THE L^p-BUSEMANN-PETTY CENTROID INEQUALITY

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Abstract. The ratio between the volume of the p-centroid body of a convex body K in $\mathbb{R}^n$ and the volume of K attains its minimum value if and only if K is an origin symmetric ellipsoid. This result, the L^p-Busemann-Petty centroid inequality, was recently proved by Lutwak, Yang and Zhang. In this paper we show that all the intrinsic volumes of the p-centroid body of K are convex functions of a time-like parameter when K is moved by shifting all the chords parallel to a fixed direction. The L^p-Busemann-Petty centroid inequality is a consequence of this general fact.

1. Introduction.

This article deals with a family of affine isoperimetric inequalities which compare the volume of a convex (or star-shaped) body in $\mathbb{R}^n$ with the one of its p-centroid body. One of the member of such a family is the classical Busemann-Petty centroid inequality which plays a central role in the framework of the affine isoperimetric inequalities (see the survey article by Lutwak [L2]). In order to describe the inequalities we are interested in, let us recall the definition of p-centroid body in terms of its *support function.* For each convex compact set K in $\mathbb{R}^n$, the support function h_K is the real-valued function defined by

$$h_K(u) = \max\{\langle z, u\rangle : z \in K\}\,,\ u \in \mathbb{R}^n\,,$$

where $\langle , \rangle$ denotes the standard inner product.

Let C be a compact subset of $\mathbb{R}^n$ with nonempty interior and denote by $V(C)$ its n-dimensional volume. According to the definition given by Lutwak and Zhang [LZ], for each real number $p \geq 1$, the *p-centroid body of C*, $\Gamma_p C$, is the convex body (i.e. a compact convex set with nonempty interior) whose support function is

$$h_{\Gamma_p C}(u) = \left\{\frac{1}{c_{n,p} V(C)} \int_C |\langle u, z\rangle|^p \, dz\right\}^{\frac{1}{p}}\ ,\ u \in \mathbb{R}^n\,, \tag{1}$$

where the integration is with respect to Lebesgue measure and

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}\,,$$

with

$$\omega_k = \pi^{\frac{k}{2}}/\Gamma(1 + \frac{k}{2})\,.$$

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Thus ω_n is the n-dimensional volume of the unit ball B of $\mathbb{R}^n$. Notice that $\Gamma_p(\lambda C) = \lambda\Gamma_p(C)$, for every $\lambda > 0$, and the constant $c_{n,p}$ is chosen so that $\Gamma_p B = B$, for every n and p.

For $p = 1$, (1) defines the body $\Gamma C = \Gamma_1 C$, known in the literature as the *centroid body* of C. Centroid bodies were first defined and investigated by Petty [P1], but the concept had previously appeared in work of Dupin, in connection with problems for floating bodies (see the books of Gardner [G], Chapter 9, and Schneider [Sc], Section 7.4, for references). When C is an origin symmetric body, the boundary of ΓC is the locus of the centroids of all the halves of C obtained by cutting C with hyperplanes through the origin.

As limits, in the Hausdorff metric, of Minkowski sums of segments, centroid bodies belong to the class of zonoids, sets appearing in many different contexts of convex geometry (see, e.g., Schneider and Weil [SW], for references).

One of the basic results obtained by Petty [P1] is an integral representation of the volume of ΓC as an average of the volume of all the simplices whose vertices are at the origin and at n points taken randomly from C. Such a representation and the Busemann random simplex inequality for convex bodies (see, e.g., [G], Theorem 9.2.6) lead to the well known *Busemann-Petty centroid inequality*, conjectured by Blaschke [B1]:

If K is a convex body in $\mathbb{R}^n$, then

$$V(\Gamma K) \geq V(K)\,, \tag{2}$$

where equality holds if and only if K is an origin symmetric ellipsoid.

Petty [P2] proved that the Busemann-Petty centroid inequality implies the *Petty projection inequality*:

If K is a convex body in $\mathbb{R}^n$, then

$$V(K)^{n-1}V(\Pi^*K) \leq \omega_n^n\,, \tag{3}$$

where equality holds if and only if K is an ellipsoid.

Here Π^*K is the polar of the projection body ΠK of K, namely

$$\Pi^*K = \{z \in \mathbb{R}^n : \frac{1}{2\omega_{n-1}} \int_{\partial K} |\langle z, v\rangle|\, dv \leq 1\}\,,$$

where the integral is done with respect to $(n-1)$-Hausdorff measure.

