# ON THE REVERSE $L^p$-BUSEMANN-PETTY CENTROID INEQUALITY

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Abstract. The volume of the $L^p$-centroid body of a convex body $K \subset \mathbb{R}^d$ is a convex function of a time-like parameter when each chord of $K$ parallel to a fixed direction moves with constant speed. This fact is used to study extrema of some affine invariant functionals involving the volume of the $L^p$-centroid body and related to classical open problems like the slicing problem. Some variants of the $L^p$-Busemann-Petty centroid inequality are established. The reverse form of these inequalities is proved in the two-dimensional case.

## 1. Introduction.

This paper deals with some functionals involving the volume of $L^p$-centroid bodies of $d$-dimensional convex sets.

Let $K$ be a convex compact set in $\mathbb{R}^d$ and let $h_K$ be its support function defined by

$$h_K(u) = \max\{\langle z, u\rangle : z \in K\}\,,\ u \in \mathbb{R}^d\,,$$

where $\langle , \rangle$ denotes the standard inner product.

Assume that $K$ is a convex body in $\mathbb{R}^d$, i.e. a convex compact set with nonempty interior and denote by $V(K)$ its $d$-dimensional volume. For each real number $p \geq 1$, let $\Gamma_p K$ be the *$p$-centroid body of $K$*, that is, the convex body whose support function is

$$h_{\Gamma_p K}(u) = \left\{ \frac{1}{c_{d,p} V(K)} \int_K |\langle u, z\rangle|^p \, dz \right\}^{\frac{1}{p}}\ ,\ u \in \mathbb{R}^d\,, \tag{1}$$

where the integration is with respect to Lebesgue measure and

$$c_{d,p} = \frac{\kappa_{d+p}}{\kappa_2 \kappa_d \kappa_{p-1}}\,,$$

with

$$\kappa_r = \pi^{\frac{r}{2}} / \Gamma(1 + \frac{r}{2})\,.$$

Notice that $\kappa_d$ is the volume of the unit ball $B^d$ of $\mathbb{R}^d$ and the constant $c_{d,p}$ is such that $\Gamma_p B^d = B^d$, for every $d$ and $p$.

The above definition is due to Lutwak and Zhang [LZ] but the concept of $L^p$-centroid body, at least for $p = 1, 2$, goes back to the nineteenth century.

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The set $\Gamma_1 K$, up to the constant $c_{d,1}$ is known in the literature as the *centroid body* $\Gamma K$ of $K$ (see the books of Gardner [G], Chapter 9, and Schneider [Sc], Section 7.4, for references). If $K$ is an origin symmetric body, it turns out that $\Gamma K$ is bounded by the locus of the centroids of all the halves of $K$ obtained by cutting $K$ with hyperplanes through the origin.

For $p = 2$, $\Gamma_2 K$ is homothetic to the *Legendre ellipsoid* of $K$, which arises in classical mechanics in connection with the moments of inertia of $K$ (see, e.g., Milman and Pajor [MP] and Lindenstrauss and Milman [LM] for references).

If $\Gamma_\infty K$ is interpreted as a limit of (1), as $p \to \infty$, then $\Gamma_\infty K = conv(K \cup (-K))$, where *conv* stands for the convex hull. Such a body was investigated by Fáry and Rédei [FR] in the framework of affine inequalities related to the geometry of numbers (see also Fáry [Fa]).

The notion of $L^p$-centroid body contributes to the viewpoint whereby the classical Brunn-Minkowski theory is a special instance of a more general $L^p$-theory for convex bodies. This idea is due to Lutwak [L] who developed the definition of Firey [Fi] of the $L^p$-Minkowski addition for sets. In the $L^p$ setting new families of affine inequalities of isoperimetric type can be obtained. As a remarkable example, Lutwak and Zhang [LZ] proved that, for $1 \leq p \leq \infty$,

$$V(K)V(\Gamma_p^* K) \leq \kappa_d^2 \,, \tag{2}$$

where $\Gamma_p^* K$ is the polar body of $\Gamma_p K$ and equality holds if and only if $K$ is an origin symmetric ellipsoid. For $p = \infty$ and $K$ origin symmetric (2) reduces to the well known Blaschke-Santalò inequality

