VOLUME INEQUALITIES FOR SETS ASSOCIATED WITH CONVEX BODIES

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Abstract. This paper deals with inequalities for the volume of a convex body and the volume of the projection body, the L^p-centroid body, and their polars. Examples are the Blaschke-Santaló inequality, the Petty and Zhang projection inequalities, the Busemann-Petty inequality. Other inequalities of the same type are still at the stage of conjectures.

The use of special continuous movements of convex bodies provides a general approach to this subject. A family of inequalities, depending on a parameter $p \geq 1$ and proved by Lutwak for $p = 1$ and $p = 2$, is obtained.

1. Introduction and preliminaries

This paper is devoted to some classical inequalities of Convex Geometry involving the volume of an n-dimensional convex body and the volume of a further body associated to the given one. More precisely, our attention is focused on the projection body, the L^p-centroid body and their polar bodies.

Our approach comes from the idea that the most part of results connected with these inequalities can be deduced by the same general method, which is based on the use of special continuous movements of the bodies we are dealing with.

Let K be a convex body in $\mathbb{R}^n$, that is a n-dimensional compact convex set, and assume that the origin is an interior point of K.

The support function of the convex body K is defined as

$$h_K(u) = \max_{x \in K} \langle u, x \rangle, \text{ for } u \in \mathbb{R}^n ,$$

where $\langle \cdot\ , \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, and the radial function of K as

$$\rho_K(u) = \max\{r \in \mathbb{R} : ru \in K\}\, , \text{ for } u \in \mathbb{R}^n\, .$$

The n-dimensional volume $V(K)$ of K can be expressed in terms of the radial function by

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(z)\, dz\, ,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

The *polar body* K^* of K can be defined as

$$K^* = \{x \in \mathbb{R}^n | \langle x, y \rangle \leq 1, \forall y \in K\}\, .$$

The polar body of K depends on the location of the origin. It is easy to see that $(K^*)^* = K$ and that

$$\rho_{K^*}(u) = \frac{1}{h_K(u)}, \text{ for } u \in \mathbb{R}^n .$$

The *projection body* of K is the convex body ΠK such that

$$h_{\Pi K}(u) = \frac{1}{2}\int_{\partial K} |\langle u, v\rangle| \, dv, \text{ for } u \in \mathbb{R}^n ,$$

where dv denotes the area element at the point on ∂K whose outer unit normal is v. It is clear from the definition that $h_{\Pi K}(u)$ is the $(n-1)$-dimensional volume of the projection of K orthogonal to u.

For every Borel subset ω of S^{n-1}, we define the *area measure* $\sigma_K(\omega)$ of K as the $(n-1)$-dimensional Hausdorff measure of the reverse image of ω through the Gauss map. Recall that the Gauss map sends each point on ∂K to the set of outward unit normal vectors to ∂K at that point. Therefore, the support function of ΠK can be rewritten as

$$h_{\Pi K}(u) = \frac{1}{2}\int_{S^{n-1}} |\langle u, v\rangle| \, d\sigma_K(v), \text{ for } u \in \mathbb{R}^n .$$

If ∂K is sufficiently smooth and its Gauss curvature is strictly positive, then

$$h_{\Pi K}(u) = \frac{1}{2}\int_{S^{n-1}} \frac{|\langle u, v\rangle|}{\gamma_K(v)} \, dv, \text{ for } u \in \mathbb{R}^n ,$$

being $\gamma_K(v)$ the Gauss curvature of ∂K at the point where the outward unit normal vector is v. Such a representation shows that $h_{\Pi K}$ is the *cosine transform* of $\frac{1}{\gamma_K}$.

Each projection body is a *zonoid*, namely it is the limit, in the Hausdorff metric, of a sum of segments. Here by sum it is meant the Minkowski addition of subsets of $\mathbb{R}^n$

$$A + B = \{x \in \mathbb{R}^n : x = a + b, a \in A, b \in B\} .$$

Zonoids play an important role in Convex Geometry, as well as in different areas of Mathematics. In particular, every full-dimensional zonoid turns out to be the projection body of a convex body.

