VOLUME INEQUALITIES FOR L_p-ZONOTOPES

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Abstract. The classical Minkowski sum of convex sets is defined by the sum of the corresponding support functions. The L_p-extension of such a definition makes use of the sum of the p-th power of the support functions. An L_p-zonotope Z_p is the p-sum of finitely many segments and is isometric to the unit ball of a subspace of ℓ_q, where $\frac{1}{p}+\frac{1}{q}=1$.

In this paper we give a sharp upper estimate of the volume of Z_p in terms of the volume of Z_1, as well as a sharp lower estimate of the volume of the polar of Z_p in terms of the same quantity.

In particular, for $p=1$, the latter result provides a new approach to Reisner's inequality for the Mahler conjecture in the class of zonoids.

1. Introduction

A *zonotope* is a convex polytope of $\mathbb{R}^d$ defined as the vector sum of a finite number of segments. The simplest zonotope is a parallelotope, the sum of d affinely independent segments, that is an affine image of a d-cube. Conversely, by increasing the number of segments, zonotopes can approximate the unit ball of $\mathbb{R}^d$. A set which is a limit, in the Hausdorff metric, of a sequence of zonotopes is called a *zonoid*. Zonoids play a basic role in the Brunn-Minkowski theory of convex bodies and appear in different contexts of the mathematical literature. We refer to [20] for an exhaustive review on this topic.

Note that the class of centrally symmetric convex sets and the one of zonoids coincide only in dimension two.

A well known problem, which is solved in the class of zonoids but not for general convex bodies, is the Mahler conjecture.

Given a convex body K in $\mathbb{R}^d$, that is a d-dimensional compact convex set, if the origin is an interior point of K, then the *polar body* K^* of K is

$$K^* = \{x \in \mathbb{R}^d | \langle x, y\rangle \le 1, \forall y \in K\},$$

where $\langle \cdot\, , \cdot\rangle$ is the usual scalar product in $\mathbb{R}^d$. Obviously $(K^*)^* = K$.

The analytical link between K and K^* is given by

$$h_K(u) = \frac{1}{\rho_{K^*}(u)},\ \forall u \in \mathbb{R}^d,$$

where

$$h_K(u) = \max_{x\in K}\langle u, x\rangle,\ \forall u \in \mathbb{R}^d$$

To Rolf, with gratitude.

is the support function of K and

$$\rho_K(u) = \max\{r \in \mathbb{R} : ru \in K\}\,, \forall u \in \mathbb{R}^d$$

is the radial function of K.

If K is an origin-symmetric convex body, then the linear invariant product of volumes

$$V(K)V(K^*)$$

is called the *volume product* of K.

While the Blaschke-Santaló inequality characterizes ellipsoids as the only maximizers, Mahler [13] conjectured that parallelotopes and their polars are minimizers of the volume product.

The conjecture is proved only for $d = 2$ (see Mahler [14] and Reisner [17]). Reisner [16], [17] proved that the conjecture is true in the restricted class of zonoids. For a simpler proof see the paper by Gordon, Meyer and Reisner [7].

For a general convex body K, the volume product is defined as the minimum, for $x \in K$, of $V(K)V((K-x)^*)$. In this case Mahler conjectured that simplices are the only minimizers. The characterization of simplices, or parallelotopes in the symmetric cases, as extremal bodies of classic functionals is a central problem in convex geometry.

For more details and results related to Mahler's problem, see [8], [15], [2], [4], [1].

In this paper we deal with classes of sets which are L_p-extensions of the one of zonotopes. In recent years, many authors devoted their attention to the L_p-Brunn-Minkowski theory, as a central part of convexity. For a detailed list of references on this subject, see, for instance, [10]. On one hand, for many notions, the L_p-setting sounds as the natural one, as in the case of the L_p-analogs of centroid bodies, projection bodies, and curvatures. On the other hand, the L_p-theory represents a useful bridge between geometric and analytic inequalities. This is the case, for example, of affine isoperimetric inequalities and Sobolev type inequalities. At this regard, see [9], [12].