$$V(K)V(K^*) \leq \kappa_d^2 \,. \tag{3}$$

Inequality (2) can be in turn obtained via (3) from the following affine inequality, which was recently proved by Lutwak, Yang and Zhang [LYZ] and, in a different way, by the authors [CG]:

$$V(\Gamma_p K) \geq V(K) \,, \tag{4}$$

where equality holds if and only if $K$ is an origin symmetric ellipsoid. This inequality, for $p = 1$, is the well known Busemann-Petty centroid inequality (see, e.g., [G], Theorem 9.2.6). For $p = 2$, inequality (4) was proved by Blaschke [B] for $d = 3$ and by John [J1] in higher dimensions (see also Petty [P1]).

The $L^p$-Busemann-Petty centroid inequality (4) states that the minimum of the functional $F(K) = V(\Gamma_p K)/V(K)$ is one. Notice that $F$ is invariant under linear transformations, but it is not translation invariant. In particular, $F$ is unbounded in the class of convex bodies. Natural restrictions which make $F$ bounded are, for example, to consider only convex bodies containing the origin or having the origin as their barycenter. In the present paper we deal with extremal problems for $F$ in these classes of convex bodies.

Precisely, let us consider the function

$$\phi_p(K, x) = \frac{V(\Gamma_p(K - x))}{V(K)} \,, \tag{5}$$

which expresses the dependence of the volume of $\Gamma_p K$ on the location of $K$. For every $K$, $\phi_p(K, x)$ is a strictly convex function of $x$ (see Theorem 2.1).

Inspired by the articles of Rogers and Shephard [RS] and [Sh], the results contained in this paper rely on the convexity of $\phi_p(K, \cdot)$ with respect to a time-like parameter under parallel chord movements of $K$, in which each chord of $K$ parallel to a fixed direction slides with constant speed (see Section 2). The same approach was used by Colesanti and the authors in [CCG] for studying functionals of Sylvester type and in [CG] for proving inequality (4).

In Section 3 we deal with the maximum of $\phi_p(K, x)$ for $x \in K$. Such a functional, denoted by $M_p$, is affine invariant in the class of convex bodies. As a maximum of convex functions, $M_p$ is still convex under parallel chord movements. Consequently we obtain that ellipsoids are the only minimizers of $M_p$. Moreover we show that, for $d = 2$, the maximum of $M_p$ is attained on triangles. This result is an extension of the one proved in the cases $p = 1, 2$ by Bisztriczky and Böröczky in [BB], where in addition they characterize triangles as unique maximizers. We prove also that parallelograms provide the maximum of $M_p$ in the class of all origin symmetric convex figures.

The search of maximizers of $M_p$ in all dimensions is related to classical open problems, like the slicing problem: Is there a constant $c_1$ independent of $d$ such that for every origin symmetric convex body $K$

$$V(K)^{\frac{d-1}{d}} \le c_1 \max_{u \neq 0} V(K \cap u^{\perp})\,? \tag{6}$$

Here $u^{\perp} = \{z \in \mathbb{R}^d : \langle u, z \rangle = 0\}$. As noted, for instance, in [BB] and [MP], (6) is equivalent to the existence of a constant $c_2$, independent of $d$, such that for every origin symmetric convex body $K$

$$V(\Gamma K)^{\frac{1}{d}} < c_2 V(K)^{\frac{1}{d}}\,. \tag{7}$$

An equivalent formulation involving $\Gamma_2 K$ can be found in [MP]. For a general discussion on the slicing problem and related results see [G], Note 9.6.