A shorter way for showing that (2) implies (3) can be found in [L2]. Such a way employs the class reduction technique introduced by Lutwak in [L1]. By this technique, affine isoperimetric inequalities proved in a small class of bodies (e.g. zonoids) can be almost automatically extended to a larger class of bodies (e.g. star-shaped sets). In [L1] it is proved that, in turn, the Petty projection inequality (3) implies the Busemann-Petty centroid inequality (2) and that this one can be extended to all compact star-shaped (about the origin) sets.

For $p = 2$, the body defined by (1) is also well known. Indeed, up to a constant, $\Gamma_2 C$ is the *ellipsoid of inertia* (or Legendre ellipsoid) of C, i.e. the ellipsoid having the same moments of inertia as C about every axis. Many results concerning such a body, whose concept is basic in classical mechanics, can be found in literature (see, e.g., Milman and Pajor [MP] and Lindenstrauss and Milman [LM] for references). We fix our attention on the following fundamental inequality which goes back, at least, to Blaschke [B2], for $n = 3$:

If K is a convex body in $\mathbb{R}^n$, then

$$V(\Gamma_2 K) \geq V(K)\,, \tag{4}$$

where equality holds if and only if K is an origin symmetric ellipsoid.

For general n, (4) was proved by John [J] (see also [P1]). Another proof of (4) was recently given by Lutwak, Yang and Zhang [LYZ1].

As conjectured by Lutwak and Zhang [LZ], inequalities (2) and (4) are special instances of the following L^p-type affine isoperimetric inequality:

Theorem 1.1. *If K is a convex body in $\mathbb{R}^n$, then for $1 \leq p < \infty$*

$$V(\Gamma_p K) \geq V(K)\,, \tag{5}$$

where equality holds if and only if K is an origin symmetric ellipsoid.

A first proof of Theorem 1.1 was recently given by Lutwak, Yang and Zhang [LYZ2]. A completely different proof of that theorem is the object of the present paper.

Theorem 1.1 is a further contribution to the L^p-extension of the classical Brunn-Minkowski theory (as well as of the dual one) for convex bodies. The first step in this direction is the paper by Lutwak [L3], in which the idea of Firey [F] of the p-Minkowski addition for sets is widely developed.

The study of affine isoperimetric inequalities in the L^p-setting provides a better understanding of links and possible hierarchies between such inequalities and makes easier to see geometric inequalities and analytic inequalities from a common point of view.

Inequality (5) strengthens the following already strong inequality proved by Lutwak and Zhang [LZ]:

If K is a convex body in $\mathbb{R}^n$, then for $1 \leq p \leq \infty$

$$V(K)V(\Gamma_p^* K) \leq \omega_n^2\,, \tag{6}$$

where equality holds if and only if K is an origin symmetric ellipsoid.

Here $\Gamma_p^* K$ is the polar body of $\Gamma_p K$ of K, i.e. $\Gamma_p^* K = \{z \in \mathbb{R}^n : h_{\Gamma_p K}(z) \leq 1\}$. If $\Gamma_\infty K$ is interpreted as a limit of (1), as $p \to \infty$, then $\Gamma_\infty K = conv(K \cup (-K))$, where *conv* stands for the convex hull. Thus, for every origin symmetric convex body K, the body $\Gamma_\infty^* K$ is just K^*, the polar body of K. In this case, for $p = \infty$, inequality (6) reduces to the well known Blaschke-Santaló inequality:

$$V(K)V(K^*) \leq \omega_n^2\,, \tag{7}$$

with equality if and only if K is an origin symmetric ellipsoid. On the other hand, by applying (7) to $\Gamma_p K$, (5) immediately gives (6).

By using Lutwak's class reduction technique, inequality (6) can be extended also to all compact star-shaped (about the origin) subsets of $\mathbb{R}^n$. Such a result is used in [LZ] to obtain the functional version of (6): For real $p \geq 1$ and continuous positive functions f_1, f_2 defined on $S^{n-1} = \partial B$,

$$\int_{S^{n-1}} \int_{S^{n-1}} |\langle u, v\rangle|^p \, f_1(u) f_2(v) \geq c_{n-2,p} \|f_1\|_{L^{\frac{n}{n+p}}(S^{n-1})} \|f_2\|_{L^{\frac{n}{n+p}}(S^{n-1})}\,,$$

with equality if and only if f_1 and f_2 are of the form $c_1|\phi(u)|^{-(n+p)}$ and $c_2|\phi^{-t}(u)|^{-(n+p)}$ respectively, being $\phi \in GL(n)$ and ϕ^{-t} the inverse of the transpose of ϕ.