Given a convex body K, for each real number $p \geq 1$, the *L^p-centroid body* $\Gamma_p K$ of K is the convex body whose support function is

$$h_{\Gamma_p K}(u) = \left\{ \frac{1}{c_{n,p} V(K)} \int_K |\langle u, z\rangle|^p \, dz \right\}^{\frac{1}{p}} , \text{ for } u \in \mathbb{R}^n ,$$

where

$$c_{n,p} = \frac{\kappa_{n+p}}{\pi \kappa_n \kappa_{p-1}} ,$$

and

$$\kappa_r = \pi^{\frac{r}{2}} / \Gamma(1 + \frac{r}{2}) .$$

Notice that κ_n is the volume of the unit ball B^n of $\mathbb{R}^n$ and the constant $c_{n,p}$ is such that $\Gamma_p B^n = B^n$.

The above definition can be extended to compact bodies in $\mathbb{R}^n$.

Up to constants, $\Gamma_1 K$ is known in the literature as the *centroid body* ΓK of K and $\Gamma_2 K$ as the *Legendre ellipsoid* of K. If K is an origin-symmetric convex body, it turns out that the boundary of ΓK is the locus of the centroids of all the halves of K obtained by cutting K with hyperplanes through the origin.

By using polar coordinates in integration, one has that

$$h_{\Gamma K}(u) = \frac{1}{n c_{n,1} V(K)} \int_{S^{n-1}} |\langle u, z\rangle| \rho_K^{n+1}(z)\, dz\,,$$

which shows that the centroid body ΓK is, up to a constant, the projection body of ΛK, the *curvature image* of K, i.e. the convex body such that $\gamma_{\Lambda K}(z) = \rho_K^{-n-1}(z)$, for every $z \in S^{n-1}$. The Minkowski Theorem guarantees the existence of such a body (see, for example, [32], Ch. 7.1).

If $\Gamma_\infty K$ is interpreted as a limit of (1), as $p \to \infty$, then

$$\Gamma_\infty K = conv(K \cup (-K))\,,$$

where *conv* stands for the convex hull.

For further details related to the content of this preliminary section, we refer to the books of Gardner [11] and Schneider [32] and to the articles by Goodey and Weil [12], Lutwak [17], Lutwak and Zhang [19], Milman and Pajor [23].

As a final remark, we notice that there are other bodies that can be associated to a given one: the *intersection body*, the *cross-section body*, the *Blaschke body*, etc.. As Richard Gardner writes in the Introduction of his book [11], *geometric tomography houses a zoo of strange geometric bodies, powerful integral transforms, and exotic but highly effective inequalities.*

2. Main volume inequalities

In this section we list the main inequalities which involve the volume of the bodies introduced in Section 1. At present, some of them are still conjectures. Further information and details on inequalities of this type can be found in [17].

2.1. **The Blaschke-Santaló inequality.** For every convex body K, assume that the origin is chosen so that $V(K^*)$ is minimum. The Blaschke-Santaló inequality states that

$$V(K)V(K^*) \le \kappa_n^2\,, \tag{1}$$

where equality holds if and only if K is an origin-symmetric ellipsoid. The quantity on the left-hand side of (1) is called *volume product* of K. Inequality (1) was proved for $n \le 3$ by Blaschke [1], [2] and for all n by Santaló [31]. The equality conditions were proved by Saint Raymond [30] in the symmetric case and by Petty [26] in the general case.

2.2. **Mahler's conjecture.** It was conjectured by Mahler [20] that the minimum of the volume product is attained when K is a simplex, that is

$$V(K)V(K^*) \geq \frac{(n+1)^{n+1}}{(n!)^2}\,. \tag{2}$$

Mahler [21] proved that (2) holds if $n = 2$ and Meyer [22] that in this case equality occurs only for triangles.