Inequality (7) could be interpreted as an estimate from above for the minimum with respect to $x$ of $\phi_1(K, x)$, with $K$ origin symmetric. Denoting by $m_p(K)$ the minimum of $\phi_p(K, x)$, a characterization of ellipsoids as minimizers of $m_p$ is provided by inequality (4). The parallel chord movements method cannot be applied directly to $m_p$, nevertheless we can employ it to obtain the same results as for $M_p$. If $K$ is centrally symmetric, then

$$m_p(K) = \phi_p(K, c_K)\,, \tag{8}$$

where $c_K$ is the barycenter of $K$. In Section 4 we prove that (8) holds also for an arbitrary convex body $K$, when $p = 2$. Besides we show that the affine invariant functional $C_p(K) = \phi_p(K, c_K)$ is convex under parallel chord movements. We deduce that, for $d = 2$, triangles are maximizers of $C_p$ among all convex figures. Since for a simplex $C_p$ and $m_p$ coincide, we conclude that triangles are maximizers also of $m_p$, for $d = 2$. In the restricted class of centrally symmetric plane convex sets, parallelograms are maximizers of $C_p$. A characterization of parallelograms as the only maximizers of $C_p$, for $p = 1, 2$, is given in [BB].

## 2. Preliminary results.

In this paper we shall make use of the following notion of continuous movements of convex bodies introduced by Rogers and Shephard (see [RS] and [Sh]). A *shadow system* along the direction $v$ is a family of convex sets $K_t \subset \mathbb{R}^d$ that can be defined by

$$K_t = conv\{x + \alpha(x)t\, v : x \in A \subset \mathbb{R}^d\}\,,$$

where $A$ is an arbitrary bounded set of points, $\alpha$ is a bounded function on $A$ and $t$ belongs to an interval of the real axis.

As proved by Rogers and Shephard [RS], *the volume $V(K_t)$ of a shadow system is a convex function of the parameter $t$.*

A *parallel chord movement* along the direction $v$ is a shadow system defined by

$$K_t = \{x + \beta(x|v^{\perp})t\, v : x \in K\}\,, \tag{9}$$

where $K$ is a convex body in $\mathbb{R}^d$ and $\beta$ is a continuous real function on the projection $K|v^{\perp}$ of $K$ onto $v^{\perp}$. Thus, a parallel chord movement can be constructed by assigning to each chord of $K = K_0$ parallel to $v$ a speed vector $\beta(z)v$, where $z$ is the projection of the chord onto $v^{\perp}$. The function $\beta$ has to be given in such a way that at time $t$ the union $K_t$ of the moving chords is convex. It is easy to verify that, if $\beta(z) = \langle z,\, u\rangle + k$, for some vector $u \in v^{\perp}$ and real constant $k$, then $K_t$ is an affine image of $K$.

The notion of parallel chord movement is strongly related to Steiner symmetrization. Fix a direction $v$ and let

$$K = \{x + yv \in \mathbb{R}^d : x \in K|v^{\perp}, y \in \mathbb{R}, f_v(x) \le y \le g_v(x)\}\,, \tag{10}$$

where $f_v$ and $-g_v$ are convex functions on $K|v^{\perp}$. If one takes $\beta(x) = -(f_v(x) + g_v(x))$ and $t \in [0,1]$ in (9), then $K_0 = K$, $K_1 = K^v$, the reflection of $K$ in the hyperplane $v^{\perp}$, and $K_{1/2}$ is the Steiner symmetral of $K$ with respect to $v^{\perp}$.

In [CG] the authors proved that, if $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction $v$, then $\{\Gamma_p K_t : t \in [0,1]\}$ is a shadow system along the same direction. Hence *the volume of $\Gamma_p K_t$ is a convex function of $t$.* Furthermore, they proved that *the volume of $\Gamma_p K_t$ is strictly convex unless $\beta$ is linear*, that is $\beta(x) = \langle x,\, u\rangle$ for some vector $u \in v^{\perp}$ and for every $x \in K_0|v^{\perp}$.

Let us apply these results to $\phi_p(K,x)$, defined by (5).

