u^\perp) = \frac{1}{2}\int_{S^{n-1}} |\langle u, v\rangle|\, dS_m(K;v)\,. \tag{6}$$

In what follows we shall denote by $e_1, e_2, \dots, e_n$ the standard orthonormal basis of $\mathbb{R}^n$. A basic result of interest from different mathematical points of view is the following inequality of Loomis and Whitney [27]: For any bounded Borel set A in $\mathbb{R}^n$,

$$\lambda_n(A)^{n-1} \le \prod_{i=1}^{n} \lambda_{n-1}(A|e_i^\perp)\,. \tag{7}$$

As noted by the authors, inequality (7) is of isoperimetric type. Indeed, denoted by ∂A the boundary of A, we have $\lambda_{n-1}(\partial A) \ge 2\lambda_{n-1}(A|e_i^\perp)$, for every i. Therefore (7) implies

$$\lambda_n(A)^{n-1} \le 2^{-n}\lambda_{n-1}(\partial A)^n\,,$$

an isoperimetric inequality without the best constant.

Clearly, in (7) equality holds if A is a box. By a box we mean a rectangular parallelotope with facets parallel to the coordinate hyperplanes. The argument used in [27] can be adapted to show that in the class of convex bodies, only boxes give equality in (7). The same result can be found, for instance, in [12], where characterizations of convex bodies of maximal volume with given brightness in finitely many directions are provided.

By using the inequality between geometric and arithmetic mean, from (7) one deduces that

$$\lambda_n(A)^{n-1} \le \left[\frac{1}{n}\sum_{i=1}^{n} \lambda_{n-1}(A|e_i^\perp)\right]^n\,, \tag{8}$$

where for convex sets equality holds if and only if A is a cubic box.

A generalization of (7) involving the projections onto all the m-dimensional subspaces spanned by $e_1, e_2, \dots, e_n$ was given by Burago and Zalgaller [11, p. 95].

A further generalization is due to Bollobas and Thomason [9], who showed that, given a bounded Borel set A in $\mathbb{R}^n$, there exists a box Z such that

$$\lambda_n(Z) = \lambda_n(A) \quad \text{and} \quad \lambda_k(Z|S) \le \lambda_k(A|S)\,,$$

for every coordinate subspace S, where k is the dimension of S.

For convex bodies Ball [2] generalized (7) to the case of projections along a set of directions satisfying John's condition and pointed out the connection between the Loomis-Whitney inequality and the Brascamp-Lieb inequality (see [3], [4], [10], [26]).

Zhang [36] extended Ball's result to compact sets, obtaining in such a way, as a functional counterpart, a more general version of the Gagliardo-Nirenberg inequality (see [18] and [28]).

More recently, further functional extensions and generalizations of the Loomis-Whitney inequality were given, for instance, by Bennett, Carbery and Wright [5], Bennett, Carbery and Tao [6] and Bobkov and Nazarov [8].

A second type of estimate can be related to the Cauchy formula for the surface area $S(K)$ of K (see, for example, [19, A.49]):

$$S(K) = \frac{1}{\kappa_{n-1}} \int_{S^{n-1}} \lambda_{n-1}(K|u^{\perp})\, du\,. \tag{9}$$

Thus, by (5) one can expect to estimate the surface area of K in terms of finitely many values of $h_{\Pi K}(u)$. This is just what Betke and McMullen [7] did in the following result. Denote by Z the zonotope

$$Z = \sum_{i=1}^{N} a_i\, [-u_i, u_i]\,,$$

where the a_i's are given positive numbers, and by $r(Z)$ and $R(Z)$ the inradius and the circumradius of Z, respectively.

If K is a convex body in $\mathbb{R}^n$ then

$$r(Z)V_{n-1}(K) \leq \sum_{i=1}^{N} a_i V_{n-1}(K|u_i^{\perp}) \leq R(Z)V_{n-1}(K)\,.$$

Equality on the left-hand (right-hand) side occurs precisely when the support of the $(n-1)$-area measure of K is contained in the subset of S^{n-1} where the support function of Z is minimal (maximal, respectively).

If the u_i's are the coordinate vectors $e_1, e_2, \dots, e_n$ and $a_i = 1$, for every i, then the previous left-hand side inequality reduces to

$$V_{n-1}(K) \leq \sum_{i=1}^{n} V_{n-1}(K|e_i^{\perp})\,, \tag{10}$$

with equality if and only if K is a box.

Inequalities (7) and (10) can be considered as the starting point of the present paper.

In Section 2 we consider the class of all convex bodies whose projections along finitely many directions spanning $\mathbb{R}^n$ have the same surface areas as a given convex body with nonempty interior. We show that, if there exists in this class an element of maximal surface area, then it has to be a polytope with all facets orthogonal to special directions. Furthermore, we supply a sufficient condition for the existence, in the same class, of elements of maximal surface area and we show that uniqueness is in general not guaranteed. All these results, if applied to the particular case of the coordinate projections, retrieve (10).

Cauchy's formula (9) for the surface area can be extended to any intrinsic volume of a convex body K through Kubota's formulas (see, for example, [19, A.48]):

$$V_m(K) = \frac{\kappa_{n-m-1}}{(n-m)\kappa_{n-m}\kappa_{n-1}} \int_{S^{n-1}} V_m(K|u^\perp)\,du\ , \quad 1 \le m \le n-1\,, \tag{11}$$

which in turn suggests that one can expect an estimate of $V_m(K)$ in terms of $\sum_{i=1}^n V_m(K|e_i^\perp)$.