For origin-symmetric convex bodies it has been conjectured that n-parallelotopes (and their polars, i.e. cross-polytopes) minimize the volume product, hence

$$V(K)V(K^*) \geq \frac{4^n}{n!}\,. \tag{3}$$

For $n = 2$, inequality (3) was proved by Mahler [21] and Reisner [28] showed that parallelograms are the only minimizers. For $n > 3$ there are bodies, different than parallelotopes and their polars, for which (3) is an equality, as shown by Saint Raymond [30]. Reisner [27], [28] and Saint Raymond [30] proved (3) for special classes of convex bodies, namely for zonoids and for the affine images of convex sets symmetric with respect to the coordinate hyperplanes. A simpler proof of Reisner's result was given in [13].

Bourgain and Milman [4] proved that there exists a constant c, not depending on the dimension, such that

$$V(K)V(K^*) \geq c^n \kappa_n^2\,.$$

2.3. **The projection body conjectures.** The problem of finding minimizers and maximizers of the functional $V(K)^{1-n}V(\Pi K)$ in the class of all convex bodies is open.

In 1972 Petty [25] conjectured that ellipsoids are the only minimizers.

As far as the maximum of $V(K)^{1-n}V(\Pi K)$ is concerned, Brannen [5] conjectured that, in the class of all convex bodies, simplices are maximizers. The same author conjectured that the largest centrally symmetric subset of a simplex gives a sharp upper bound of the functional in the class of centrally symmetric convex bodies.

2.4. **The Petty projection inequality.** Petty [25] proved that

$$V(K)^{n-1}V(\Pi^* K) \leq \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n, \tag{4}$$

with equality if and only if K is an ellipsoid. Notice that Petty's conjecture in 2.3 and the Blaschke-Santaló inequality (1) imply (4).

2.5. **The Zhang projection inequality.** The reverse inequality of (4) is due to Zhang [34]:

$$V(K)^{n-1}V(\Pi^* K) \geq \binom{2n}{n} n^{-n}\,, \tag{5}$$

with equality if and only if K is a simplex.

2.6. **The L^p-Busemann-Petty inequality.** Let $p \geq 1$. Then

$$V(\Gamma_p K)V(K)^{-1} \geq 1\,, \tag{6}$$

with equality if and only if K is an origin-symmetric ellipsoid. For $p = 1$, inequality (6) was proved by Petty [24] via the Busemann random simplex inequality (see [11], Theorem 9.2.6). For $p = 2$, Blaschke [3] proved (6) when $n = 3$. For $p = 2$ and general n, (6) was proved by John [14]. For general p, inequality (6) was proved by Lutwak, Yang and Zhang [18] and, in a different way, by Campi and Gronchi [8].

Concerning a reverse version of (6), let us notice that the functional we are considering is not bounded from above in the class of all convex bodies. A natural assumption is to restrict ourselves to the bodies containing the origin. It has been conjectured that in such a class simplices with one vertex at the origin provide the maximum. In [9] this conjecture is proved for $n = 2$ (see also Section 4).

2.7. **The L^p-Blaschke-Santaló inequality.** Let $p \geq 1$. Then

$$V(\Gamma_p^* K)V(K) \leq \kappa_n^2\,, \tag{7}$$

with equality if and only if K is an origin-symmetric ellipsoid, as proved by Lutwak and Zhang in [19]. The name of this inequality comes from the fact that if K is origin-symmetric and p tends to infinity, then (7) gives (1).

Even in this case, it has been conjectured that suitable simplices are minimizers of the left-hand side of (7) in the class of all convex bodies containing the origin. A result in the two-dimensional case is contained in Subsection 4.3.

3. The Rogers and Shephard method

Most of the results described in the previous section can be obtained by the use of *shadow systems*, which were introduced by Rogers and Shephard in [29] and [33].

A *shadow system* along the direction v is a family of convex bodies $K_t \subset \mathbb{R}^n$ that can be defined by

$$K_t = conv\{z + \alpha(z)t\,v : z \in A \subset \mathbb{R}^n\}\,, \tag{8}$$

where A is an arbitrary bounded set of points in $\mathbb{R}^n$, α is a real bounded function on A, and t runs in an interval of the real axis. The function α in (8) is called the *speed function* of the shadow system.

As proved in [29], *the volume of K_t is a convex function of t*. The proof is based on the fact that the length of each chord of K_t parallel to v turns out to be a convex function of t. This convexity result was extended by Shephard [33] to mixed volumes of shadow systems along the same direction v.

