ON VOLUME PRODUCT INEQUALITIES FOR CONVEX SETS

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Abstract.
The volume of the polar body of a symmetric convex set K of $\mathbb{R}^d$ is investigated. It is shown that its reciprocal is a convex function of the time t along movements, in which every point of K moves with constant speed parallel to a fixed direction.
This result is applied to find reverse forms of the L^p-Blaschke-Santaló inequality for two-dimensional convex sets.

1. Introduction.

Let K be a convex body in $\mathbb{R}^d$, that is a d-dimensional compact convex set, and assume that the origin is an interior point of K.

The support function of the convex body K is defined as

$$h_K(u) = \max_{x \in K} \langle u, x \rangle, \ \forall u \in \mathbb{R}^d ,$$

where $\langle \cdot \, , \, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^d$, and the radial function of K as

$$\rho_K(u) = \max\{r \in \mathbb{R} : ru \in K\} \, , \forall u \in \mathbb{R}^d \, .$$

The d-dimensional volume $V(K)$ of K can be expressed in terms of the radial function by

$$V(K) = \frac{1}{d} \int_{S^{d-1}} \rho_K^d(z) \, dz \, ,$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$.

In this paper, we are interested in the volume of the *polar body* of K and, in particular, in its behavior under special continuous transformations of K.

The polar body K^* of K can be defined as

$$K^* = \{x \in \mathbb{R}^d | \langle x, y \rangle \le 1, \forall y \in K\} \, .$$

Notice that the polar body of K strongly depends on the location of the origin. It follows from the definition that $(K^*)^* = K$ and that

$$\rho_{K^*}(u) = \frac{1}{h_K(u)}, \ \forall u \in \mathbb{R}^d \, .$$

If K is an origin symmetric convex body, then the product

$$V(K)V(K^*)$$

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is called the *volume product* of K and it is invariant under linear transformations.

For a general convex body K, the volume product is defined as the minimum, for $x \in K$, of $V(K)V((K-x)^*)$. The unique point where such a minimum is attained is called the *Santaló point* of K. More precisely, the quantity $V((K-x)^*)^{-\frac{1}{d}}$ turns out to be a strictly concave function of x, as shown by A. D. Aleksandrov in [A2]. In that paper, the volume product and other averages of powers of support functions are studied, in connection with estimates of solutions of the Dirichlet problem for elliptic partial differential equations (see also [A1]). Volume product inequalities are also used by Talenti [T] for the Monge-Ampére equation.

One of the main questions still open in convex geometry is the problem of finding a sharp lower estimate for the volume product of a convex body (see the survey article [L]).

A sharp upper estimate of the volume product is given by the Blaschke-Santaló inequality: For every convex body K in $\mathbb{R}^d$

$$V(K)V(K^*) \leq \kappa_d^2\,, \tag{1}$$

where κ_d is the volume of the unit ball in $\mathbb{R}^d$. Equality holds if and only if K is an ellipsoid centered at the origin (see again [L]).

A sharpening of this inequality was proved by Meyer and Pajor [MP]. It says that for every convex body K and every affine hyperplane H dividing K into K^+ and K^- (both of non zero volume) there exists a point z from the relative interior of $K \cap H$ such that

$$\frac{4V(K^+)V(K^-)V((K-z)^*)}{V(K)} \leq \kappa_d^2\,.$$

It was conjectured by Mahler [Ma1] that the minimum of the volume product is attained when K is a simplex, that is

$$V(K)V(K^*) \geq \frac{(d+1)^{d+1}}{(d!)^2}\,. \tag{2}$$

In 1939, Mahler [Ma2] proved that (2) holds if $d=2$ and, in 1991, Meyer [Me] showed that equality occurs only for triangles.