In Section 3 we prove a sharp estimate of this type, for $m = 1$.

Section 4 is devoted to study the case of $V_m(K)$, for any m between 1 and $n-1$. We show that

$$V_m(K) \le \frac{1}{n-m} \sum_{i=1}^n V_m(K|e_i^\perp)\,,$$

provided the mth intrinsic volumes of the projections of K satisfy the equilibrium condition

$$V_m(K|e_j^\perp) \le \frac{1}{n-m} \sum_{i=1}^n V_m(K|e_i^\perp)\,,\ \text{for every } 1 \le j \le n\,,$$

which is trivial for $m = n-1$ and is proved in Section 3 for $m = 1$. Moreover, we show that the above inequalities are satisfied by bodies of dimension m as well as by zonoids. We conjecture the same holds true for a general convex body.

Finally we prove that, if K fulfills the above equilibrium condition, then there exists a box Z whose coordinate projections have the same mth intrinsic volume as K, and that $V_{m+1}(K) \le V_{m+1}(Z)$. For $m = n-1$, this is nothing but the Loomis-Whitney inequality for convex sets. For $m = 1$, we obtain a sharp upper bound for $V_2(K)$ in terms of the mean widths of the coordinate projections of K.

2. Rearranging the $(n-1)$-area measure

Let U be a set of unit vectors $u_1, u_2, \ldots, u_N$ spanning $\mathbb{R}^n$. The hyperplanes $u_1^\perp, u_2^\perp, \ldots, u_N^\perp$ divide S^{n-1} into the spherical closed polytopes $\omega_1, \omega_2, \ldots, \omega_p$ having disjoint interiors. For brevity, we call a vertex of one of these polytopes a *node*. Given a convex body K in $\mathbb{R}^n$, let $\Phi(K;U)$ be the class of all convex bodies H such that

$$V_{n-1}(H|u_i^\perp) = V_{n-1}(K|u_i^\perp)\ , \quad \text{for every } i = 1, 2, \ldots, N\,.$$

In [12] (see also [19, Th. 4.4.2]) it is proved that in $\Phi(K;U)$ there exists a unique element of maximal volume, which is a centrally symmetric polytope, having each facet orthogonal to some node.

The technique used in [12] can be applied in searching for elements of maximal surface area in $\Phi(K;U)$.

Theorem 2.1. *Let K be a convex body in $\mathbb{R}^n$ with nonempty interior. If K is not a polytope having each facet orthogonal to some node, then there exists a centrally symmetric polytope P in $\Phi(K;U)$, with each facet orthogonal to some node, such that*

$$V_{n-1}(K) < V_{n-1}(P)\,.$$

Proof. First we assume that K is a polytope with r facets. Let μ_1, μ_2, ..., μ_r be the outward normal vectors to the facets of K, with $\|\mu_i\|$ equal to the $(n-1)$-measure of the ith facet, for every i. Note that, by Minkowski's theorem (see [30, p. 390]), the sum of these vectors equals the zero vector.

Without loss of generality, we can assume that $\mu_1/\|\mu_1\|$ does not coincide with any node and is contained in ω_1. Hence $\mu_1 = \sum_{i=1}^q \lambda_i v_{1,i}$, where the $v_{1,i}$'s are the vertices of ω_1 and the λ_i's are nonnegative numbers. Note that such a decomposition may not be unique. The vectors $\lambda_1 v_{1,1}, \lambda_2 v_{1,2}, \ldots, \lambda_q v_{1,q}$, μ_2, ..., μ_r span $\mathbb{R}^n$ and their sum is the zero vector. Therefore, by Minkowski's theorem, there exists a polytope $\tilde{K}$ whose facets are orthogonal to those vectors and have the same $(n-1)$-measures as the norms of the vectors (possibly adding up vectors along the same direction). For every $u \in S^{n-1}$, by (5) we have that

$$V_{n-1}(\tilde{K}|u^\perp) - V_{n-1}(K|u^\perp) = \frac{1}{2}\sum_{i=1}^q \lambda_i\,|\langle v_{1,i}, u\rangle| - \frac{1}{2}|\langle \sum_{i=1}^q \lambda_i v_{1,i}, u\rangle| \geq 0\,. \tag{12}$$

Note that the above difference is strictly positive if and only if the sign of $\langle v_{1,i}, u\rangle$ is not constant with respect to i. In particular, if $u^\perp$ does not intersect the interior of ω_1, then $\tilde{K}$ and K have the same brightness along u. Consequently, $\tilde{K}$ is in $\Phi(K;U)$ and, by the Cauchy formula (9), $V_{n-1}(\tilde{K}) \geq V_{n-1}(K)$.

If $V_{n-1}(\tilde{K}) = V_{n-1}(K)$, then, by the Cauchy formula (9), (12) and the continuity of the brightness function of a convex body, equality holds in (12) for every $u \in S^{n-1}$. Consequently (see [19, Th. 3.3.2]), the even parts of the $(n-1)$-area measures of K and $\tilde{K}$ are the same. This contradicts the assumption that $\mu_1/\|\mu_1\|$ does not coincide with any node.

If $\tilde{K}$ is not centrally symmetric, then we can replace it by its Blaschke body $\nabla\tilde{K}$ (see [19, p. 116]), which is also in $\Phi(K;U)$, satisfies $V_{n-1}(\nabla\tilde{K}) = V_{n-1}(\tilde{K})$ and is centrally symmetric.