We recall that the mixed volume of the convex bodies K_1, K_2, ..., K_n can be defined by

$$V(K_1, K_2, \ldots, K_n) = \frac{1}{n!}\sum_{j=1}^{n}(-1)^{n+j}\sum_{i_1<\cdots<i_j} V(K_{i_1} + K_{i_2} + \cdots + K_{i_j})\,. \tag{9}$$

As proved by Minkowski (see [32], Ch. 5), mixed volumes are the coefficients of the homogeneous polynomial $V(t_1K_1 + \cdots + t_nK_n)$ of degree n in the variables t_1, t_2, $\ldots$, t_n. Special instances of mixed volumes are the *quermassintegrals* $W_i(K)$ of a convex set K, which are defined by (9), for $i = 0, 1, \ldots, n$, by taking $(n-i)$ copies of K and i copies of the unit ball B^n. Notice that every mixed volume can be expressed by an integral. An example that we shall use later is given by

$$V(K,\ldots,K,L) = V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(v)\, d\sigma_K(v). \tag{10}$$

Shephard's proof of the convexity of mixed volumes under shadow systems is based on the remark that every shadow system K_t along v can be seen as the projection of an $(n+1)$-dimensional convex body onto a fixed hyperplane orthogonal to w with respect to the direction $w - tv$.

A special type of shadow system is naturally related to the well known Steiner process of symmetrization. Precisely, fix a direction v and let

$$K = \{x + yv \in \mathbb{R}^n : x \in K|v^\perp, y \in \mathbb{R}, f(x) \le y \le g(x)\}\,;$$

here $K|v^\perp$ denotes the orthogonal projection of K onto the hyperplane $v^\perp$ through the origin, orthogonal to v, and f and $-g$ are convex functions on $K|v^\perp$. The shadow system with speed function $\alpha(x) = -(f(x)+g(x))$ and $t \in [0,1]$ is such that $K_0 = K$, $K_1 = K^v$, the reflection of K in the hyperplane $v^\perp$, and $K_{1/2}$ is the Steiner symmetral of K with respect to $v^\perp$. In this process, the volume is trivially constant, while the behavior of the other quermassintegrals is described by Shephard's result on mixed volumes.

Thus the Steiner symmetral of a convex body K can be obtained through a shadow system along a direction v in which the speed function is constant on each chord parallel to v. Shadow systems of this type are called *parallel chord movements*. So, a parallel chord movement along the direction v is a family of convex bodies K_t in $\mathbb{R}^n$ defined by

$$K_t = \{z + \alpha(x)t\, v : z \in K, x = z - \langle z, v\rangle v\}\,,$$

where α is a continuous real function on $v^\perp$. Notice that the above definition requires that α must be chosen so that, for every t, the set K_t is convex.

If the speed function α is an affine function (that is, $\alpha(x) = \langle x, u\rangle + k$, for some vector u and real constant k), then it is easy to see that K_t is an affine image of K, for every t in the range of the movement.

Shadow systems can be applied successfully according to the following Shephard argument (see [33]): *If a functional defined in the class of all convex sets is continuous, invariant under reflections and convex with respect to the parameter t of any parallel chord movement, then it attains its minimum at the ball among all sets of given volume.* Here the continuity refers to the Hausdorff metric.

This statement follows from the well-known property of the Steiner symmetrization, that, if suitably repeated, changes every convex set in a ball.

Consequences of this procedure are, for example, classical isoperimetric type inequalities for quermassintegrals (see [6], p. 144, [11], p. 372). Other geometric functionals have the same convex behavior under shadow systems; see, for example, [33], [7] and the next section.

A question is whether the same convexity property of the functionals can help in finding reverse inequalities.