For centrally symmetric convex bodies the inequality

$$V(K)V(K^*) \geq \frac{4^d}{d!} \tag{3}$$

is a conjecture as well, where the value on the right-hand side is the volume product of a d-parallelotope. It was proved in dimension two by Mahler [Ma2]. Reisner [Re2] showed that parallelograms are the only minimizers, for $d=2$. Saint Raymond [SR], for $d>3$, exhibited convex bodies, different than parallelotopes and their polar bodies, for which (3) is an equality. He also proved, for every d, that (3) holds true for all the affine images of convex sets symmetric with respect to the coordinate hyperplanes. Inequality (3) was proved by Reisner [Re1], [Re2] for all zonoids. Such a class can be defined as the closure, in the Hausdorff metric, of finite vector sums of segments. A simpler proof of Reisner's result was given in [GMR].

Bourgain and Milman [BM] proved that there exists a constant c, not depending on the dimension, such that

$$V(K)V(K^*) \geq c^d \kappa_d^2 \,.$$

In the present paper we shall consider movements of a given origin symmetric convex body K, obtained by assigning to each point of K a speed, independent of the time t, which is parallel to a fixed direction (see Section 2). Denoting by K_t the convex hull of the resulting points at time t, we shall prove that $V^{-1}(K_t^*)$ is a convex function of t (Theorem 1), where $K_t^* = (K_t)^*$.

The main tool in the proof is the so-called Borell-Brascamp-Lieb inequality, which deals with the p-means of functions and their integrals. Such an inequality is an extension of the Prékopa-Leindler inequality and can be considered as an inverse Hölder inequality. The importance of the Borell-Brascamp-Lieb inequality and its links with other famous inequalities (e.g., the isoperimetric inequality) is widely described in the survey article by Gardner [G2].

By using Theorem 1, one can rediscover, for origin symmetric convex bodies, the Blaschke-Santaló inequality as well as Mahler's inequality (3) in the two dimensional case.

In Section 3 we apply Theorem 1 to the study of the functional

$$V(\Gamma_p^* K)V(K)\,, \tag{4}$$

where $\Gamma_p K$ is the L^p-centroid body of K. We obtain an alternative proof of a result by Lutwak and Zhang [LZ], stating that, in the class of all convex bodies of $\mathbb{R}^d$, the quantity (4) attains its maximum when K is an ellipsoid centered at the origin. Such a result is called the L^p-Blaschke-Santaló inequality. This name comes from the fact that, when K is origin symmetric and p goes to infinity, (4) tends to the volume product.

We shall deal with the problem of obtaining reverse form of the L^p-Blaschke-Santaló inequality and we solve it, for $d = 2$, by exploiting Theorem 1. Namely, we are able to show that, among all convex figures containing the origin, (4) is minimum when K is a triangle with a vertex at the origin (Theorem 3). Triangles are still minimizers of (4) in the class of all convex bodies with their barycenter at the origin (Theorem 4). Finally, if one takes the maximum value of (4) with respect to all possible locations of K in the plane, then such a value is minimum for triangles (Theorem 5).

The same results can be rephrased for parallelograms, instead of triangles, if we restrict ourselves to the class of centrally symmetric convex figures.

2. Polar bodies and shadow systems: the main result.

A *shadow system* along the direction $v \in S^{d-1}$ is a family of convex sets $K_t \subset \mathbb{R}^d$ that can be defined by

$$K_t = conv\{x + \alpha(x)t\, v : x \in A \subset \mathbb{R}^d\}\,,$$

where *conv* stands for convex hull, A is an arbitrary bounded set of points, α is a bounded function on A and t belongs to an interval of the real axis. The notion of shadow system was introduced by Rogers and Shephard ([RS] and [Sh]), who proved the basic fact that *the volume of K_t is a convex function of t.* This result is a powerful tool for obtaining geometric inequalities of isoperimetric type.

As suggested by Shephard in [Sh], a shadow system can be seen as the family of projections of a $(d+1)$-dimensional convex set. Namely, let $e_1, e_2, \ldots, e_{d+1}$ be an orthonormal basis of $\mathbb{R}^{d+1}$ and let $\tilde{K}$ be the $(d+1)$-dimensional convex set defined by

$$\tilde{K} = conv\{x + \alpha(x)e_{d+1} : x \in A \subset e_{d+1}^{\perp}\},$$

where $e_{d+1}^{\perp} = \{w \in \mathbb{R}^{d+1} : \langle w, e_{d+1} \rangle = 0\}$.