Applying the same argument to all the μ_i's proves the theorem when K is a polytope.

To conclude the proof, assume now that K is an arbitrary convex body with nonempty interior and take a sequence $\{K_i\}$ of polytopes converging to K in the Hausdorff metric. For each K_i there exists in $\Phi(K_i;U)$ a centrally symmetric polytope P_i, with each facet orthogonal to some node, such that $V_{n-1}(P_i) \geq V_{n-1}(K_i)$. Since the $(n-1)$-area measure of each P_i is discrete and is concentrated at the nodes, up to subsequences, the $(n-1)$-area measure of P_i converges to an even measure σ, which cannot be concentrated on any great sphere. Indeed, if for some $w \in S^{n-1}$ the support of σ is contained in $w^\perp \cap S^{n-1}$, then, by (12)

$$0 = \lim_{i\to\infty} V_{n-1}(P_i|w^\perp) \geq \lim_{i\to\infty} V_{n-1}(K_i|w^\perp) = V_{n-1}(K|w^\perp)\,,$$

which is impossible, since K has nonempty interior. Therefore, by Minkowski's theorem, there exists a centrally symmetric polytope P whose $(n-1)$-area measure is σ. Clearly, P belongs to $\Phi(K;U)$ and $V_{n-1}(P) \geq V_{n-1}(K)$. Now assume that $V_{n-1}(P) = V_{n-1}(K)$. Since, for every $u \in S^{n-1}$,

$$V_{n-1}(P|u^\perp) = \lim_{i\to\infty} V_{n-1}(P_i|u^\perp) \geq \lim_{i\to\infty} V_{n-1}(K_i|u^\perp) = V_{n-1}(K|u^\perp)\,,$$

we deduce, again by the Cauchy formula (9), that K and P have the same brightness function. We conclude by [19, Th. 3.3.2] that the $(n-1)$-area measure of K is concentrated on the nodes.

This concludes the proof.

□

Theorem 2.1 suggests that looking for bodies of maximal surface area in $\Phi(K;U)$ can be reduced to a finite dimensional problem. Unlike the case of the volume, it can happen that in $\Phi(K;U)$ no body of maximal surface area exists. To see this, consider the polytope

$$L = \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \leq V_{n-1}(K|u_i^\perp),\ \text{for } i = 1, 2, \ldots, N\}\,. \tag{13}$$

All the projection bodies of elements from $\Phi(K;U)$ are inscribed in L, i.e. are contained in L and touch the faces (possibly lower-dimensional) of L orthogonal to the u_i's. In view of (2), (5) and (9), a body in $\Phi(K;U)$ maximizes the surface area if and only if its projection body maximizes the mean width among the projection bodies inscribed in L. All the cases of non-existence of maximizers can be related to sequences of projection bodies inscribed in L converging to a zonoid Z such that $1 < \dim Z < n$. Indeed, a zonoid whose dimension is between 1 and n is not a projection body in $\mathbb{R}^n$. A non-existence example can be obtained by taking K to be the unit ball in $\mathbb{R}^3$ and $u_1 = (\sin\theta, 0, \cos\theta)$, $u_2 = (0, \sin\theta, \cos\theta)$, $u_3 = (-\sin\theta, 0, \cos\theta)$ and $u_4 = (0, -\sin\theta, \cos\theta)$. Consequently, L is an octahedron. If $\theta < \theta_0$ (the value for which L is a regular octahedron), then straightforward computations show that the zonotope inscribed in L with maximal mean width is the square Q that is the horizontal central section of L. Hence, there exists a sequence of coordinate boxes Z_i inscribed in L and converging to Q. For every i, let C_i be the origin-symmetric coordinate box in $\Phi(K;U)$ such that $\Pi C_i = Z_i$. We have $V_{n-1}(H) < \lim_{i\to\infty} V_{n-1}(C_i) = \frac{n}{2\kappa_{n-1}} V_1(Q)$, for every $H \in \Phi(K;U)$.

The above observations suggest sufficient conditions for the existence of a maximizer. For instance, if no hyperplane intersecting all the facets of L exists, then there exists a surface area maximizer in $\Phi(K;U)$.

As far as the uniqueness (up to translations) is concerned, while in $\Phi(K;U)$ there exists only one element of maximal volume, the uniqueness of the surface area maximizer is not guaranteed. More precisely, if a centrally symmetric polytope P of maximal surface area in $\Phi(K;U)$ is not a parallelotope, then there exist infinitely many polytopes in $\Phi(K;U)$ with the same surface area as P. Indeed, in such a case one can redistribute the $(n-1)$-area measure on the nodes without changing the even part of the $(n-1)$-area measure (see [20]).

In the special case $U = \{e_1, e_2, \ldots, e_n\}$, the polytope L defined by (13) is a parallelotope. Hence, the zonoid with maximal mean width inscribed in L is L itself. Such a zonoid is the projection body of a unique convex body, which is a coordinate box. Thus, we obtain (10), with equality if and only if K is a box.

Note that the technique of rearranging the $(n-1)$-area measure of K to find maximizers of V_n or V_{n-1} in $\Phi(K;U)$ does not work in general for mth intrinsic volumes with $m < n-1$. A counterexample in the case $m = 1$ was given by the authors in [14].