We are able to give a positive answer only in the case $n = 2$. In order to do this, it is convenient to introduce the class Ω_n of n-dimensional convex bodies K with the following property: *If K_t is a parallel chord movement, $t \in [-1, 1]$, and $K = K_0$, then the speed of the movement is an affine function.*

Assume now that $F(K)$ is a functional defined in the class of all n-dimensional convex bodies and suppose F has a maximum in that class. If F is convex under parallel chord movements, then a maximizer of F must belong to Ω_n. Notice that, if F is bounded from above and convex under parallel chord movements, then F must be invariant under affine transformations. Moreover, if F is *strictly* convex, i.e. $F(K_t)$ is strictly convex unless the speed of K_t is affine, then all maximizers of F belong to Ω_n. As shown in [7], all simplices are in Ω_n, triangles are the only polygons in Ω_2, and, for $n \geq 2$, in Ω_n there are other bodies than simplices. For $n = 2$, by an approximation argument, we can conclude that triangles are maximizers of all continuous functionals which are convex under parallel chord movements (provided the maximum exists).

For $n = 2$, if one considers only centrally symmetric convex sets, then parallelograms play the same role as triangles in the general case.

The same method can be applied also to *linear* invariant functionals which are not bounded from above. Since these are the cases we shall deal with in the next section, we give here all the details.

A linear invariant continuous functional is bounded in the class of bodies containing the origin. Indeed, by John's theorem ([15], Theorem III), we can restrict ourselves to bodies containing a ball of radius one and contained in a ball of radius n, with the same center.

If F is a convex functional under parallel chord movements, then it is convex under translations, hence every maximizer of F has the origin as an extreme point. In particular, if P is a polytope, we can assume, without decreasing the value of F, that the origin is one of its vertices.

Let F be a functional defined in the class of all convex bodies containing the origin, which is continuous with respect to the Hausdorff metric, linear invariant and convex under parallel chord movements. Let P be a polygon and let $0, v_1, v_2, \ldots, v_m$ be its vertices clockwise ordered.

Let us consider the shadow system $\{P_t : t \in [t_0, t_1]\}$, $t_0 < 0 < t_1$, along v_2, with speed 1 at v_1 and 0 at the other vertices. If t_0 and t_1 are sufficiently close to 0, then only the triangle $0v_1v_2$ moves, while the remaining part of P keeps still. Let us choose $[t_0, t_1]$ as the largest interval such that the area of P_t is constant for all $t \in [t_0, t_1]$. Hence, $\{P_t : t \in [t_0, t_1]\}$ is just a parallel chord movement and P_{t_0} and P_{t_1} have exactly $m-1$ vertices. By the convexity of F,

$$F(P) \leq \max\{F(P_{t_0}), F(P_{t_1})\}\,.$$

If $m > 4$, iterations of this argument lead to the conclusion that $F(P) \leq F(T)$, where T is a triangle with a vertex at the origin. The continuity of F implies that T is a maximizer in the whole class of plane convex figures containing the origin.

4. Applications

This section contains some applications of two results involving shadow systems and related ideas.

For $v \in S^{n-1}$, let H_v^+ be the halfspace bounded by $v^\perp$ and containing v.

Theorem 4.1. *If K_t, $0 \leq t \leq 1$, is a shadow system along the direction v, then $V(K_t^* \cap H_v^+)^{-1}$ is a convex function of t.*

The proof (see [10]) is based on a generalized form of the Prekopa-Leindler inequality. Notice that the theorem deals only with one of the halves of K^* cut off by $v^\perp$. Clearly, if K_t is a family of origin-symmetric convex bodies, then $V(K_t^*)^{-1}$ is a convex function of t.

Theorem 4.2. *If K_t, $0 \leq t \leq 1$, is a parallel chord movement along the direction v, then $\Gamma_p K_t$, for $p \geq 1$, is a shadow system along v. Hence, $V(\Gamma_p K)$ is convex under parallel chord movement. Moreover, it is strictly convex unless the speed function is linear.*

For the proof of Theorem 4.2 see [8].

4.1. By applying Theorem 4.1 to the reciprocal of the volume product $V(K)V(K^*)$, we obtain the Blaschke-Santaló inequality (1) for origin-symmetric convex bodies (without the characterization of ellipsoids as unique maximizers).

For $n = 2$, we get the Mahler inequality (3) (again without characterization).

4.2. Theorem 4.2 immediately gives the L^p-Busemann-Petty centroid inequality (6), with the characterization of minimizers.