The proof of Theorem 1.1 given in [LYZ2] involves the L^p-analog of the Petty projection inequality. To state it, one has to introduce the L^p-projection body of a convex body K, for $p > 1$. This is done by defining a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} which is the L^p-analog of the classical surface area measure of K. The L^p-projection body $\Pi_p K$ of K is defined as the convex body whose support function is the spherical L^p-cosine transform of $S_p(K, \cdot)$. In [LYZ2] the following inequality is proved:

If K is a convex body in $\mathbb{R}^n$, then for $1 \leq p < \infty$

$$V(K)^{\frac{n-p}{p}} V(\Pi_p^* K) \leq \omega_n^{n/p}\,, \tag{8}$$

where equality holds if and only if K is an origin symmetric ellipsoid.

As well as in the case $p = 1$, by the class reduction technique, it is shown in [LYZ2] that inequality (8) implies Theorem 1.1 and that (5) is valid also for all the star-shaped (about the origin) bodies. Conversely, Theorem 1.1 implies the L^p-Petty projection inequality (8).

In this paper a direct proof of Theorem 1.1 is given. The L^p-Busemann-Petty centroid inequality is obtained here as a consequence of a general fact concerning the behaviour of $\Gamma_p K$ under special transformations acting on K. Namely, if each chord of K parallel to a fixed direction moves with a constant speed, depending continuously on the chord, then the volume of the corresponding p-centroid bodies is a convex function of the time-like parameter. This fact, if used in the case of the Steiner symmetrization, leads to the conclusion.

Notice that (5) is valid also for $p = \infty$. In this case inequality (5) becomes trivial and equality holds if and only if K is origin symmetric. Significant results about the ratio $V(\Gamma_\infty K)/V(K)$ were obtained by Fáry and Rédei [FR].

The techniques applied here were already used by the authors and A. Colesanti [CCG] for Sylvester's type functionals. They may be developed to obtain inequalities similar to that of Theorem 1.1. These will be the subject of a forthcoming study.

2. Description of results.

According to the definition of Rogers and Shephard (see [RS] and [Sh]), a *shadow system* (or a *linear parameter system*) along the direction v is a family of convex bodies $K_t \subset \mathbb{R}^n$ that can be defined by

$$K_t = conv\{z + \alpha(z) t\, v : z \in A \subset \mathbb{R}^n\}\,, \tag{9}$$

where A is an arbitrary bounded set of points, α is a real bounded function on A, and the parameter t runs in an interval of the real axis.

Notice that the orthogonal projection $K_t|v^\perp$ of K_t onto $v^\perp = \{z \in \mathbb{R}^n : \langle v, z\rangle = 0\}$ is independent of t.

As proved by Shephard [Sh], *every mixed volume involving n shadow systems along the same direction is a convex function of the parameter. In particular, the volume $V(K_t)$ and all quermassintegrals $W_i(K_t)$, $i = 1, 2, \ldots, n$, of a shadow system are convex functions of t.*

For the sake of completeness, we shall sketch here the elegant proof given in [Sh]. For definitions and properties of mixed volumes, we refer the reader to [Sc].

Define a convex body $\tilde{K}$ in $\mathbb{R}^{n+1}$ as follows. Let $(O; e_1, e_2, \ldots, e_{n+1})$ be an orthonormal system of $\mathbb{R}^{n+1}$. To every $z = (z_1, z_2, \ldots, z_n) \in A$ we associate the point $U(z) = (z_1,\, z_2,\, \ldots,\, z_n,\, \alpha(z))$ in $\mathbb{R}^{n+1}$. Set $\tilde{K} = conv\{U(z) : z \in A\}$. It is easily verified that K_t, as defined in (9), turns out to be the projection of $\tilde{K}$ onto $e_{n+1}^{\perp}$ along the direction $e_{n+1} - tv$. Because of this property we say that $\tilde{K}$ originates the shadow system $\{K_t : t \in [0,1]\}$. Furthermore the volume of K_t can be expressed as the mixed volume $V(\tilde{K}, \tilde{K}, \ldots, \tilde{K}, [O, e_{n+1} - tv])$, where $[O, e_{n+1} - tv]$ denotes the segment connecting the origin and $e_{n+1} - tv$. The multilinearity and monotonicity of mixed volumes provide the convexity of the volume of K_t with respect to t. This argument can be extended to every mixed volume involving n shadow systems. Indeed, one can verify that $V(K_t^{(1)}, K_t^{(2)}, \ldots, K_t^{(n)}) = V(\tilde{K}^{(1)}, \tilde{K}^{(2)}, \ldots, \tilde{K}^{(n)}, [O, e_{n+1} - tv])$, which implies that all these functionals are convex in t.