3. An estimate for the mean width

Inequality (10) and the related equality conditions can be also obtained by basic properties of mixed volumes in the following way.

Let C be the origin-symmetric cube $[-1,1]^n$ in $\mathbb{R}^n$. Thus $C = \sum_{i=1}^n [-e_i, e_i]$. Since $B \subset C$, the monotonicity of mixed volumes implies

$$V(\underbrace{K,\dots,K}_{n-1},B) \le V(\underbrace{K,\dots,K}_{n-1},C)\,. \tag{14}$$

Note that

$$V(\underbrace{K,\dots,K}_{n-1},C) = \frac{1}{n}\int_{S^{n-1}} h_C(v)\,dS_{n-1}(K;v)\,.$$

Hence, by (2), equality in (14) holds if and only if the support of the $(n-1)$-area measure of K is concentrated on the coordinate axes, i.e. if and only if K is a box.

Therefore, by (1) and (4) with $m = n-1$ we obtain

$$V_{n-1}(K) = \frac{n}{2}V(\underbrace{K,\dots,K}_{n-1},B) \le \frac{n}{2}V(\underbrace{K,\dots,K}_{n-1},\sum_{i=1}^n[-e_i,e_i]) = \sum_{i=1}^n V_{n-1}(K|e_i^\perp)\,.$$

We can follow an analogous procedure in dealing with the first intrinsic volume of K, which can be expressed by (1) as

$$V_1(K) = \frac{n}{\kappa_{n-1}}V(\underbrace{B,\dots,B}_{n-1},K)\,. \tag{15}$$

If $z \in \mathbb{R}^n$, then

$$z = \frac{1}{n-1}\sum_{i=1}^n (z|e_i^\perp)\,. \tag{16}$$

Thus, for every convex body K,

$$K \subset \frac{1}{n-1}\sum_{i=1}^n (K|e_i^\perp)\,. \tag{17}$$

Therefore, by (15) and (17), we obtain

$$V_1(K) = \frac{n}{\kappa_{n-1}}V(\underbrace{B,\dots,B}_{n-1},K) \le \frac{n}{(n-1)\kappa_{n-1}}V\left(\underbrace{B,\dots,B}_{n-1},\sum_{i=1}^n(K|e_i^\perp)\right) = \frac{1}{n-1}\sum_{i=1}^n V_1(K|e_i^\perp)\,.$$

We show that equality holds if and only if K is a box. By (2) and (17),

$$V(\underbrace{B,\dots,B}_{n-1},K)=\frac{1}{n-1}V\left(\underbrace{B,\dots,B}_{n-1},\sum_{i=1}^{n}(K|e_i^{\perp})\right)$$

if and only if, for every $z\in\mathbb{R}^n$,

$$h_K(z)=\frac{1}{n-1}\sum_{i=1}^{n}h_{K|e_i^{\perp}}(z)\,,$$

or equivalently, since $h_{K|u^{\perp}}(z)=h_K(z|u^{\perp})$, if and only if

$$h_K(z)=\frac{1}{n-1}\sum_{i=1}^{n}h_K(z|e_i^{\perp})\,. \tag{18}$$

By (3) and (16), equality (18) holds if and only if, for every $z\in\mathbb{R}^n$, there exists $p\in K$ such that $h_K(z)=\langle p,z\rangle$ and $h_K(z|e_i^{\perp})=\langle p,z|e_i^{\perp}\rangle$, for every $i=1,2,\dots,n$.

Such a condition means that the normal cone of K at p contains all the projections onto the coordinate hyperplanes of its interior points. Hence, the normal cone at every vertex of K is a union of orthants. Consequently, boxes are the unique bodies satisfying (18).

Thus we have proved the following result.

Theorem 3.1. *For every convex body K in $\mathbb{R}^n$,*

$$V_1(K)\le\frac{1}{n-1}\sum_{i=1}^{n}V_1(K|e_i^{\perp})\,, \tag{19}$$

with equality if and only if K is a box.

Inequalities (10) and (19) and the related equality conditions suggest the following natural question.

Given a convex body K, does there exist a box Z such that, for every $i=1,2,\dots,n$,

$$V_{n-1}(K|e_i^{\perp})=V_{n-1}(Z|e_i^{\perp})\,, \tag{20}$$

or

$$V_1(K|e_i^{\perp})=V_1(Z|e_i^{\perp})\,? \tag{21}$$

In both cases the answer is positive.

The existence of a box satisfying (20) is algebraically trivial, while the existence of a box satisfying (21) will follow from a result contained in the next section, where we deal with intrinsic volumes of any order.

4. Estimates for mth intrinsic volumes

The existence of a box whose coordinate projections have the same mth intrinsic volumes as those of a given convex body K is the object of the following theorem. Note that condition (22) given below trivially holds when $m = n-1$.