First, by definitions (1) and (5), we see easily that $\phi_p(K,x)$, for every fixed $x$, is continuous with respect to the Hausdorff metric. Moreover, an easy consequence of (1) is that

$$h_{\Gamma_p(LK)}(u) = h_{\Gamma_p K}(L^* u)\,,$$

for every $L \in GL(d)$, where $L^*$ is the transpose of $L$. The definition of support function implies that

$$h_{\Gamma_p K}(L^* u) = h_{L\Gamma_p K}(u)\,,$$

hence $\Gamma_p(LK) = L\Gamma_p K$. Therefore

$$\phi_p(LK, Lx) = \frac{V(\Gamma_p(LK - Lx))}{V(LK)} = \frac{V(L(\Gamma_p(K - x))}{V(LK)} = \phi_p(K,x)\,, \tag{11}$$

for every $L \in GL(d)$. Obviously $\phi_p(TK, Tx) = \phi_p(K,x)$, for every translation $T$, hence (11) holds true for every affine map $L$.

**Theorem 2.1.** *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction $v$ with speed function $\beta$, then $\phi_p(K_t, x)$ is a convex function of $t$ and a strictly convex function of $x$. Moreover, it is strictly convex in $t$ unless $\beta(z) = \langle z - x,\, u\rangle$ for some $u \in v^\perp$.*

*Proof.* By Fubini's theorem $V(K_t)$ is constant. For every $x$, $\{K_t - x : t \in [0,1]\}$ is a parallel chord movement of $K - x$, so the convexity of $\phi_p(K_t, x)$ with respect to $t$ immediately follows and strict convexity holds unless the speed of the movement $\{K_t - x : t \in [0,1]\}$ is linear, that is unless $\beta(z + x) = \langle z,\, u\rangle$, for some $u \in v^\perp$.

Let $t$ be fixed and take $x_1, x_2 \in \mathbb{R}^d$. Then $\{K_t - ((1-\lambda)x_1 + \lambda x_2) : \lambda \in [0,1]\}$ is a parallel chord movement along $x_2 - x_1$ with constant speed function. This implies the strict convexity of $\phi_p(K_t, x)$ with respect to $x$. $\square$

**Remark.** If $\beta(z) = \langle z - x,\, u\rangle$ for some $u \in v^\perp$, then $\phi_p(K_t, x)$ is constant for every $t \in [0,1]$. This follows from (11), by taking $Lz = z + \langle z,\, u\rangle tv$.

## 3. The functional $M_p$.

Let

$$M_p(K) = \max_{x \in K} \phi_p(K, x)\,.$$

From Theorem 2.1 we deduce that

$$M_p(K) = \max_{x \in \partial K} \phi_p(K, x)\,,$$

where $\partial K$ is the boundary of $K$.

To prove that $M_p$ is continuous in the Hausdorff metric, take a sequence of convex bodies $\{K_n\}$ converging to $K$ and choose $y_n \in \partial K_n$ such that $M_p(K_n) = \phi_p(K_n, y_n)$. Up to a subsequence we may assume that $y_n$ converges to $y \in K$ and then

$$\lim_{n\to\infty} M_p(K_n) = \lim_{n\to\infty} \phi_p(K_n - y_n, 0) = \phi_p(K - y, 0) \le M_p(K)\,.$$

On the other hand, let $z \in \partial K$ be such that $M_p(K) = \phi_p(K, z)$. Denote by $z_n$ the closest point to $z$ in $K_n$; clearly the sequence $\{K_n - z_n + z\}$ converges to $K$. Therefore

$$M_p(K) = \lim_{n\to\infty} \phi_p(K_n - z_n + z, z) = \lim_{n\to\infty} \phi_p(K_n, z_n) \le \lim_{n\to\infty} M_p(K_n)\,.$$

From (11) it follows that the functional $M_p$ is affine invariant. As a consequence of this fact, $M_p$ has maximum and minimum in the class of all convex bodies as well as in the class of all centrally symmetric convex bodies. Indeed, by John's theorem ([J2], Theorem III), we can restrict ourselves to the class of (symmetric) convex bodies containing the unit ball $B^d$ and contained in $dB^d$. A standard compactness argument provides the existence of the extrema of $M_p$.

A basic tool in looking for such extrema is the following theorem.