The same theorem, for $n = 2$, implies also a reverse form of the inequality.

Namely, *all triangles with a vertex at the origin maximize $V(\Gamma_p K)V(K)^{-1}$ in the class of all plane convex bodies containing the origin.*

4.3. Combining Theorem 4.1 with Theorem 4.2 provides the L^p-Blaschke-Santaló inequality (7) (without characterization) and the following reverse form for $n = 2$:

In the class of all plane convex bodies K containing the origin, triangles with a vertex at the origin are minimizers of $V(\Gamma_p^* K)V(K)$.

If one deals with plane origin-symmetric convex bodies, then centered parallelograms are maximizers and minimizers for the functionals in 4.2 and 4.3, respectively.

4.4. **Double entry functionals.** In this subsection we consider geometric functionals depending on two different convex bodies. We already noticed in Section 3 that mixed volumes are convex along shadow systems; for example, if K_t and L_t, $0 \leq t \leq 1$, are shadow systems along the same direction v, then $V_1(K_t, L_t)$ is a convex function of t.

Let us consider now the functional

$$G_p(K,L) = \frac{V_1(K,\Gamma_p L)}{V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}}\,, \tag{11}$$

where $p \geq 1$.

Theorem 4.3. *If K_t and L_t, $0 \leq t \leq 1$, are parallel chord movements along the same direction v, then $G_p(K_t,L_t)$, for $p \geq 1$, is a convex function of t.*

The theorem follows from Theorem 4.2 and from the above mentioned property of mixed volumes.

Theorem 4.4. *For every pair of convex bodies K and L,*

$$G_p(K,L) \geq 1\,, \tag{12}$$

where equality holds if and only if K and L are homothetic ellipsoids, with L origin-symmetric.

Proof. The theorem is a consequence of the Minkowski inequality (see [32], p. 317) and the L^p-Busemann-Petty centroid inequality (6). The equality conditions for these two inequalities imply the second part of the statement. □

The functional $G_p(K,L)$ is not bounded from above. Indeed, if L is moved far from the origin, then $\Gamma_p L$ becomes larger and larger. Moreover, if K and L approach to a couple of $(n-1)$-dimensional sets lying on non parallel hyperplanes, then the volumes of K and L tend to zero, while $V_1(K,L)$ tends to a positive number. Nevertheless, Theorem 4.3 provides a two-dimensional reverse form of inequality (12) when $K = L$.

Theorem 4.5. *For $n = 2$, if K contains the origin, then $G_p(K,K)$ attains its maximum when K is a triangle with a vertex at the origin.*

Let us show that inequality (12), for $p = 1$, implies the Petty projection inequality (4). We have

$$\begin{aligned}
1 \;&\leq\; \min_L G_1(K,L) = \min_L \frac{(n+1)\kappa_n \int_{S^{n-1}} \int_L |\langle x,u\rangle|\, dx d\sigma_K(u)}{2n\kappa_{n-1} V(K)^{\frac{n-1}{n}} V(L)^{\frac{n+1}{n}}} \\
&=\; \min_L \frac{(n+1)\kappa_n \int_L \int_{S^{n-1}} |\langle x,u\rangle|\, d\sigma_K(u) dx}{2n\kappa_{n-1} V(K)^{\frac{n-1}{n}} V(L)^{\frac{n+1}{n}}} \\
&=\; \min_L \frac{(n+1)\kappa_n \int_L h_{\Pi K}(x)\, dx}{n\kappa_{n-1} V(K)^{\frac{n-1}{n}} V(L)^{\frac{n+1}{n}}} \\
&=\; \frac{(n+1)\kappa_n \int_{\Pi^* K} h_{\Pi K}(x)\, dx}{n\kappa_{n-1} V(K)^{\frac{n-1}{n}} V(\Pi^* K)^{\frac{n+1}{n}}} = \frac{\kappa_n}{\kappa_{n-1} V(K)^{\frac{n-1}{n}} V(\Pi^* K)^{\frac{1}{n}}}\,,
\end{aligned}$$

where we used the fact that the integral of a support function h_M on a set L of given volume is minimum when L is just a level set of h_M, that is, when L is homothetic to M^*.