Theorem 4.1. *Let $1 \le m \le n-1$ and let K be a convex body in $\mathbb{R}^n$. There exists a box Z such that*

$$V_m(K|e_i^\perp) = V_m(Z|e_i^\perp)\,,\ i = 1, 2, \dots, n$$

if and only if

$$V_m(K|e_j^\perp) \le \frac{1}{n-m}\sum_{i=1}^n V_m(K|e_i^\perp)\,,\ \text{for every } 1 \le j \le n\,. \tag{22}$$

Proof. Let $Z = \sum_{i=1}^n a_i[-e_i, e_i]$. By (4) and the multilinearity of mixed volumes, we have that

$$\begin{aligned} V_m(Z|e_i^\perp) &= \tfrac{n\binom{n-1}{m}m!}{2\kappa_{n-m-1}} \sum_{1\le i_1<\dots<i_m\le n} a_{i_1}\dots a_{i_m} V([-e_{i_1}, e_{i_1}], \dots, [-e_{i_m}, e_{i_m}], \underbrace{B, \dots, B}_{n-m-1}, [-e_i, e_i]) \\ &= 2^m \sum_{1\le i_1<\dots<i_m\le n} a_{i_1}\dots a_{i_m}\,. \end{aligned}$$

Hence, setting $q_j = V_m(K|e_j^\perp)$, $q = (q_1, q_2, \dots, q_n)$ and denoting by $\sigma_m(a_1, \dots, a_n)$ the elementary symmetric polynomial of degree m in the variables a_i, the statement of the theorem can be rephrased equivalently as follows.

The system

$$\begin{cases} 2^m\sigma_m(a_2, a_3, \dots, a_n) = q_1 \\ 2^m\sigma_m(a_1, a_3, \dots, a_n) = q_2 \\ \quad\vdots \\ 2^m\sigma_m(a_1, a_2, \dots, a_{n-1}) = q_n \end{cases} \tag{23}$$

has a solution $a = (a_1, a_2, \dots, a_n)$ with nonnegative entries if and only if

$$q_j \le \frac{1}{n-m}\sum_{i=1}^n q_i\,, \tag{24}$$

for all $j = 1, 2, \dots, n$.

First assume that system (23) has a nonnegative solution and let $1 \le j \le n$. Adding all the equations in (23) and subtracting $(n-m)$ times the jth equation gives

$$(n-m)2^m a_j\sigma_{m-1}(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n) = \sum_{i=1}^n q_i - (n-m)q_j\,.$$

Since all a_i's are nonnegative, the left-hand side is nonnegative and so (24) is satisfied.

Now assume that (24) holds and that $q_i > 0$ for $i = 1, 2, \dots, n$. Consider the function $f(a) = 2^m\sigma_{m+1}(a_1, \dots, a_n)$ on the polytope P obtained as intersection of the nonnegative orthant and the hyperplane $\langle a, q\rangle = 1$. The function f is analytic and the polytope P is compact. Hence, an absolute maximum is attained, say at x.

If x is a point in the relative interior of P, then $\nabla f(x) = \lambda q$, with $\lambda > 0$. By the homogeneity of f we conclude that a multiple of x satisfies (23).

If x is on the boundary of P, then we can assume, due to the symmetry of P and f, that $x_1 = x_2 = \cdots = x_h = 0$ and $x_i > 0$ for $i > h \geq 1$. Note that the maximum $f(x)$ is surely positive; consequently $h < n - m - 1$. Moreover, $\nabla f(x)$ is a linear combination of q, e_1, e_2, $\ldots$, e_h, namely

$$\left\{\begin{array}{c} 2^m\sigma_m(x_2, x_3, \ldots, x_n) = \lambda_0 q_1 + \lambda_1 \\ 2^m\sigma_m(x_1, x_3, \ldots, x_n) = \lambda_0 q_2 + \lambda_2 \\ \vdots \\ 2^m\sigma_m(x_1, x_2, \ldots, x_{n-1}) = \lambda_0 q_n + \lambda_n \end{array}\right. , \tag{25}$$

where $\lambda_i = 0$ for $i > h$.

Multiplying the ith equation by x_i, $i = 1, 2, \ldots, n$, and summing yields

$$2^m(m+1)\sigma_{m+1}(x_1, \ldots, x_n) = \lambda_0\,,$$

where we used the relation $\langle x, q\rangle = 1$ and $x_i\lambda_i = 0$ for all i. Hence λ_0 is nonnegative.

For all $1 \leq j \leq h$, we have that $\lambda_j \leq 0$. Indeed, if we consider the vector

$$v_j = q - \frac{\sum_{i=h+1}^n q_i^2}{q_j} e_j - \sum_{i=1}^h q_i e_i\,,$$

then $\langle q, v_j\rangle = 0$ and $\langle e_i, v_j\rangle \leq 0$ for all $1 \leq i \leq h$. Since the derivative of f at x in the directions v_j has to be positive, we infer that

$$0 \leq \langle \lambda_0 q + \sum_{i=1}^h \lambda_i e_i, v_j\rangle = -\lambda_j \frac{\sum_{i=h+1}^n q_i^2}{q_j}\,,$$

which implies $\lambda_j \leq 0$.

Summing all equations in (25) and subtracting $(n-m)$ times the jth equation yields

$$2^m(n-m)x_j\sigma_{m-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = \lambda_0(\sum_{i=1}^n q_i \ - (n-m)q_j) + \sum_{i=1}^h \lambda_i \ - (n-m)\lambda_j\,.$$

If in turn we add all these equations for $1 \leq j \leq h$, then we get

$$0 = \lambda_0\left(h\sum_{i=1}^n q_i - (n-m)\sum_{i=1}^h q_i\right) - (n-m-h)\sum_{i=1}^h \lambda_i\,. \tag{26}$$

The coefficient of λ_0 in (26) is nonnegative. Indeed, summing both sides of (24) from 1 to h gives

$$\sum_{i=1}^h q_i \leq \frac{h}{n-m}\sum_{i=1}^n q_i\,.$$

Since $\lambda_i \leq 0$ for $1 \leq i \leq h$ and $n > m + h$, from (26) we deduce $\lambda_i = 0$ also for $1 \leq i \leq h$. By the homogeneity of f we conclude by (25) that a multiple of x satisfies (23).