**Theorem 3.1.** *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement with speed function $\beta$, then $M_p(K_t)$ is a convex function of $t$. Moreover, it is strictly convex unless $\beta$ is affine.*

*Proof.* Let $v$ be the direction of the movement $\{K_t : t \in [0,1]\}$. We have

$$\begin{aligned} M_p(K_t) &= \max_{y \in K_t} \phi_p(K_t, y) \\ &= \max_{x \in K_0} \phi_p(K_t, x + \beta(x|v^\perp)tv) \\ &= \max_{x \in K_0} \phi_p(K_t - \beta(x|v^\perp)tv, x)\,. \end{aligned}$$

For every $x \in K_0$, $\{K_t - \beta(x|v^\perp)tv : t \in [0,1]\}$ is still a parallel chord movement along $v$. Therefore Theorem 2.1 implies that $M_p(K_t)$, as a maximum of convex functions, is convex with respect to $t$. The same theorem provides the conditions for the strict convexity. Indeed, if $M_p(K_t)$ is not a strictly convex function of $t$, then there exist $x \in K_0$ and $u \in v^\perp$ such that $\beta(z) - \beta(x|v^\perp) = \langle z - x,\, u\rangle$, for every $z \in v^\perp$. Hence $\beta(z) = \langle z,\, u\rangle + c$, for some constant $c$. □

A first consequence of Theorem 3.1 is the following characterization of minimizers.

**Corollary 3.2.** *The minimum of $M_p(K)$ in the class of all convex bodies is attained if and only if $K$ is an ellipsoid.*

*Proof.* Let us fix a direction $v$. We have seen in Section 2 that the Steiner process of symmetrization of the convex body $K$ with respect to $v^\perp$ is described by a parallel chord movement whose endpoints are $K$ and $K^v$. The affine invariant functional $M_p$ takes the same value at $K$ and $K^v$. Therefore, Theorem 3.1 implies that $M_p$ is not increased if one replaces $K$ by its Steiner symmetral and that the value of $M_p$ does not change after the symmetrization only if the speed of the movement is an affine function. By the representation (10) of $K$, the speed of the movement is $-(f_v + g_v)$. Hence it is an affine function if and only if all the midpoints of the chords of $K$ parallel to $v$ lie on a hyperplane.

It is well known (see, e.g., [P2]) that ellipsoids are the only bodies enjoying this property for every direction $v$. □

Clearly, if $B$ is a ball then $\phi_p(B, x)$ has constant value when $x$ belongs to the boundary of $B$ and that value is just $M_p(B)$. By the affine invariance of $\phi_p$, the same happens for ellipsoids.

Let us turn now to the problem of finding maximizers of the functional $M_p$. We solve here the problem for $d = 2$.

**Theorem 3.3.** *For $d = 2$, the maximum of $M_p(K)$ in the class of all convex bodies is attained if $K$ is a triangle.*

*Proof.* By the continuity of $M_p$, an approximation argument makes it sufficient to show that triangles give the maximum of $M_p$ in the class of all polygons. Let $P$ be a polygon with $n$ vertices, $n > 3$, and assume that $v_1$, $v_2$, $v_3$ are three consecutive vertices of $P$. Take a direction $u$ parallel to the line through $v_1$ and $v_3$ and consider the shadow system $\{P_t : t \in [t_0, t_1]\}$, $t_0 < 0 < t_1$, along $u$, with speed 1 at $v_2$ and 0 at the other vertices. If $t_0$ and $t_1$ are sufficiently close to 0, then only the triangle $v_1v_2v_3$ moves, while the remaining part of $P$ keeps still. Let us choose $[t_0, t_1]$ as the largest interval such that the area of $P_t$ is constant for all $t \in [t_0, t_1]$. Hence $\{P_t : t \in [t_0, t_1]\}$ is just a parallel chord movement and $P_{t_0}$ and $P_{t_1}$ have exactly $n - 1$ vertices. By Theorem 3.1,

$$M_p(P) < max\{M_p(P_{t_0}), M_p(P_{t_1})\}\,,$$

where the inequality is strict because the assumption $n > 3$ implies that the speed of the movement is not affine. If $n > 4$, iterations of this argument lead to the conclusion. $\square$

Theorem 3.3 implies that, if the origin belongs to $K$, then

$$V(\Gamma_p K) \leq V(\Gamma_p T)\,,$$

where $T$ is a triangle with the same area as $K$ having a vertex at the origin.