For $p \geq 1$, an L_p-*zonotope* is the Firey p-sum of a finite number of segments. Precisely, the function

$$h_p(u) = \left[\sum_{i=1}^{s} |\langle u, v_i\rangle|^p\right]^{1/p}, \text{ for } u \in \mathbb{R}^d$$

is the support function of a convex set Z_p, which is the p-sum of the segments $[-v_i, v_i]$, $i = 1, 2, \ldots, s$.

The notion of L_p-zonoid was already introduced in [20] by Schneider and Weil. Note that L_2-zonoids are ellipsoids, which are related to the Legendre ellipsoid of the given set of segments. Moreover, L_p-zonoids (or L_p-zonotopes) are, up to isometry, unit balls of finite dimensional subspaces of L_q (or ℓ_q, respectively), where $q = \frac{p-1}{p}$. In [10], Lutwak, Yang and Zhang find sharp upper and lower estimates for the volume of Z_p and Z_p^*, under the assumption that the set of segments is isotropic, that is Z_2 is the unit ball of $\mathbb{R}^d$. Actually, they deal with L_p-zonoids, defined through integrals with respect to even not necessarily discrete measures, and prove that extremals occur for the uniformly distributed measure on the unit sphere and for the atomic measure concentrated on the vectors of an orthonormal basis and their opposite.

Here, we deal with the same problem under a different constraint. Namely, we consider the linearly invariant functionals

$$\frac{V(Z_1)}{V(Z_p)} \quad \text{and} \quad V(Z_p^*)V(Z_1)\,,$$

where V stands for d-dimensional volume, and we prove that their minimum is attained when Z_1 is a parallelotope (Theorems 4.3 and 4.4).

As noted in Section 2, for $p > 1$, both functionals are not bounded from above. When $p = 1$, the second functional is nothing but the volume product, and then Reisner's theorem is a special case of Theorem 4.4.

When p tends to infinity, Z_p tends to the convex hull of the segments $[-v_i, v_i]$, and Theorems 4.3 and 4.4 are still valid. Lutwak, Yang and Zhang recently showed in [11] that, in the case of isotropic measures on the unit sphere with their centroid at the origin, the regular simplex of $\mathbb{R}^d$ minimizes the volume of Z_∞ and maximizes the volume of Z_∞^*.

The technique we use for proving our results is based on a method introduced by Rogers and Shephard in [18]. This method consists in moving each point of a set with a constant in time speed, parallel to a fixed direction. Here, we are able to find a movement keeping fixed the volume of Z_1. Under such a movement, by using previous results proved by the authors in [3], [4], it turns out that each functional we are dealing with is the reciprocal of a convex in time function.

2. L_p-zonotopes and their volume

Given a finite set $\Lambda = \{v_1, v_2, \ldots, v_s\}$ of vectors spanning $\mathbb{R}^d$ and $p \geq 1$, the function

$$h_p(u;\Lambda) = \left[\sum_{i=1}^{s} |\langle u, v_i\rangle|^p\right]^{1/p}, \text{ for } u \in \mathbb{R}^d\,, \tag{1}$$

is the support function of a convex body in $\mathbb{R}^d$, that we shall denote by $Z_p(\Lambda)$ and call the L_p-zonotope of the given set of vectors.

If $p = 1$, then (1) defines the support function of the Minkowski sum of the segments $[-v_i, v_i]$, a polytope known in the literature as the zonotope generated by those segments (see for example [19], p. 182). For $p > 1$, $Z_p(\Lambda)$ is the Firey p-sum of the same segments (see [6]). We recall that the support function of the Firey p-sum of two convex sets A and B is defined by

$$h^p_{A+_pB}(x) = h^p_A(x) + h^p_B(x)\,.$$

If p tends to infinity, then $Z_p(\Lambda)$ tends to $Z_\infty(\Lambda)$, the convex hull of Λ and its reflection about the origin. As usual, the distance in the space of convex sets is given by the Hausdorff metric. To measure the distance between finite sets of vectors, we choose the product of the standard metric in $\mathbb{R}^d$.