The projections of $\tilde{K}$ onto a hyperplane orthogonal to e_{d+1} along the directions $e_{d+1} - t\,v$ are just the family K_t. This interpretation permits of finding that not only the volume but also other geometric quantities are convex functions of the parameter t of a shadow system. Indeed, projecting all the sets of a shadow system onto a linear space along a fixed direction produces another shadow system. So, for example, the brightness function of K_t is a convex function of t. Recall that the brightness of a convex set, as a function of $u \in S^{d-1}$ is the $(d-1)$-dimensional volume of its orthogonal projection onto $u^{\perp}$. Besides, by the Cauchy formula, the surface area of a convex set is the average of its brightness function (see, for example [Sc], p. 295). Therefore, the surface area of K_t is still a convex function with respect to t. Analogously, by taking the projections of K_t onto lower dimensional linear spaces, it turns out that all the so-called intrinsic volumes of K_t (see [Sc], p. 210) are convex functions with respect to the parameter t.

The Steiner process of symmetrization can be seen as originated from a particular shadow system. Indeed, if one moves each chord of K parallel to v so that, at $t = 1$, such a chord is in the reflected position with respect to $v^{\perp}$, then, at $t = \frac{1}{2}$, the union of all the chords is nothing but the Steiner symmetral of K with respect to v. Precisely, let K be represented by

$$K = \{x + yv \in \mathbb{R}^d : x \in K|v^{\perp}, y \in \mathbb{R}, f(x) \leq y \leq g(x)\},$$

where $\cdot|v^{\perp}$ denotes the orthogonal projection onto $v^{\perp}$ and f and $-g$ are convex functions on $K|v^{\perp}$. The shadow system along v, with speed $\alpha(x) = -f(x|v^{\perp}) - g(x|v^{\perp})$, is such that $K_{1/2}$ coincides with the Steiner symmetral of K about $v^{\perp}$.

A shadow system with a speed function constant on each chord parallel to the direction of the movement is called a *parallel chord movement.*

A way of exploiting convexity is the one suggested by the following Shephard's argument, contained in [Sh]: *If a functional defined in the class of all convex sets is continuous, invariant under reflections and convex with respect to the parameter t of any shadow system, then it attains its minimum at the ball among all sets of given volume.* Here the continuity refers to the Hausdorff metric.

This statement follows by the use of Steiner symmetrization. It is worth to recall that such a symmetrization keeps the volume unchanged and leads, if suitably repeated, to a ball.

Consequences of this procedure are classical isoperimetric type inequalities for intrinsic volumes (see [BZ], p. 144, [G1], p. 372). Moreover, it was found out that other functionals of geometric flavor enjoy the same convexity property under shadow systems (see, for example, [Sh], [CCG], [CG1]).

Definition 1. *Let $p \neq 0$. A nonnegative function f on $\mathbb{R}^d$ is called p-concave on a convex set A if*

$$f((1-\lambda)x+\lambda y)\geq[(1-\lambda)f(x)^p+\lambda f(y)^p]^{1/p}$$

for all $x,\ y\in A$ and $0<\lambda<1$.

Notice that, when $p<0$, f is p-concave if and only if f^p is convex. The above definition can be extended to the case $p=0$ by continuity (see [G2]).

In what follows, we are interested in the behavior of integrals of a family of p-concave functions, which can be deduced from the following result.

Borell-Brascamp-Lieb inequality. *Let $0<\lambda<1$, let $-1/d\leq p\leq\infty$, and let f, g, and h be nonnegative integrable functions on $\mathbb{R}^d$ satisfying*

$$h((1-\lambda)x+\lambda y)\geq[(1-\lambda)f(x)^p+\lambda g(y)^p]^{1/p}\,,$$

for all $x,\ y\in\mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d}h(x)\,dx\geq\left[[(1-\lambda)\left(\int_{\mathbb{R}^d}f(x)\,dx\right)^{p/(dp+1)}+\lambda\left(\int_{\mathbb{R}^d}g(x)\,dx\right)^{p/(dp+1)}\right]^{(dp+1)/p}.$$

For an exhaustive description of references about the above result see again [G2].