We are also interested in a particular type of shadow system we shall call *parallel chord movements*. A parallel chord movement along the direction v is a family of convex bodies K_t in $\mathbb{R}^n$ defined by

$$K_t = \{z + \beta(x)t\, v : z \in K, x = z - \langle z, v\rangle v\}\,, \tag{10}$$

where K is a convex body in $\mathbb{R}^n$, β is a continuous real function on $v^{\perp}$ and the parameter t runs in an interval of the real axis, say $t \in [0,1]$. In other words, to each chord of $K = K_0$ parallel to v we assign a speed vector $\beta(x)v$, where x is the projection of the chord onto $v^{\perp}$; then we let the chords move for a time t and denote by K_t their union. Such an union has to be convex; this is the only restriction we have on defining the speed function β.

Notice that if $\{K_t : t \in [0,1]\}$ is a parallel chord movement, then via Fubini's theorem one deduces that the volume of K_t is independent of t.

If the speed function β of the movement is an affine function (that is, $\beta(x) = \langle x, u\rangle + k$, for some vector u and real constant k), then it is easy to see that K_t is an affine image of K, for every t in the range of the movement.

Another special instance is the movement related to Steiner symmetrization. Precisely, fix a direction v and let

$$K = \{x + yv \in \mathbb{R}^n : x \in K|v^{\perp}, y \in \mathbb{R}, f(x) \le y \le g(x)\}\,;$$

here f and $-g$ are convex functions on $K|v^{\perp}$. The parallel chord movement with speed function $\beta(x) = -(f(x)+g(x))$ and $t \in [0,1]$ is such that $K_0 = K$, $K_1 = K^v$, the reflection of K in the hyperplane $v^{\perp}$, and $K_{1/2}$ is the Steiner symmetral of K with respect to $v^{\perp}$.

Suppose now that a general parallel chord movement is applied to a convex body K. What happens to the corresponding p-centroid bodies?

The answer to this question is given by the following theorem.

Theorem 2.1. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then $\Gamma_p K_t$ is a shadow system along the same direction v.*

By the above-mentioned Shephard's result, Theorem 2.1 implies that the volume of $\Gamma_p K_t$ is a convex function of t. It is easy to verify that $\Gamma_p(K^v) = (\Gamma_p K)^v$, for every direction v and real $p \geq 1$. If $\{K_t : t \in [0,1]\}$ is the parallel chord movement related to Steiner symmetrization along v, then

$$V(\Gamma_p K_{1/2}) \leq \frac{1}{2} V(\Gamma_p K_0) + \frac{1}{2} V(\Gamma_p K_1) = V(\Gamma_p K)\,,$$

that is, the volume of the p-centroid body is not increased after a Steiner symmetrization. It is well known that every convex body can be transformed into a ball through a sequence of suitable Steiner symmetrizations. Clearly the ratio $V(\Gamma_p K)/V(K)$ is continuous in the Hausdorff metric. Therefore it attains its minimum value when K is a ball.

The following result allows to characterize all the minimizers.

Theorem 2.2. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement with speed function β, then the volume of $\Gamma_p K_t$ is a strictly convex function of t unless β is linear, that is $\beta(x) = \langle x, u\rangle$ for some vector u.*

If K is not an origin symmetric ellipsoid, it is well known (see, e.g., Petty [P3]) that there exists a direction v such that the Steiner symmetral of K along the direction v is not an image of K under a linear transformation. Therefore, $V(\Gamma_p K)/V(K)$ attains its minimum value if and only if K is an origin symmetric ellipsoid (Theorem 1.1).

3. Proofs.

As a first remark, we notice that if $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then the orthogonal projection of $\Gamma_p K_t$ onto $v^\perp$ is independent of t. Indeed, by (1) and (10),

$$\begin{aligned} h_{\Gamma_p K_t}(u) &= \left\{ \frac{1}{c_{n,p} V(K_t)} \int_{K_t} |\langle u, z\rangle|^p \, dz \right\}^{\frac{1}{p}} \\ &= \left\{ \frac{1}{c_{n,p} V(K_0)} \int_{K_0} \left|\langle u, z + \beta(z|v^\perp) t v\rangle\right|^p \, dz \right\}^{\frac{1}{p}} \\ &= \left\{ \frac{1}{c_{n,p} V(K_0)} \int_{K_0} \left|\langle u, z\rangle + \beta(z|v^\perp) t \langle u, v\rangle\right|^p \, dz \right\}^{\frac{1}{p}} \;; \end{aligned} \tag{11}$$

thus, for every $u \in v^\perp$, $h_{\Gamma_p K_t}(u) = h_{\Gamma_p K_0}(u)$.

This fact is not sufficient by itself for concluding that $\{\Gamma_p K_t : t \in [0,1]\}$ is a shadow system. The following lemma provides necessary and sufficient conditions in order that a family of convex bodies having constant orthogonal projection onto a fixed hyperplane is actually a shadow system.