Note that, for any p, the map $\Lambda \to Z_p(\Lambda)$ is continuous. The same map is not injective. In the sequel we shall consider only sets not containing parallel vectors. When a set contains two vectors v_i, $v_j = \lambda v_i$, we replace them with $(1 + |\lambda|)v_i$. This rule creates discontinuities for the map $\Lambda \to Z_p(\Lambda)$, when $p \neq 1$. Nevertheless, this map is semicontinuous in the following sense:

$$\lim_{n\to\infty} Z_p(\Lambda_n) \subset Z_p(\lim_{n\to\infty} \Lambda_n)\,.$$

If $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is a one-to-one linear map, then

$$Z_p(\varphi\Lambda) = \varphi Z_p(\Lambda)\,.$$

Indeed,

$$h_p^p(u;\varphi\Lambda) = \sum_{i=1}^{s} |\langle u, \varphi v_i\rangle|^p = \sum_{i=1}^{s} |\langle \varphi^T u, v_i\rangle|^p = h_p^p(\varphi^T u;\Lambda)\,,$$

where φ^T denotes the transpose of φ.

We are interested in the functionals

$$\frac{V(Z_1(\Lambda))}{V(Z_p(\Lambda))} \quad \text{and} \quad V(Z_p^*(\Lambda))V(Z_1(\Lambda))\,,$$

where $Z_p^*(\Lambda)$ is the polar body of $Z_p(\Lambda)$ and Λ varies in the class of all finite sets of vectors spanning $\mathbb{R}^d$.

Both functionals are invariant under linear transformations.

Clearly, for $p = 1$, the first functional is constant, and the second one is the volume product of Z_1.

For $p > 1$, we observe that both functionals have no maximum. To see this, let us fix an integer k and consider a set Ω of kd pairwise not parallel vectors $w_{i,j}$ from $\mathbb{R}^d$ of the form $w_{i,j} = \frac{e_i}{k} + \varepsilon_j$, where $E = \{e_1,\, e_2,\, \dots,\, e_d\}$ is an orthonormal basis and $\|\varepsilon_j\| \leq \frac{1}{k^2}$, for $j = 1, 2, \dots, k$.

Thus,

$$h_1(u;\Omega) = \sum_{i=1}^{d}\sum_{j=1}^{k} |\langle u, w_{i,j}\rangle| \geq \sum_{i=1}^{d} |\langle u, e_i\rangle| - \frac{d}{k}\|u\| = h_1(u;E) - \frac{d}{k}\|u\|\,. \tag{2}$$

On the other hand,

$$\begin{aligned} h_p^p(u;\Omega) = \sum_{i=1}^{d}\sum_{j=1}^{k} |\langle u, w_{i,j}\rangle|^p &\leq \sum_{i=1}^{d} 2^{p-1}\left(\frac{|\langle u, e_i\rangle|^p}{k^{p-1}} + \frac{\|u\|^p}{k^{2p-1}}\right) \\ &= \left(\frac{2}{k}\right)^{p-1}\left(h_p^p(u;E) + \frac{d}{k^p}\|u\|^p\right)\,. \end{aligned} \tag{3}$$

Estimate (2) implies that $V(Z_1(\Omega))$ tends to 2^d as k tends to infinity, while (3) implies that $V(Z_p(\Omega))$ tends to 0 and $V(Z_p^*(\Omega))$ tends to infinity.

The problem of finding the minimum of each functional is studied in Section 4.

3. Reduction of zonotopes

In this section we present special continuous transformations of zonotopes, that keep unchanged the volume and simplify the structure. In such a reduction process, each point moves with constant speed, parallel to a fixed direction. Thus, according to a definition by Rogers and Shephard [18], [21], we are using a *shadow system*, and we can profit of the relevant properties.