Corollary. *Let $F(x,y)$ be a nonnegative p-concave function on $\mathbb{R}^n\times\mathbb{R}^m$, $p\geq-1/n$. If, for every y in $\mathbb{R}^m$, the integral*

$$\int_{\mathbb{R}^n}F(x,y)\,dx$$

exists, then it is a $\frac{p}{np+1}$-concave function of y.

Proof. Take $y_0,\ y_1\in\mathbb{R}^m$ and fix $\lambda\in(0,1)$. Let $y_\lambda=(1-\lambda)y_0+\lambda y_1$, and

$$f(x)=F(x,y_0)\ ,\quad g(x)=F(x,y_1)\ ,\quad h(x)=F(x,y_\lambda)\,.$$

For every $x_0,\ x_1\in\mathbb{R}^n$, we have that

$$h^p((1-\lambda)x_0+\lambda x_1)=F^p((1-\lambda)x_0+\lambda x_1,y_\lambda)\geq(1-\lambda)f^p(x_0)+\lambda g^p(x_1)\,,$$

where we used the p-concavity of F.

Thus the Borell-Brascamp-Lieb inequality leads to the conclusion. □

We are now ready to state and prove the following theorem.

Theorem 1. *If K_t, $t\in[0,1]$, is a shadow system of origin symmetric convex bodies in $\mathbb{R}^d$, then $V(K_t^*)^{-1}$ is a convex function of t.*

Proof. Let K_t be a shadow system as in the statement. Therefore, there exists a bounded function α on K_0 such that

$$K_t=conv\{x+\alpha(x)t\,v:x\in K_0\}\,.$$

Let us introduce the $(d+1)$-dimensional convex body

$$\tilde{K}=conv\{x+\alpha(x)e_{d+1}:x\in K_0\}\,,$$

so that K_t can be thought of as the projection along the direction $e_{d+1} - tv$ of $\tilde{K}$ onto $e_{d+1}^{\perp}$.

There is a connection between the support functions h_{K_t}, $t \in [0,1]$, and the support function of $\tilde{K}$. Precisely, for $u \in e_{d+1}^{\perp}$,

$$h_{K_t}(u) = \max_{x \in K_t} \langle u, x \rangle = \max_{x \in K_0} \langle u, x + \alpha(x) tv \rangle \,.$$

Notice that $\langle u, x + \alpha(x)tv \rangle = \langle u + t\langle u, v\rangle e_{d+1}, x + \alpha(x) e_{d+1} \rangle$; moreover, as x runs in K_0, $x + \alpha(x)tv$ covers all the extreme points of $\tilde{K}$. Recall that an extreme point of a convex set is a point which can not be expressed as convex linear combination of two different points of the set. Hence,

$$h_{K_t}(u) = \max_{y \in \tilde{K}} \langle u + t\langle u, v\rangle e_{d+1}, y \rangle = h_{\tilde{K}}(u + t\langle u, v\rangle e_{d+1}) \,. \tag{5}$$

We know that

$$V(K_t^*) = \frac{1}{d} \int_{S^{d-1}} h_{K_t}^{-d}(z) \, dz \,. \tag{6}$$

Let $D^{d-1} = \{x \in v^{\perp} : \|x\| \le 1\}$; thus $S_+^{d-1} = \{z \in S^{d-1} : \langle z, v \rangle \ge 0\}$ can be seen as the graph of the function $\sqrt{1 - \|x\|^2}$, $x \in D^{d-1}$. Consequently,

$$\int_{S^{d-1}} h_{K_t}^{-d}(z) \, dz = 2 \int_{D^{d-1}} \frac{h_{K_t}^{-d}(x + \sqrt{1 - \|x\|^2} v)}{\sqrt{1 - \|x\|^2}} \, dx \,, \tag{7}$$

where we took into account that K_t is origin symmetric.