Lemma 3.1. *Let $\{H_t : t \in [0,1]\}$, be a one-parameter family of convex bodies such that $H_t|v^\perp$ is independent of t. Assume the bodies H_t are defined by*

$$H_t = \{x + yv : x \in H_t|v^\perp, y \in \mathbb{R}, f_t(x) \leq y \leq g_t(x)\}\,,\ \forall t \in [0,1]\,,$$

for suitable functions g_t, f_t. Then $\{H_t : t \in [0,1]\}$ is a shadow system along the direction v if and only if for every $x \in H_0|v^\perp$,

(i) $g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in $[0,1]$,

(ii) $f_{\lambda t_1+(1-\lambda)t_2}(x) \le \lambda g_{t_1}(x) + (1-\lambda) f_{t_2}(x) \le g_{\lambda t_1+(1-\lambda)t_2}(x)$, *for every* $t_1, t_2, \lambda \in [0,1]$.

Proof. Let us first prove that every shadow system $\{H_t : t \in [0,1]\}$ satisfies conditions (i) and (ii). Following the argument of Shephard, we can regard H_t as the projection of some convex body $\tilde{H}$ in $\mathbb{R}^{n+1}$ along $e_{n+1} - tv$ onto $e_{n+1}^\perp$. For a fixed $x \in H_0|v^\perp$, the function $g_t(x)$ depends only on the points of $\tilde{H}$ contained in the two-dimensional section by the plane through x and parallel to both v and e_{n+1}. For every point P from this section of $\tilde{H}$, let $P'(t)$ be its projection along $e_{n+1} - tv$ onto $e_{n+1}^\perp$. The function $\langle P'(t) - O, v\rangle$ is clearly a linear function of t. So $g_t(x)$ is a maximum of linear functions of t and then it is convex. Similarly, $f_t(x)$, as a minimum of linear functions of t, is concave.

The second condition requires that $Q = x + \lambda g_{t_1}(x)v + (1-\lambda) f_{t_2}(x)v$ belongs to $H_{\lambda t_1+(1-\lambda)t_2}$, for every $t_1, t_2, \lambda \in [0,1]$. This follows from the convexity of the body $\tilde{H}$ originating our movement. Indeed, from $x + g_{t_1}(x)v \in H_{t_1}$ and $x + f_{t_2}(x)v \in H_{t_2}$, we deduce the existence of real numbers y_1 and y_2 such that $P_1 = x + g_{t_1}(x)v - y_1 e_{n+1} + y_1 t_1 v \in \tilde{H}$ and $P_2 = x + f_{t_2}(x)v - y_2 e_{n+1} + y_2 t_2 v \in \tilde{H}$. If we consider the projections of these two points of $\tilde{H}$ onto $e_{n+1}^\perp$ along the directions $e_{n+1} - t_1 v$ and $e_{n+1} - t_2 v$, then, by definition of the functions f_t and g_t, we infer

$$
\begin{aligned}
g_{t_1}(x) &\ge f_{t_2}(x) + y_2(t_2 - t_1)\,, \\
f_{t_2}(x) &\le g_{t_1}(x) + y_1(t_1 - t_2)\,.
\end{aligned}
\tag{12}
$$

Consider the projections of P_1 and P_2 onto $e_{n+1}^\perp$ along the direction $e_{n+1} - [\lambda t_1 + (1-\lambda)t_2]v$; they are the points $x + g_{t_1}(x)v + (1-\lambda)y_1(t_1 - t_2)v$ and $x + f_{t_2}(x)v + \lambda y_2(t_2 - t_1)v$. In order to verify condition (ii) it is enough to show that Q lies between such points. This is a straightforward consequence of (12).

We now prove that (i) and (ii) are sufficient conditions. To do this, we consider our family $\{H_t : t \in [0,1]\}$ as lying in the hyperplane $e_{n+1}^\perp$ in $\mathbb{R}^{n+1}$ and we construct a convex body $\tilde{H}$ so that its projection onto $e_{n+1}^\perp$ along the direction $e_{n+1} - tv$ coincides with H_t, for every $t \in [0,1]$.

Take a point $Q \in H_0$ and suppose that $Q = x + yv$, where $x \in H_0|v^\perp$ and $y \in [f_0(x), g_0(x)]$. Let

$$
\gamma(Q) = \inf_{t \in (0,1]} \frac{g_t(x) - y}{t}\,,
$$

$$
\phi(Q) = \sup_{t \in (0,1]} \frac{f_t(x) - y}{t}\,.
$$

From (ii) we can deduce by contradiction that $\gamma(Q) \ge \phi(Q)$, for every $Q \in H_0$. Indeed, if we assume that $\gamma(Q) < \phi(Q)$, then there exist s_1 and s_2 in $(0,1]$ such that

$$
\frac{g_{s_1}(x) - y}{s_1} < \frac{f_{s_2}(x) - y}{s_2}\,.
$$

Suppose that $s_1 > s_2 = (1-\lambda)s_1$, for some $\lambda \in [0,1]$; then

$$
(1-\lambda) g_{s_1}(x) + \lambda y < f_{(1-\lambda)s_1}(x)
$$

and, since $y \ge f_0(x)$,

$$
(1-\lambda) g_{s_1}(x) + \lambda f_0(x) < f_{(1-\lambda)s_1}(x)\,,
$$

which contradicts (ii). A similar argument also leads to a contradiction in the case $s_2 > s_1$.