Precisely, a shadow system X_t of points (vectors) from $\mathbb{R}^d$ is a family of sets, which can be represented as follows:

$$X_t = \{x_i + ta_i v\}_{i\in I} ,$$

where $t \in [t_0, t_1]$, $x_i, v \in \mathbb{R}^d$, $a_i \in \mathbb{R}$, and I is an arbitrary set of indices. Here, t can be seen as a time-like parameter and, consequently, the number a_i as the speed of the point x_i along the direction v. By definition, shadow systems of convex bodies are convex hulls of shadow systems of points.

Let $\Lambda = \{v_1, v_2, \dots, v_s\}$ be a finite set of vectors spanning $\mathbb{R}^d$, with $s > d$. Assume that $v_2, v_3, \dots, v_s$ span $\mathbb{R}^d$ and define $\Lambda_t = \{w_1, w_2, \dots, w_s\}$, where

$$\left\{ \begin{array}{ll} w_1 & = (1+at)v_1 \\ w_i & = v_i - tv_1 \frac{\langle v_1, v_i\rangle}{\|v_1\|^2} . \end{array} \right. \tag{4}$$

Here, a is a positive parameter to be chosen later and t varies in $[-\frac{1}{a}, 1]$. Clearly $\Lambda_0 = \Lambda$.

First, we note that $Z_1(\Lambda_t)$ is a shadow system of convex sets. Indeed, $Z_1(\Lambda_t)$ is the Minkowski sum of segments and each of them is a shadow system (see [5], Lemma 1).

At the endpoints of the movement, the zonotope $Z_1(\Lambda_{-\frac{1}{a}})$ is the sum of $s-1$ segments, while $Z_1(\Lambda_1)$ is a cylinder, the sum of $s-1$ segments orthogonal to v_1 and a segment parallel to v_1. Recall that, in case Λ_1 contains parallel vectors, we replace them with a parallel vector whose length is the sum of their lengths. We continue to denote by Λ_1 this possible new set.

Let us consider now the volume of $Z_1(\Lambda_t)$. The formula for the volume of a zonotope (see [19], p.297) yields

$$V(Z_1(\Lambda_t)) = 2^d \sum_{1\le i_1<\dots<i_d\le s} |[w_{i_1}, w_{i_2}, \dots, w_{i_d}]| , \tag{5}$$

where $[w_{i_1}, w_{i_2}, \dots, w_{i_d}]$ denotes the determinant of the matrix whose rows are w_{i_1}, w_{i_2}, $\dots$, w_{i_d}. Therefore, by (4), the volume of $Z_1(\Lambda_t)$ is a second degree polynomial in t. Actually, the coefficient of t^2 vanishes, since each determinant containing t^2 comes from a matrix with two rows multiple of v_1. So we have,

$$\begin{aligned} 2^{-d}V(Z_1(\Lambda_t)) &= \sum_{2\le i_2<\dots<i_d\le s} |[w_1, w_{i_2}, \dots, w_{i_d}]| + \sum_{2\le i_1<\dots<i_d\le s} |[w_{i_1}, w_{i_2}, \dots, w_{i_d}]| \\ &= |1+at| \sum_{2\le i_2<\dots<i_d\le s} |[v_1, v_{i_2}, \dots, v_{i_d}]| + \\ + \sum_{2\le i_1<\dots<i_d\le s} &|[v_{i_1}, \quad v_{i_2} \quad, \dots, v_{i_d}] - \frac{t}{\|v_1\|^2} \sum_{j=1}^{d} \langle v_1, v_{i_j}\rangle [v_{i_1}, \dots, v_{i_{j-1}}, v_1, v_{i_{j+1}}, \dots, v_{i_d}]| \,. \end{aligned}$$

Let us focus our attention on the last term. Linear algebra gives

$$\begin{aligned} \sum_{j=1}^{d} \langle v_1, v_{i_j}\rangle [v_{i_1}, \dots, v_{i_{j-1}}, v_1, v_{i_{j+1}}, \dots, v_{i_d}] &= \langle v_1, \sum_{j=1}^{d} v_{i_j} [v_{i_1}, \dots, v_{i_{j-1}}, v_1, v_{i_{j+1}}, \dots, v_{i_d}]\rangle \\ &= \langle v_1, v_1[v_{i_1}, v_{i_2}, \dots, v_{i_d}]\rangle \,. \end{aligned}$$