Notice that $G_1(K,L)$ can be expressed as

$$G_1(K,L) = \frac{(n+1)\kappa_n \int_{\partial K}\int_L |\langle x,u\rangle|\,dxdu}{2n\kappa_{n-1}V(K)^{\frac{n-1}{n}}V(L)^{\frac{n+1}{n}}}\,.$$

The last expression reminds some functionals studied by Lutwak in [16]. Precisely, he showed that Petty's projection body conjecture (see Subsection 2.3) is equivalent to say that the functional

$$\frac{\int_{\partial K}\int_{\partial L}|\langle x,u\rangle|\,dxdu}{V(K)^{\frac{n-1}{n}}V(L)^{\frac{n-1}{n}}}$$

attains its minimum, in the class of all pairs of convex bodies, if and only if K and L are homothetic to origin-symmetric polar reciprocal ellipsoids.

In the same paper Lutwak proved also that, for $p=1$ and $p=2$, the functional

$$\frac{\int_K\int_L|\langle x,u\rangle|^p\,dxdu}{V(K)^{\frac{n+p}{n}}V(L)^{\frac{n+p}{n}}}$$

attains its minimum if and only if K and L are homothetic to origin-symmetric polar reciprocal ellipsoids.

We are able to extend such a result to the case of an arbitrary $p \geq 1$.

Theorem 4.6. *Let $p \geq 1$, then*

$$\int_K\int_L|\langle x,u\rangle|^p\,dxdu \geq \frac{n\kappa_{n+p}}{\pi(n+p)\kappa_{p-1}\kappa_n^{\frac{n+2p}{n}}}V(K)^{\frac{n+p}{n}}V(L)^{\frac{n+p}{n}}\,,$$

with equality if and only if K and L are homothetic to origin-symmetric polar reciprocal ellipsoids.

Proof. By the definition of L^p-centroid body, we have that

$$\frac{\int_K\int_L|\langle x,u\rangle|^p\,dxdu}{V(K)^{\frac{n+p}{n}}V(L)^{\frac{n+p}{n}}} = \frac{c_{n,p}V(L)}{V(K)^{\frac{n+p}{n}}V(L)^{\frac{n+p}{n}}}\int_K h^p_{\Gamma_pL}(v)\,dv\,.$$

Therefore, for any fixed L, the minimum of the functional we are dealing with is attained if and only if K is homothetic to a level set of h_{Γ_pL}. Since the functional is invariant under rescaling, the minimum is attained when $K=\Gamma_p^*L$. Thus,

$$\frac{\int_K\int_L|\langle x,u\rangle|^p\,dxdu}{V(K)^{\frac{n+p}{n}}V(L)^{\frac{n+p}{n}}} \geq \frac{nc_{n,p}}{n+p}[V(L)V(\Gamma_p^*L)]^{-\frac{p}{n}} \geq \frac{n\kappa_{n+p}}{\pi(n+p)\kappa_{p-1}\kappa_n^{\frac{n+2p}{n}}}\,,$$

where we used the L^p-Blaschke-Santaló inequality (7).

The above argument provides also the equality conditions. □

References

[1] W. Blaschke, Über affine Geometrie VII: Neue Extremeigenschaften von Ellipse und Ellipsoid, *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl.* **69** (1917), 306–318.

[2] W. Blaschke, Affine Geometrie IX: Verschiedene Bemerkungen und Aufgaben, *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl.* **69** (1917), 412–420.

[3] W. Blaschke, Affine Geometrie XIV: Eine Minimumaufgabe für Legendres Trägheitsellipsoid, *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl.* **70** (1918), 72–75.

[4] J. Bourgain and V. Milman, New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$, *Invent. Math.* **88** (1987), 319–340.

[5] N. S. Brannen, Volumes of projection bodies, *Mathematika* **43** (1996), 255–264.

[6] Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer-Verlag, Berlin Heidelberg, 1988.

[7] S. Campi, A. Colesanti and P. Gronchi, A note on Sylvester's problem for random polytopes in a convex body, *Rend. Ist. Mat. Univ. Trieste* **31** (1999), 79–94.