From the convexity of the functions $f_t(x)$ and $-g_t(x)$ with respect to x, we deduce that $-\gamma$ and ϕ are convex functions of Q. Therefore the set

$$\tilde{H} = \{z + re_{n+1} : z \in H_0, r \in \mathbb{R}, \phi(z) \le r \le \gamma(z)\} ,$$

is a convex body in $\mathbb{R}^{n+1}$. In order to complete the proof it is enough to verify that H_t is just the projection of $\tilde{H}$ onto $e_{n+1}^{\perp}$ along the direction $e_{n+1} - tv$. For simplicity we shall denote by L_t such a projection.

Let $z_0 + r_0 e_{n+1}$, $z_0 \in H_0$, $r_0 \in \mathbb{R}$, be a point from $\tilde{H}$; its projection onto $e_{n+1}^{\perp}$ along the direction $e_{n+1} - tv$ is the point $z_0 + r_0 tv$ and it belongs to H_t if and only if

$$f_t(z_0|v^{\perp}) \le \langle v, z_0 + r_0 tv\rangle \le g_t(z_0|v^{\perp}) ,$$

or equivalently

$$\frac{f_t(z_0|v^{\perp}) - \langle v, z_0\rangle}{t} \le r_0 \le \frac{g_t(z_0|v^{\perp}) - \langle v, z_0\rangle}{t} .$$

The previous inequalities easily follow from $\phi(z_0) \le r_0 \le \gamma(z_0)$ and then $L_t \subset H_t$.

Conversely, let P be an extreme point of H_t and assume that P can be written as $x_0 + g_t(x_0)v$, with $x_0 \in H_0|v^{\perp}$ (if $P = x_0 + f_t(x_0)v$, the proof can be trivially adapted). The convexity of $g_s(x_0)$ with respect to s ensures the existence of the left-hand side derivative with respect to s of $g_s(x_0)$ at t; call ν such a derivative and consider the point $z_0 = x_0 + (g_t(x_0) - \nu t)v$. We want to show that z_0 belongs to H_0, or equivalently that $f_0(x_0) \le g_t(x_0) - \nu t \le g_0(x_0)$. While the second inequality follows from the convexity of $g_s(x_0)$ with respect to s, the first one is a consequence of (ii). Indeed, from $g_{(1-\lambda)t}(x_0) \ge \lambda f_0(x_0) + (1-\lambda)g_t(x_0)$, we infer

$$\frac{g_t(x_0) - g_{(1-\lambda)t}(x_0)}{\lambda t} \le \frac{-f_0(x_0) + g_t(x_0)}{t} , \ \forall \lambda \in (0,1) ,$$

which, for $\lambda \to 0$, reduces to the desired inequality.

From the convexity of $g_s(x_0)$ with respect to s, we deduce that $\gamma(z_0) = \nu$; therefore $z_0 + \nu e_{n+1}$ belongs to $\tilde{H}$ and $z_0 + \nu tv = P$ belongs to L_t.

In conclusion, we have showed that L_t contains all the extreme points of H_t and the lemma is proved. □

Proof of Theorem 2.1. Let $\{K_t : t \in [0,1]\}$, be a parallel chord movement along the direction v. By (11)

$$\begin{aligned} h_{\Gamma_p K_t}(u) &= \left\{ \frac{1}{c_{n,p} V(K_0)} \int_{K_0} \left|\langle u, z\rangle + \beta(z|v^{\perp}) t \langle u, v\rangle\right|^p \, dz \right\}^{\frac{1}{p}} \\ &= \|\langle u, \cdot\rangle + \beta(\cdot|v^{\perp}) t \langle u, v\rangle\|_p , \ u \in \mathbb{R}^n , \end{aligned} \tag{13}$$

where, for simplicity, $\|q(\cdot)\|_p$ stands for $\left\{\frac{1}{c_{n,p}V(K_0)} \int_{K_0} |q(z)|^p \, dz\right\}^{\frac{1}{p}}$. From the Minkowski inequality for p-norms we deduce that $h_{\Gamma_p K_t}(u)$ is a convex function of t, for every $u \in \mathbb{R}^n$. Notice also that $h_{\Gamma_p K_t}(u)$ is a Lipschitz function of t, with Lipschitz constant $\|\beta(\cdot|v^{\perp})\|_p$, independently of u.