Therefore,

$$2^{-d}V(Z_1(\Lambda_t)) = |1+at| \sum_{2\le i_2<\dots<i_d\le s} |[v_1, v_{i_2}, \dots, v_{i_d}]| + |1-t| \sum_{2\le i_1<\dots<i_d\le s} |[v_{i_1}, v_{i_2}, \dots, v_{i_d}]| \,.$$

Now, let us choose

$$a = \frac{\sum\limits_{2\le i_1<\dots<i_d\le s} |[v_{i_1}, v_{i_2}, \dots, v_{i_d}]|}{\sum\limits_{2\le i_2<\dots<i_d\le s} |[v_1, v_{i_2}, \dots, v_{i_d}]|} \,.$$

It follows that

$$V(Z_1(\Lambda_t)) = V(Z_1(\Lambda)) \,, \text{ for every } t \in [-\frac{1}{a}, 1] \,.$$

Our choice of the parameter a has a geometric meaning. Namely, one can check that

$$a + 1 = \frac{V(Z_1(\Lambda))}{\|v_1\| V(Z_1(\Lambda)|v_1^{\perp})} \,,$$

where $V(Z_1(\Lambda)|v_1^{\perp})$ denotes the $(d-1)$-volume of the orthogonal projection of $Z_1(\Lambda)$ onto a hyperplane orthogonal to v_1.

The above process can be applied to every zonotope which is not an affine image of a d-cube. By performing finitely many times the process to each of the zonotopes corresponding to the endpoints of the movement, we arrive at an affine image of a d-cube. Actually, Λ_1 may have the same number of vectors as Λ. Nevertheless, after at most d steps, the number of vectors is surely reduced.

4. Main results

We have already seen that, if Λ_t is a shadow system of vectors, then $Z_1(\Lambda_t)$ is a shadow system of convex sets. We want to show that the same holds for $Z_p(\Lambda_t)$, $p > 1$.

To do this we need a characterization of shadow systems proved in [3] (Lemma 3.1), that we restate here for completeness.

Proposition 4.1. *Let H_t, $t \in [t_0, t_1]$, be a one-parameter family of convex bodies such that $H_t|v^\perp$ is independent of t. Assume the bodies H_t are defined by*

$$H_t = \{x + yv : x \in H_t|v^\perp, y \in \mathbb{R}, f_t(x) \le y \le g_t(x)\}, \ \forall t \in [t_0, t_1],$$

for suitable functions g_t, f_t. Then H_t, $t \in [t_0, t_1]$, is a shadow system of convex sets along the direction v if and only if for every $x \in H_t|v^\perp$,
(i) $g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in $[t_0, t_1]$,
(ii) $f_{\lambda r+(1-\lambda)s}(x) \le \lambda g_r(x) + (1-\lambda) f_s(x) \le g_{\lambda r+(1-\lambda)s}(x)$, for every $r, s \in [t_0, t_1], \lambda \in [0, 1]$.

The following theorem is a discrete version of Theorem 2.1 in [3].

Theorem 4.2. *If Λ_t, $t \in [t_0, t_1]$, is a shadow system of vectors along the direction v and $1 \le p \le \infty$, then $Z_p(\Lambda_t)$ is a shadow system of convex bodies along the same direction.*

Proof. Each vector of Λ_t has constant speed. Let a_i be the speed of v_i. By (1), for every $u \in \mathbb{R}^d$, we have

$$h_p(u; \Lambda_t) = \left[\sum_{i=1}^{s} |\langle u, v_i + t a_i v \rangle|^p \right]^{1/p} = \|\langle u, v_i \rangle + t a_i \langle u, v \rangle\|_p, \tag{6}$$

where we used the usual shortening for ℓ_p norms. The Minkowski inequality for p-norms yields that $h_p(u; \Lambda_t)$, as a function of t, is convex, for every $u \in \mathbb{R}^d$. Moreover, $h_p(u; \Lambda_t)$ is a Lipschitz function of t, with Lipschitz constant $\|\langle u, v \rangle a_i\|_p$, for every $u \in \mathbb{R}^d$.