To conclude the proof we have to focus on the case where some of the q_i's vanish. If, say $q_1 = 0$, then the body K is such that $K|e_1^\perp$ has dimension less than m. If $\dim K < m$, then the theorem holds trivially. Otherwise, if $\dim K = m$, we consider a sequence of n-dimensional convex bodies K_i converging to K in the Hausdorff metric. For each K_i we proved the existence of a box Z_i whose coordinate projections have the same mth intrinsic volume as those of K_i. Up to a subsequence, Z_i converges to a box Z with the properties we seek. □

If K has dimension greater than m, then the box whose existence is claimed in Theorem 4.1 is unique. We omit the details here, but note that this can be shown by using the strict concavity of the function f introduced in the previous proof or via the Aleksandrov-Fenchel inequality as used in the proof of Theorem 4.5 below.

Theorem 4.1 can be used to prove the following result.

Theorem 4.2. *If a convex body K in $\mathbb{R}^n$ satisfies*

$$V_m(K) \leq \frac{1}{n-m} \sum_{i=1}^{n} V_m(K|e_i^\perp)\,, \tag{27}$$

then there exists a box Z such that

$$V_m(K|e_i^\perp) = V_m(Z|e_i^\perp)\,,\ i = 1, 2, \ldots, n\,.$$

Such a body Z is a maximizer of V_m under the previous constraints.

Proof. We claim that if K satisfies inequality (27), then it also satisfies (22). Indeed, it suffices to prove

$$V_m(K|e_j^\perp) \leq V_m(K)\,,\ \text{for every } j.$$

To see this, note that the body K can be represented in the form

$$K = \{x + y \in \mathbb{R}^n | x \in e_j^\perp,\ f(x) \leq y \leq g(x)\}\,,$$

where f and g are suitable functions. Define

$$K(t) = \{x + y \in \mathbb{R}^n | x \in e_j^\perp,\ (1-t)f(x) - tg(x) \leq y \leq (1-t)g(x) - tf(x)\}\,,$$

for every $t \in [0, 1]$. Note that $K(0) = K$ and $K(1)$ is the reflection of K in the hyperplane $e_j^\perp$. The family $\{K(t)\}_{t\in[0,1]}$ is a shadow system and it is known (see, for instance, [31] and [13]) that the mth intrinsic volume of $K(t)$ is a convex function of t. Since $V_m(K) = V_m(K(1))$ and $K|e_j^\perp$ is contained in $K(1/2)$ (the Steiner symmetral of K along the direction e_j), we have

$$V_m(K|e_j^\perp) \leq V_m(K(1/2)) \leq V_m(K)\,.$$

Thus the claim is proved and the existence of a box Z is guaranteed by Theorem 4.1.

To prove that $V_m(Z)$ is maximal among all bodies whose coordinate projections have the same mth intrinsic volumes as those of K, it is sufficient to observe that

$$V_m(Z) = \frac{1}{n-m} \sum_{i=1}^{n} V_m(Z|e_i^\perp)\,.$$

□

We now focus on inequality (27). We already know that it holds when $m = n-1$ and $m = 1$, with equality in both cases if and only if K is a box. Does it hold for every m and for every convex body K?

A weaker inequality than (27) can be obtained as follows. Every $u \in S^{n-1}$ can be written as

$$u = \sum_{i=1}^{n} |\langle u, e_i\rangle| (sign\langle u, e_i\rangle) e_i\,.$$

Therefore, by the sublinearity of support functions and the fact that $\Pi_m K$ is origin symmetric,

$$h_{\Pi_m K}(u) \le \sum_{i=1}^{n} |\langle u, e_i\rangle| h_{\Pi_m K}((sign\langle u, e_i\rangle) e_i) = \sum_{i=1}^{n} |\langle u, e_i\rangle| h_{\Pi_m K}(e_i)\,.$$

Thus, by the left-hand equality in (6),

$$V_m(K|u^\perp) \le \sum_{i=1}^{n} |\langle u, e_i\rangle| V_m(K|e_i^\perp)\,.$$

(Compare the inequality of Firey [17].)

Integrating both sides of the above inequality on S^{n-1}, by Kubota's formulas (11) we obtain

$$V_m(K) \le \frac{2\kappa_{n-m-1}}{(n-m)\kappa_{n-m}} \sum_{i=1}^{n} V_m(K|e_i^\perp)\,,$$

where we used that, for every i,

$$\int_{S^{n-1}} |\langle u, e_i\rangle|\, du = 2\kappa_{n-1}\,.$$

(See (5) with $K = B$.) Note that

$$1 \le \frac{2\kappa_{n-m-1}}{\kappa_{n-m}}\,,$$

with equality only if $m = n-1$.