Since the orthogonal projection of $\Gamma_p K_t$ onto $v^\perp$ is independent of t, it is sufficient to show that the family $\Gamma_p K_t$ satisfies conditions (i) and (ii) of Lemma 3.1.

As $\Gamma_p K_t$ is an origin symmetric convex body, for every $t \in [0,1]$, it can be represented by

$$\Gamma_p K_t = \{x + yv : x \in (\Gamma_p K_0)|v^\perp, -g_t(-x) \le y \le g_t(x)\},$$

where g_t is a suitable concave function defined on $(\Gamma_p K_0)|v^\perp$.

Since $z \in \Gamma_p K_t$ if and only if $\langle z, u\rangle \le h_{\Gamma_p K_t}(u)$, for every $u \in \mathbb{R}^n$, we can write

$$\begin{aligned} g_t(x) &= \sup\{\lambda \in \mathbb{R} : \langle x + \lambda v, u\rangle \le h_{\Gamma_p K_t}(u),\ \forall u \in \mathbb{R}^n\} \\ &= \sup\{\lambda \in \mathbb{R} : \lambda\langle v, u\rangle \le h_{\Gamma_p K_t}(u) - \langle x, u\rangle,\ \forall u \in \mathbb{R}^n\}, \end{aligned} \tag{14}$$

for every $x \in (\Gamma_p K_0)|v^\perp$.

The inner product and support functions are both homogeneous of degree 1. Thus in (14) we need consider only the vectors u such that $|\langle u, v\rangle| = 1$. Furthermore, the vectors u with a non-positive scalar product with v provide no bounds for λ. Therefore we get

$$\begin{aligned} g_t(x) &= \sup\{\lambda \in \mathbb{R} : \lambda \le h_{\Gamma_p K_t}(w+v) - \langle x, w+v\rangle,\ \forall w \in v^\perp\} \\ &= \inf_{w \in v^\perp}\{h_{\Gamma_p K_t}(w+v) - \langle x, w\rangle\}. \end{aligned} \tag{15}$$

Notice that $g_t(x)$ is in fact the minimum, as $w \in v^\perp$, of $\{h_{\Gamma_p K_t}(w+v) - \langle x, w\rangle\}$, unless x belongs to the boundary of $(\Gamma_p K_0)|v^\perp$. The minimum is attained when $w + v$ is directed as a normal vector to $\Gamma_p K_t$ at $x + g_t(x)v$.

As an infimum of equi-Lipschitz functions of t, $g_t(x)$ is a Lipschitz function too. Its convexity will follow from the inequality

$$2g_{\frac{t_1+t_2}{2}}(x) \le g_{t_1}(x) + g_{t_2}(x).$$

By (13) and (15) we can write

$$\begin{aligned} 2g_{\frac{t_1+t_2}{2}}(x) &= \inf_{u \in v^\perp}\{\|\langle 2u + 2v, \cdot\rangle + \beta(\cdot|v^\perp)(t_1+t_2)\|_p - \langle x, 2u\rangle\} \\ &= \inf_{u_1, u_2 \in v^\perp}\{\|\langle u_1 + u_2 + 2v, \cdot\rangle + \beta(\cdot|v^\perp)(t_1+t_2)\|_p - \langle x, u_1 + u_2\rangle\} \\ &\le \inf_{u_1, u_2 \in v^\perp}\{\|\langle u_1 + v, \cdot\rangle + \beta(\cdot|v^\perp)t_1\|_p + \|\langle u_2 + v, \cdot\rangle + \beta(\cdot|v^\perp)t_2\|_p - \\ &\qquad \langle x, u_1\rangle - \langle x, u_2\rangle\} \\ &= \inf_{u_1 \in v^\perp}\{\|\langle u_1 + v, \cdot\rangle + \beta(\cdot|v^\perp)t_1\|_p - \langle x, u_1\rangle\} \\ &\quad + \inf_{u_2 \in v^\perp}\{\|\langle u_2 + v, \cdot\rangle + \beta(\cdot|v^\perp)t_2\|_p - \langle x, u_2\rangle\} \\ &= g_{t_1}(x) + g_{t_2}(x), \end{aligned}$$

where we again used the Minkowski inequality for p-norms. Hence condition (i) is verified.