By (5),

$$h_{K_t}(x + \sqrt{1 - \|x\|^2} v) = h_{\tilde{K}}(x + \sqrt{1 - \|x\|^2} v + t\sqrt{1 - \|x\|^2} e_{d+1}) \,. \tag{8}$$

Therefore, from (6), (7) and (8) we obtain that

$$V(K_t^*) = \frac{2}{d} \int_{D^{d-1}} \frac{h_{\tilde{K}}^{-d}(\frac{x}{\sqrt{1 - \|x\|^2}} + v + te_{d+1})}{(1 - \|x\|^2)^{\frac{d+1}{2}}} \, dx \,,$$

where we used also the homogeneity of the support function.

Use the change of variables $y = \frac{x}{\sqrt{1 - \|x\|^2}}$ in the latter integral. It is easy to check that the determinant of the Jacobian matrix of that map is just $(1 - \|x\|^2)^{-\frac{d+1}{2}}$. We conclude that

$$V(K_t^*) = \frac{2}{d} \int_{\mathbb{R}^{d-1}} h_{\tilde{K}}^{-d}(y + v + te_{d+1}) \, dy \,.$$

The function $h_{\tilde{K}}$ is convex in $\mathbb{R}^{d+1}$, therefore, by the corollary of the Borell-Brascamp-Lieb inequality, $V(K_t^*)$ is p-concave with $p = \frac{-1/d}{1 - (d-1)/d} = -1$. $\square$

3. Consequences.

By Theorem 1, if we apply Shephard's argument, quoted in the previous section, to the reciprocal of the volume product, we immediately deduce the Blaschke-Santaló inequality (1), for origin symmetric convex bodies.

A different argument, that we shall use explicitly in the proof of Theorem 3, leads to the Mahler's inequality (3), for plane convex figures.

Both the above results can be seen as special instances of a more general class of inequalities, which involve the L^p-centroid body of a body K.

For each real number $p \geq 1$, the L^p- *centroid body* $\Gamma_p K$ of K is the convex body with support function

$$h_{\Gamma_p K}(u) = \left\{ \frac{1}{c_{d,p} V(K)} \int_K |\langle u, z \rangle|^p \, dz \right\}^{\frac{1}{p}} , \; u \in \mathbb{R}^d , \tag{9}$$

where

$$c_{d,p} = \frac{\kappa_{d+p}}{\kappa_2 \kappa_d \kappa_{p-1}}$$

and

$$\kappa_r = \pi^{\frac{r}{2}} / \Gamma(1 + \frac{r}{2}) .$$

Notice that κ_d is the volume of the unit ball B^d of $\mathbb{R}^d$ and the constant $c_{d,p}$ is such that $\Gamma_p B^d = B^d$, for every d and p.

This definition is due to Lutwak and Zhang [LZ] and extends to $p > 1$ the concept of centroid body, corresponding to $p = 1$. For $p = 2$, formula (9) gives, up to a constant, the Legendre ellipsoid of K. If one defines $\Gamma_\infty K$ as the limit of $\Gamma_p K$ when $p \to \infty$, then $\Gamma_\infty K = conv(K \cup (-K))$.

Inequalities involving the volume of $\Gamma_p K$ can be found in [LZ], [LYZ], [CG1] and [CG2]. Here we are interested in the functional (4), with $p \geq 1$, and we set for simplicity

$$G_p(K) = V(\Gamma_p^* K) V(K) .$$

It is easy to check that G_p is continuous and invariant under reflection about hyperplanes through the origin. Moreover, for every linear map L, $\Gamma_p(LK) = L\Gamma_p K$ (see, e.g., [CG2]).

In [LZ] it is shown that the maximum of $G_p(K)$ is attained if and only if K is an origin symmetric ellipsoid. For $p = \infty$, this yields the Blaschke-Santaló inequality (1) for centrally symmetric convex bodies.

The "if" part of the Lutwak-Zhang result can be now deduced also from Shephard's argument and the following theorem.

Theorem 2. *If K_t, $t \in [0, 1]$ is a parallel chord movement, then $V(\Gamma_p^* K_t)^{-1}$ is a convex function of t.*

Theorem 2 is a consequence of Theorem 1 and the following one, proved in [CG1].