Since the orthogonal projection of $Z_p(\Lambda_t)$ onto $v^\perp$ is independent of t, it is sufficient to show that the family $Z_p(\Lambda_t)$ satisfies conditions (i) and (ii) of Proposition 4.1.

As $Z_p(\Lambda_t)$ is origin symmetric, for every $t \in [t_0, t_1]$, it can be represented by

$$Z_p(\Lambda_t) = \{x + yv : x \in (Z_p(\Lambda_{t_0}))|v^\perp, -g_t(-x) \le y \le g_t(x)\},$$

where g_t is a suitable concave function defined on $(Z_p(\Lambda_{t_0}))|v^\perp$.

Since $z \in Z_p(\Lambda_t)$ if and only if $\langle z, u \rangle \le h_p(u; \Lambda_t)$, for every $u \in \mathbb{R}^d$, we can write

$$\begin{aligned} g_t(x) &= \sup\{\lambda \in \mathbb{R} : \langle x + \lambda v, u \rangle \le h_p(u; \Lambda_t), \ \forall u \in \mathbb{R}^d\} \\ &= \sup\{\lambda \in \mathbb{R} : \lambda \langle v, u \rangle \le h_p(u; \Lambda_t) - \langle x, u \rangle, \ \forall u \in \mathbb{R}^d\}, \end{aligned} \tag{7}$$

for every $x \in (Z_p(\Lambda_{t_0}))|v^\perp$.

Scalar products and support functions are homogeneous functions of degree 1. Thus in (7) we have to consider only the vectors u such that $|\langle u, v \rangle| = 1$. Furthermore, the vectors u with a non-positive scalar product with v provide no bounds for λ. Therefore we get

$$\begin{aligned} g_t(x) &= \sup\{\lambda \in \mathbb{R} : \lambda \leq h_p(w+v;\Lambda_t) - \langle x, w+v\rangle\,,\ \forall w \in v^\perp\} \\ &= \inf_{w\in v^\perp} \{h_p(w+v;\Lambda_t) - \langle x, w\rangle\}\,. \end{aligned} \tag{8}$$

Notice that $g_t(x)$ is in fact the minimum, as $w \in v^\perp$, of $\{h_p(w+v;\Lambda_t) - \langle x, w\rangle\}$, unless x belongs to the boundary of $(Z_p(\Lambda_{t_0}))|v^\perp$. The minimum is attained when $w+v$ is directed as a normal vector to $Z_p(\Lambda_t)$ at $x + g_t(x)v$.

As an infimum of equi-Lipschitz functions of t, $g_t(x)$ is a Lipschitz function of t, and it is convex if

$$2g_{\frac{t_1+t_2}{2}}(x) \leq g_{t_1}(x) + g_{t_2}(x)$$

holds for every t_1, t_2 in its range. By (8) we can write

$$\begin{aligned} 2g_{\frac{t_1+t_2}{2}}(x) &= \inf_{u\in v^\perp}\{\|\langle 2u+2v, v_i\rangle + a_i(t_1+t_2)\|_p - \langle x, 2u\rangle\} \\ &= \inf_{u_1,u_2\in v^\perp}\{\|\langle u_1+u_2+2v, v_i\rangle + a_i(t_1+t_2)\|_p - \langle x, u_1+u_2\rangle\} \\ &\leq \inf_{u_1,u_2\in v^\perp}\{\|\langle u_1+v, v_i\rangle + a_it_1\|_p + \|\langle u_2+v, v_i\rangle + a_it_2\|_p - \langle x, u_1\rangle - \langle x, u_2\rangle\} \\ &= \inf_{u_1\in v^\perp}\{\|\langle u_1+v, v_i\rangle + a_it_1\|_p - \langle x, u_1\rangle\} + \inf_{u_2\in v^\perp}\{\|\langle u_2+v, v_i\rangle + a_it_2\|_p - \langle x, u_2\rangle\} \\ &= g_{t_1}(x) + g_{t_2}(x)\,, \end{aligned}$$

where we again used the Minkowski inequality for p-norms. Hence condition (i) is verified.