By (6) and the left-hand equality in (2), inequality (27) can be rewritten as follows:

$$\int_{S^{n-1}} \left(1 - \frac{\kappa_{n-m}}{2\kappa_{n-m-1}} \sum_{i=1}^{n} |\langle u, e_i\rangle|\right) dS_m(K; u) \le 0\,. \tag{28}$$

The quantity

$$1-\frac{\kappa_{n-m}}{2\kappa_{n-m-1}}\sum_{i=1}^{n}|\langle u,e_i\rangle|$$

is the difference between the support function of the unit ball and the support function of the origin-symmetric cube with edge length $\kappa_{n-m}/\kappa_{n-m-1}$ and, for $m<n-1$, it assumes positive values on a subset of S^{n-1}. Only in the case $m=n-1$ the cube entirely contains the ball, so the inequality

$$1-\frac{\kappa_{n-m}}{2\kappa_{n-m-1}}\sum_{i=1}^{n}|\langle u,e_i\rangle|\le 0$$

holds for every u only when $m=n-1$. Thus, if $m<n-1$, the inequality

$$\int_{S^{n-1}}\left(1-\frac{\kappa_{n-m}}{2\kappa_{n-m-1}}\sum_{i=1}^{n}|\langle u,e_i\rangle|\right)d\mu(u)\le 0$$

does not hold in general for an arbitrary positive measure μ such that $\int_{S^{n-1}}u\,d\mu(u)=0$. Therefore, proving inequality (28) should require the use of suitable features of $S_m(K;\cdot)$.

Note that

$$\frac{\kappa_{n-m}}{\kappa_{n-m-1}}=\sqrt{\pi}\,\frac{\Gamma(\frac{n-m+1}{2})}{\Gamma(\frac{n-m+2}{2})}$$

and then, if $p=n-m$,

$$\frac{\kappa_p}{\kappa_{p-1}}\sim\sqrt{\frac{\pi}{1+p/2}}\,.$$

On the other hand, the length of the diagonal of the cube is $\sqrt{n}\kappa_{n-m}/\kappa_{n-m-1}$, which tends to $\sqrt{2\pi}$ as n tends to ∞. Notice also that, for fixed n, the smallest cube corresponds to $m=1$.

Inequality (27) turns out to be true for special classes of sets. A first example is given by the following result.

Lemma 4.3. *Let E be a compact set in $\mathbb{R}^n$ contained in a subspace of dimension m. Then*

$$\lambda_m(E)\le\frac{1}{n-m}\sum_{i=1}^{n}\lambda_m(E|e_i^{\perp})\,,$$

where equality holds if and only if E is contained in a coordinate m-dimensional subspace.

Proof. Let u_1, u_2, …, u_m be an orthonormal system of the subspace containing E. Note that, if $C=\sum_{i=1}^{m}[0,u_i]$, then $\lambda_m(E|e_i^{\perp})=\lambda_m(E)V_m(C|e_i^{\perp})$ for every $i=1,2,\dots,n$. Hence

$$\lambda_m(E|e_i^{\perp})=\lambda_m(E)\prod_{j=1}^{m}\sqrt{1-\langle u_j,e_i\rangle^2}\tag{29}$$

and we want to prove the inequality

$$\sum_{i=1}^{n}\prod_{j=1}^{m}\sqrt{1-\langle e_i,u_j\rangle^2}\geq n-m\,, \tag{30}$$

with equality if and only if $\{u_1,u_2,\dots,u_m\}\subset\{e_1,e_2,\dots,e_n\}$.

If σ_k again denotes the kth elementary symmetric polynomial, then expanding the product gives

$$\prod_{j=1}^{m}\big(1-\langle e_i,u_j\rangle^2\big)=1-\sum_{j=1}^{m}\langle e_i,u_j\rangle^2+\sum_{k=2}^{m}(-1)^k\sigma_k(\langle e_i,u_1\rangle^2,\dots,\langle e_i,u_m\rangle^2)\,.$$

Since $\sigma_1(\langle e_i,u_1\rangle^2,\dots,\langle e_i,u_m\rangle^2)\leq\|e_i\|=1$ and

$$\sigma_{2k+1}(x_1,\dots,x_m)\leq\sigma_{2k}(x_1,\dots,x_m)\sigma_1(x_1,\dots,x_m)\,,$$

for every choice of nonnegative numbers x_1, x_2, $\dots$, x_m, we infer

$$\prod_{j=1}^{m}\big(1-\langle e_i,u_j\rangle^2\big)\geq 1-\sum_{j=1}^{m}\langle e_i,u_j\rangle^2\,. \tag{31}$$

Now we can conclude that

$$\sum_{i=1}^{n}\prod_{j=1}^{m}\sqrt{1-\langle e_i,u_j\rangle^2}\geq\sum_{i=1}^{n}\prod_{j=1}^{m}\big(1-\langle e_i,u_j\rangle^2\big)\geq n-\sum_{j=1}^{m}\sum_{i=1}^{n}\langle e_i,u_j\rangle^2=n-m\,.$$

In order to have equality, all $\langle e_i,u_j\rangle$ must equal 0 or 1 and moreover we need m of these scalar products equal to 1. This means that each u_i must coincide with one of the e_i's. □

Theorem 4.3 can be used to prove that (27) also holds for zonoids.

Theorem 4.4. *Let Z be a zonoid in $\mathbb{R}^n$. Then*

$$V_m(Z)\leq\frac{1}{n-m}\sum_{i=1}^{n}V_m(Z|e_i^\perp)\,,$$

where equality holds if and only if Z is a box.