Let us now turn to (ii). It is enough to prove the first inequality; the second will follow by interchanging t_1 with t_2 and x with $-x$. We can write

$$
\begin{aligned}
(1-\lambda)g_{t_2}(x) &= \inf_{u\in v^\perp}\{\|(1-\lambda)\langle u+v,\cdot\rangle+\beta(\cdot|v^\perp)(1-\lambda)t_2\|_p-\langle x,(1-\lambda)u\rangle\}\\
&= \inf_{u_1,u_2\in v^\perp}\{\|\langle u_2-\lambda u_1+v-\lambda v,\cdot\rangle+\beta(\cdot|v^\perp)[(1-\lambda)t_2+\lambda t_1-\lambda t_1]\|_p\\
&\qquad -\langle x,u_2-\lambda u_1\rangle\}\\
&\le \inf_{u_1,u_2\in v^\perp}\{\|\langle u_2+v,\cdot\rangle+\beta(\cdot|v^\perp)[\lambda t_1+(1-\lambda)t_2]\|_p+\\
&\qquad \|\langle -\lambda u_1-\lambda v,\cdot\rangle-\beta(\cdot|v^\perp)\lambda t_1\|_p-\langle x,u_2-\lambda u_1\rangle\}\\
&= \inf_{u_1\in v^\perp}\{\lambda\|\langle u_1+v,\cdot\rangle+\beta(\cdot|v^\perp)t_1\|_p+\lambda\langle x,u_1\rangle\}+\\
&\qquad \inf_{u_2\in v^\perp}\{\|\langle u_2+v,\cdot\rangle+\beta(\cdot|v^\perp)[\lambda t_1+(1-\lambda)t_2]\|_p-\langle x,u_2\rangle\}\\
&= \lambda g_{t_1}(-x)+g_{\lambda t_1+(1-\lambda)t_2}(x)\,.
\end{aligned}
$$

This concludes the proof. □

Proof of Theorem 2.2. By Fubini's theorem we have

$$V(\Gamma_p K_t)=\int_{(\Gamma_p K_0)|v^\perp}[g_t(x)+g_t(-x)]\,dx=2\int_{(\Gamma_p K_0)|v^\perp}g_t(x)\,dx\,,$$

where we integrate with respect to Lebesgue measure. Hence the convexity of the volume is an easy consequence of that of $g_t(x)$ with respect to t.

If $2V(\Gamma_p K_{\frac{t_1+t_2}{2}})=V(\Gamma_p K_{t_1})+V(\Gamma_p K_{t_2})$, for some t_1, $t_2\in[0,1]$, then we deduce that

$$2g_{\frac{t_1+t_2}{2}}(x)=g_{t_1}(x)+g_{t_2}(x)\,, \tag{16}$$

for almost every $x\in(\Gamma_p K_0)|v^\perp$. In fact, by the continuity of the functions g_t's, equality (16) holds everywhere. Take a point x from the interior of $(\Gamma_p K_0)|v^\perp$. We recall that in this case $g_t(x)$ is a minimum, for every t. Therefore there exist u_1, $u_2\in v^\perp$ such that

$$
\begin{aligned}
g_{t_1}(x)+g_{t_2}(x) &= h_{\Gamma_p K_{t_1}}(u_1+v)-\langle x,u_1\rangle+h_{\Gamma_p K_{t_2}}(u_2+v)-\langle x,u_2\rangle\\
&= \|\langle u_1+v,\cdot\rangle+\beta(\cdot|v^\perp)t_1\|_p+\|\langle u_2+v,\cdot\rangle+\beta(\cdot|v^\perp)t_2\|_p-\langle x,u_1\rangle-\langle x,u_2\rangle\,.
\end{aligned}
$$

By the Minkowski inequality we get

$$
\begin{aligned}
g_{t_1}(x)+g_{t_2}(x) &\ge 2\|\langle\frac{u_1+u_2}{2}+v,\cdot\rangle+\beta(\cdot|v^\perp)\frac{t_1+t_2}{2}\|_p-2\langle x,\frac{u_1+u_2}{2}\rangle\\
&= 2h_{\Gamma_p K_{\frac{t_1+t_2}{2}}}(\frac{u_1+u_2}{2}+v)-2\langle x,\frac{u_1+u_2}{2}\rangle\\
&\ge 2g_{\frac{t_1+t_2}{2}}(x)\,.
\end{aligned}
$$

Thus, by (16), the equality condition for the Minkowski inequality has to hold. Namely, there exists a constant c such that

$$\langle u_1+v,z\rangle+\beta(z|v^\perp)t_1=c\langle u_2+v,z\rangle+c\beta(z|v^\perp)t_2\,, \tag{17}$$

for every $z\in K_0$, owing to the continuity of β. If one sets $z=z'+\lambda v$ in (17), then by differentiating with respect to the parameter λ, it turns out that $c=1$.

The conclusion is that β is a linear function. □

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