[8] S. Campi and P. Gronchi, The L^p-Busemann-Petty centroid inequality, *Adv.Math.* **167** (2002), 128–141.

[9] S. Campi and P. Gronchi, On the reverse L^p-Busemann-Petty centroid inequality, *Mathematika* **49** (2002), 1–11.

[10] S. Campi and P. Gronchi, On volume product inequalities for convex sets, *preprint.*

[11] R. J. Gardner, *Geometric Tomography*, Cambridge University Press, Cambridge, 1995.

[12] P. Goodey and W. Weil, Zonoids and Generalisations, in *Handbook of Convex Geometry* (eds. P. M. Gruber and J. M. Wills), North-Holland, Amsterdam, 1993, 1297–1326.

[13] Y. Gordon, M. Meyer and S. Reisner, Zonoids with minimal volume-product – A new proof, *Proc. AMS* **104** (1988), 273–276.

[14] F. John, Polar correspondance with respect to convex regions, *Duke Math. J.* **3** (1937), 355–369.

[15] F. John, Extremum problems with inequalities as subsidiary conditions, in: Courant Anniversary Volume (Interscience, New York), 1948, pp. 187–204.

[16] E. Lutwak, On a conjectured projection inequality of Petty, *Contemp. Math.* **113** (1990), 171–182.

[17] E. Lutwak, Selected affine isoperimetric inequalities, in *Handbook of Convex Geometry* (eds. P. M. Gruber and J. M. Wills), North-Holland, Amsterdam, 1993, 151–176.

[18] E. Lutwak, D. Yang and G. Zhang, L_p affine isoperimetric inequalities, *J. Differential Geom.* **56** (2000), 111–132.

[19] E. Lutwak and G. Zhang, Blaschke-Santaló inequalities, *J. Differential Geom.* **47** (1997), 1–16.

[20] K. Mahler, Ein Übertragungsprinzip für konvexe Körper, Časopis Pěst. Mat. Fys. **68** (1939), 93–102.

[21] K. Mahler, Ein Minimalproblem für konvexe Polygone, *Mathematica (Zutphen) B* **7** (1939), 118–127.

[22] M. Meyer, Convex bodies with minimal volume product in $\mathbb{R}^2$, *Monatsh. Math.* **112** (1991), 297–301.

[23] V. D. Milman and A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, in *Geometric Aspects of Functional Analysis* (eds. J. Lindenstrauss and V. D. Milman), Lecture Notes in Mathematics **1376**, Springer, Heidelberg, 1989, 64–104.

[24] C. M. Petty, Centroid surfaces, *Pacific J. Math.* **11** (1961), 1535–1547.

[25] C. M. Petty, Isoperimetric problems, in *Proc. Conf. Convexity and Combinatorial Geomerty, Univ. Oklahoma*, (1971), 26–41.

[26] C. M. Petty, Affine isoperimetric problems, *Ann. N. Y. Acad. Sc.* **440** (1985), 113–127.

[27] S. Reisner, Random polytopes and the volume product of symmetric convex bodies, *Math. Scand.* **57** (1985), 386–392.

[28] S. Reisner, Zonoids with minimal volume product, *Math. Z.* **192** (1986), 339–346.

[29] C. A. Rogers and G. C. Shephard, Some extremal problems for convex bodies, *Mathematika* **5** (1958), 93–102.

[30] J. Saint-Raymond, Sur le volume des corps convexes symétriques, in *Séminaire Choquet - Initiation à l'Analyse 1980/81 Exp. No. 11*, Université Pierre et Marie Curie, Paris, 1981, 1–25.

[31] L. A. Santaló, Un invariante afin para los cuerpos convexos del espacio de n dimensiones, *Portugal. Math.* **8** (1949), 155–161.
[32] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.
[33] G. C. Shephard, Shadow systems of convex bodies, *Israel J. Math.* **2** (1964), 229–36.
[34] G. Zhang, Restricted chord projection and affine inequalities, *Geom. Dedicata* **39** (1991), 213–222.

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