Proof. Assume that Z is a zonotope. The general case follows by standard approximation arguments. If $Z=\sum_{i=1}^{N}[0,u_i]$, then the multilinearity of mixed volumes yields

$$V(\underbrace{Z,\dots,Z}_{m},\underbrace{B,\dots,B}_{n-m})=m!\sum_{1\leq i_1<\dots<i_m\leq N}V([0,u_{i_1}],\dots,[0,u_{i_m}],\underbrace{B,\dots,B}_{n-m})$$

and

$$V(\underbrace{Z,\dots,Z}_{m},\underbrace{B,\dots,B}_{n-m-1},[-e_j,e_j])=m!\sum_{1\leq i_1<\dots<i_m\leq N}V([0,u_{i_1}],\dots,[0,u_{i_m}],\underbrace{B,\dots,B}_{n-m-1},[-e_j,e_j])\,.$$

Therefore, by (1) and (4) we deduce that

$$V_m(Z) - \frac{1}{n-m}\sum_{i=1}^{n} V_m(Z|e_i^{\perp}) = m! \sum_{1\le i_1<\cdots<i_m\le N}\left[V_m(F_{i_1 i_2\ldots i_m}) - \frac{1}{n-m}\sum_{i=1}^{n} V_m(F_{i_1 i_2\ldots i_m}|e_i^{\perp})\right],$$

where $F_{i_1 i_2\ldots i_m}$ is the Minkowski sum of the vectors u_{i_1}, u_{i_2}, $\ldots$, u_{i_m}. Since such vector sums are m-dimensional faces of Z, the result follows from Theorem 4.3. □

Inequality (27) also implies an estimate of the $(m+1)$th intrinsic volume of a convex body in terms of the mth intrinsic volumes of its coordinate projections, as in the classical Loomis-Whitney inequality.

Theorem 4.5. *Let $1 \le m \le n-1$ and let K be a convex body in $\mathbb{R}^n$. If there exists a box Z such that*

$$V_m(K|e_i^{\perp}) = V_m(Z|e_i^{\perp}),\ i = 1, 2, \ldots, n,$$

then

$$V_{m+1}(K) \le V_{m+1}(Z),$$

with equality if and only if K is a box.

Proof. The theorem is a consequence of the Aleksandrov-Fenchel inequality. Indeed, our hypothesis on the box Z implies that

$$V(\underbrace{K,\ldots,K}_{m}, Z, B,\ldots,B) = V(\underbrace{Z,\ldots,Z}_{m+1}, B,\ldots,B)$$

and the Aleksandrov-Fenchel inequality states that

$$V(\underbrace{K,\ldots,K}_{m}, Z, B,\ldots,B)^{m+1} \ge V(\underbrace{Z,\ldots,Z}_{m+1}, B,\ldots,B)V(\underbrace{K,\ldots,K}_{m+1}, B,\ldots,B)^{m}.$$

Hence

$$V(\underbrace{K,\ldots,K}_{m+1}, B,\ldots,B) \le V(\underbrace{Z,\ldots,Z}_{m+1}, B,\ldots,B),$$

which, by (1), is equivalent to the required inequality. The equality condition follows from that for the special case of the Aleksandrov-Fenchel inequality used here (see [19, (B.19), p. 420]). □

For each value of m (and n), $V_{m+1}(Z)$ can be expressed in terms of the $V_m(Z|e_i^{\perp})$'s. Thus Theorem 4.5 yields a sharp estimate of $V_{m+1}(K)$ in terms of $V_m(K|e_1^{\perp})$, $V_m(K|e_2^{\perp})$, $\ldots$, $V_m(K|e_n^{\perp})$. From the algebraic point of view this means expressing the $(m+1)$th elementary symmetric polynomial in n variables in terms of the mth elementary symmetric polynomials in subsets of $n-1$ variables.

For example, since we know that for every convex body K there exists a box Z such that

$$V_1(K|e_i^{\perp}) = V_1(Z|e_i^{\perp}),\ i = 1, 2, \ldots, n,$$

the following theorem holds.

Theorem 4.6. *For every convex body K in $\mathbb{R}^n$,*

$$V_2(K) \leq \frac{1}{n-1}\left[\sum_{i=1}^{n} V_1(K|e_i^\perp)\right]^2 - \sum_{i=1}^{n} V_1(K|e_i^\perp)^2 ,$$

where equality holds if and only if K is a box.

As a final remark, we note that from an estimate like (27) of the mth intrinsic volume of a body in terms of the mth intrinsic volumes of its projections on the coordinate hyperplanes, one can also deduce estimates of the mth intrinsic volume in terms of the mth volumes of the projections on the coordinate subspaces of lower dimension. To be precise, assume that inequality (27) holds for every convex body K in $\mathbb{R}^n$ (actually we would only need that it holds for K and all its projections on the coordinate subspaces of lower dimension, say between $r+1$ and $n-1$). By an induction argument (on n and r), one can show that

$$V_m(K) \leq \frac{1}{\binom{n-m}{n-r}} \sum_{i=1}^{s} V_m(K|\Lambda_i) ,$$

where r is a fixed integer, $1 \leq r \leq n-1$, $s = \binom{n}{r}$ and Λ_1, Λ_2, $\ldots$, Λ_s are the coordinate r-dimensional subspaces.

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Dipartimento di Ingegneria dell'Informazione, Università degli Studi di Siena, Via Roma 56, 53100 Siena, Italy
E-mail address: campi@dii.unisi.it

Dipartimento di Matematica e Applicazioni per l'Architettura, Università degli Studi di Firenze, Piazza Ghiberti 27, 50122 Firenze, Italy
E-mail address: paolo@fi.iac.cnr.it