If K_t, $t \in [0, 1]$ is a parallel chord movement along the direction v, then $\Gamma_p K_t$ is a shadow system along the same direction v.

Let us deal now with the problem of finding lower bounds for G_p. Clearly, the functional $G_p(K)$ tends to zero as K moves away from the origin. So, natural ways for posing the problem is to restrict ourselves to convex bodies containing the origin or to bodies with their barycenter at the origin or else to consider the functional

$$M_p(K) = \max_{x\in\mathbb{R}^d} G_p(K-x)\,.$$

Notice that $G_p(K-x)^{-1}$ is a convex function of x, by Theorem 2.

In all these three cases, the Lutwak-Zhang inequality implies that the maximum is attained only at ellipsoids centered at the origin.

The search of minimizers, for $d=2$, is guided by Theorem 2, through a method already used by the authors in [CG2].

Theorem 3. *For $d=2$, the minimum of $G_p(K)$ in the class of all convex bodies containing the origin is attained if K is a triangle with one vertex at the origin.*

Proof. Let P be a polygon with n vertices, $n>3$. Since we are interested in minimizers of G_p, we can assume that the origin is at one of the vertices of P. Take three consecutive vertices v_1, v_2, v_3, different from the origin. Denote by u a direction of the line through v_1 and v_3 and consider the following shadow system. Assign speed u to v_2 and 0 to the other vertices and consider the resulting shadow system P_t, where $t\in[t_0,t_1]$, which is the largest interval such that the area of P_t is constant for all $t\in[t_0,t_1]$. Notice that P_t, $t\in[t_0,t_1]$ is a parallel chord movement such that only the triangle $v_1v_2v_3$ moves and then the origin remains in P_t, for all t. Moreover, $P_0=P$, and P_{t_0} and P_{t_1} have exactly $n-1$ vertices.

By Theorem 2,

$$G_p(P) \geq min\{G_p(P_{t_0}), G_p(P_{t_1})\}\,.$$

If $n>4$, iterations of this argument lead to the conclusion that

$$G_p(P) \geq G_p(T)\,,$$

where T is a triangle with one vertex at the origin. The linear invariance of G_p ensures that $G_p(T)$ attains the same value whichever triangle, with a vertex at the origin, we consider. By the continuity of G_p, an approximation argument ends the proof. □

Theorem 4. *For $d=2$, the minimum of $G_p(K)$ in the class of all convex bodies with their barycenter at the origin is attained if K is a triangle.*

Proof. We can follow the same outlines of the previous proof. What we need here to add is that the barycenter of P_t must remain at the origin for every t. The key property is that, along a parallel chord movement K_t, the barycenter c_{K_t} of K_t moves with constant speed parallel to the direction u of the movement. Indeed,

$$\begin{aligned} c_{K_t} &= \frac{1}{V(K_t)}\int_{K_t} x\,dx = \frac{1}{V(K_{t_0})}\int_{K_{t_0}} (x+\alpha(x)tu)\,dx \\ &= c_{K_{t_0}} + \frac{tu}{V(K_{t_0})}\int_{K_{t_0}} \alpha(x)\,dx\,. \end{aligned}$$

Therefore, the family $P_t - c_{P_t}$, $t\in[t_0,t_1]$ is a parallel chord movement of bodies with their barycenter at the origin. □

Theorem 5. *For $d = 2$, the minimum of $M_p(K)$ in the class of all convex bodies is attained if K is a triangle.*

Proof. Clearly, for every convex figure K, Theorem 4 implies that

$$M_p(K) \geq G_p(K - c_K) \geq G_p(T - c_T) = M_p(T)\,,$$

where T is an equilateral triangle and the last equality follows from the symmetry of T. $\square$

As a conclusion we can observe that the same method used in the proofs of Theorems 3, 4 and 5 can be trivially adapted to the case of centrally symmetric convex figures and leads to show that parallelograms, instead of triangles, are now minimizers.

In the same way, the Mahler inequality (3) can be obtained from Theorem 2. On the other hand, such an inequality can be also deduced from Theorem 3, by taking $p = \infty$.

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