In the special cases $p = 1$ and $p = 2$, Bisztriczky and Böröczky [BB] proved that triangles are the only maximizers of $M_p$.

**Theorem 3.4.** *For $d = 2$, the maximum of $M_p(K)$ in the class of all centrally symmetric convex bodies is attained if $K$ is a parallelogram.*

*Proof.* One can easily adapt the proof of Theorem 3.3 to the symmetric case. In particular, by assuming that the bodies are origin symmetric, the parallel chord movements employed in this case have an odd function as speed. $\square$

**Remark.** The reason why the method used in proving Theorem 3.3 is successful only for $d = 2$ can be explained by introducing the class $\mathcal{P}^d$ of all convex bodies $K$ in $\mathbb{R}^d$ with the following property: If $\{K_t : t \in [-1, 1]\}$ is a parallel chord movement and $K = K_0$, then the speed of the movement is an affine function. The class $\mathcal{P}^d$ is non-empty, for every $d$. For instance, in [CCG] it is proved that all the simplices in $\mathbb{R}^d$ belong to $\mathcal{P}^d$. From Theorem 3.1 it follows that all the maximizers of $M_p$ are in $\mathcal{P}^d$. As shown in the proof of Theorem 3.3, triangles are the only polygons in $\mathcal{P}^2$. If $d > 2$, there are in $\mathcal{P}^d$ other polytopes than simplices; for example, a prism with a triangular base belongs to $\mathcal{P}^3$ (see again [CCG]).

As far as the characterization of maximizers is concerned, the method encounters the obstacle that $\mathcal{P}^2$ contains bodies other than triangles. Given a convex set $K$, represented as in (10), a parallel chord movement $\{K_t : t \in [-1, 1]\}$ with $K = K_0$ corresponds to a speed function $\beta$ such that $f_v \pm \beta$ and $-g_v \pm \beta$ are still convex functions in $K|v^\perp$. In terms of second derivatives, if we find a non zero measure $\beta''$ such that $f_v'' \pm \beta''$ and $-g_v'' \pm \beta''$ are still positive measures, then $K$ does not belong to $\mathcal{P}^2$. If, for example, $f_v''$ and $-g_v''$ are Hausdorff measures of different dimensions, then it is impossible to find such a $\beta''$ and consequently to move $K$ along $v$. In order to exhibit an example of a non-triangular set from $\mathcal{P}^2$, one can use a measure $\mu$ on the unit circle $S^1$ of the same type as the one obtained by Rogers and Taylor in [RT] via a suitable modification to the Cantor ternary function. The key property of such a measure is the following. Let $\alpha$ be a function defined in the support of $\mu$ so that $\mu$, at every point $x$, is locally comparable with a Hausdorff $\alpha(x)$-dimensional measure. If $\alpha(x) = \alpha(y)$, then one of the arcs of $S^1$ joining $x$ and $y$ has $\mu$-measure zero. By a possible balancing of the measure $\mu$ at some point, we can assume that $\int_{S^1} z\, d\mu = 0$ and then, by Minkowski's theorem (see, e. g., [Sc], p. 392), there exists a convex set $K$ with area measure $\mu$. One can check that, for every direction $v$, the measures $f_v''$ and $-g_v''$ are nowhere comparable each other in $K|v^\perp$. Hence $K \in \mathcal{P}^2$.

## 4. The functionals $m_p$ and $C_p$.

Let

$$m_p(K) = \min_{x \in K} \phi_p(K, x)$$

and

$$C_p(K) = \phi_p(K, c_K)\,,$$

where the point $c_K$ is the barycenter of the body $K$. Let us first deal with the functional $C_p$.