Let us now turn to (ii). It is enough to prove the first inequality; the second will follow by interchanging r with s, λ with $1-\lambda$, and x with $-x$. We can write

$$\begin{aligned} (1-\lambda)g_s(x) &= \inf_{u\in v^\perp}\{\|(1-\lambda)\langle u+v, v_i\rangle + a_i(1-\lambda)s\|_p - \langle x, (1-\lambda)u\rangle\} \\ &= \inf_{u_1,u_2\in v^\perp}\{\|\langle u_2-\lambda u_1+v-\lambda v, v_i\rangle + a_i[(1-\lambda)s+\lambda r-\lambda r]\|_p \\ &\qquad -\langle x, u_2-\lambda u_1\rangle\} \\ &\leq \inf_{u_1,u_2\in v^\perp}\{\|\langle u_2+v, v_i\rangle + a_i[\lambda r+(1-\lambda)s]\|_p\ + \\ &\qquad \|\langle -\lambda u_1-\lambda v, v_i\rangle - a_i\lambda r\|_p - \langle x, u_2-\lambda u_1\rangle\} \\ &= \inf_{u_1\in v^\perp}\{\lambda\|\langle u_1+v, v_i\rangle + a_ir\|_p + \lambda\langle x, u_1\rangle\}\ + \\ &\qquad \inf_{u_2\in v^\perp}\{\|\langle u_2+v, v_i\rangle + a_i[\lambda r+(1-\lambda)s]\|_p - \langle x, u_2\rangle\} \\ &= \lambda g_r(-x) + g_{\lambda r+(1-\lambda)s}(x)\,. \end{aligned}$$

This concludes the proof. Note that the case $p=\infty$ is trivial, according to the definition of shadow system of convex bodies. □

We are able now to prove the following theorem.

Theorem 4.3. *Let $1 < p \leq \infty$. For every finite set Λ of vectors spanning $\mathbb{R}^d$,*

$$\frac{V(Z_1(\Lambda))}{V(Z_p(\Lambda))} \geq \frac{V(Z_1(E))}{V(Z_p(E))},$$

where E is an orthonormal basis of $\mathbb{R}^d$.

Proof. Let $\Lambda = \{v_1, v_2, \ldots, v_s\}$ be a finite set of vectors spanning $\mathbb{R}^d$ with $s > d$. Assume that $v_2, v_3, \ldots, v_s$ span $\mathbb{R}^d$ and define the shadow system Λ_t as in (4). By Theorem 4.2, $Z_p(\Lambda_t)$ is a shadow system. Rogers and Shephard proved in [18] that the volume of a shadow system of convex sets is a convex function of t. Therefore, since the volume of $Z_1(\Lambda_t)$ is constant along the process, the function $\frac{V(Z_1(\Lambda_t))}{V(Z_p(\Lambda_t))}$ attains in $[-\frac{1}{a}, 1]$ its minimum value at one of the endpoints.

As shown in Section 3, such a procedure can be iterated finitely many times up to obtaining a parallelotope. □

Theorem 4.4. *Let $1 \leq p \leq \infty$. For every finite set Λ of vectors spanning $\mathbb{R}^d$,*

$$V(Z_1(\Lambda))V(Z_p^*(\Lambda)) \geq V(Z_1(E))V(Z_p^*(E)),$$

where E is an orthonormal basis of $\mathbb{R}^d$.

Proof. Let us consider Λ and Λ_t as in the previous proof. As shown by the authors in [4] (Theorem 1), if K_t is a shadow system of origin symmetric convex bodies, then the volume of $(K_t)^*$ is the reciprocal of a convex function of t. Therefore, it turns out that the function $[V(Z_1(\Lambda_t))V(Z_p^*(\Lambda_t))]^{-1}$ is a convex function, too. The same argument as in the previous theorem concludes the proof. □

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