Since $c_{LK} = Lc_K$ for every affine map $L$, by (11) one has that $C_p$ is an affine invariant functional. Furthermore, the continuity of $\phi_p$ and $c_K$ immediately implies the continuity of $C_p$ with respect to the Hausdorff metric.

For some convex bodies $C_p = m_p$. If $K$ has a symmetry, then the barycenter of $K$ is a fixed point of the symmetry itself. By Theorem 2.1, the same happens to the point where the minimum of $\phi_p(K, x)$ is attained. Hence $c_K$ is the extremal point when $K$ is centrally symmetric or the group of symmetries of $K$ has a unique fixed point.

Even if $K$ has no symmetry, $C_p(K)$ is still the minimum of $\phi_p(K, x)$ when $p = 2$. To see this, we recall that the moment of inertia of a body $K$ about the axis through the origin parallel to the direction $u$ is

$$J_K(u) = \int_K (\|x\|^2 - \langle x,\, u\rangle^2)\, dx$$

and the polar moment of $K$ is

$$I_K = \int_K \|x\|^2\, dx\,.$$

Hence

$$\frac{V(K)}{d+2} h^2_{\Gamma_2 K}(u) = I_K - J_K(u) = \frac{1}{d-1}[J_K(u_1) + \cdots + J_K(u_{d-1}) - (d-2)J_K(u)]\,,$$

where $u_1, \ldots, u_{d-1}$ is an orthonormal basis of $u^\perp$.

Let us assume that $c_K$ is at the origin. If $K_t = K + tz$, then by Steiner's theorem (see, e. g., [Se], Section 3.5)

$$J_{K_t}(u) = J_K(u) + t^2\|z\|^2 - t^2\langle z,\, u\rangle^2\,.$$

Therefore, simple calculations give

$$h^2_{\Gamma_2 K_t}(u) = h^2_{\Gamma_2 K}(u) + \frac{(d+2)t^2}{V(K)}\langle z,\, u\rangle^2\,.$$

The last equality implies that $\Gamma_2 K$ is contained in the 2-centroid body of any translate of $K$. Thus

$$C_2(K) = m_2(K)\,.$$

Let us deal now with extrema of $C_p$. The same argument used in the previous section ensures that $C_p$ has maximum and minimum in the class of all convex bodies as well as in the class of the symmetric ones.

**Theorem 4.1.** *If $\{K_t : t \in [0, 1]\}$ is a parallel chord movement with speed function $\beta$, then $C_p(K_t)$ is a convex function of $t$. Moreover, it is strictly convex unless $\beta$ is affine.*

*Proof.* If $v$ is the direction of the movement $\{K_t : t \in [0,1]\}$, then the barycenter of $K_t$ moves with constant speed along the direction $v$. Indeed

$$\begin{aligned} c_{K_t} &= \frac{1}{V(K_t)} \int_{K_t} x\, dx = \frac{1}{V(K_0)} \int_{K_0} (x + \beta(x|v^\perp)tv)\, dx \\ &= c_{K_0} + \frac{tv}{V(K_0)} \int_{K_0} \beta(x|v^\perp)\, dx\,. \end{aligned}$$

Therefore $\{K_t - c_{K_t} : t \in [0,1]\}$ is still a parallel chord movement. Since $C_p(K_t) = \phi_p(K_t - c_{K_t}, 0)$, Theorem 2.1 completes the proof. $\square$

By the same proof of Corollary 3.2, Theorem 4.1 implies that in the class of all convex bodies the only minimizers of $C_p$ are ellipsoids. Such a result is already known as a consequence of the $L^p$-Busemann-Petty inequality (4). Notice that the same inequality implies that ellipsoids are the only minimizers of $m_p$ too.

Concerning the maximum, from Theorem 4.1 we deduce the following results.

**Theorem 4.2.** *For $d = 2$, the maximum of $C_p(K)$ in the class of all convex bodies is attained if $K$ is a triangle.*

*Proof.* It is enough to apply Theorem 4.1, and follow step by step the proof of Theorem 3.3. $\square$

The result just proved for $C_p$ implies the following theorem.

**Theorem 4.3.** *For $d = 2$, the maximum of $m_p(K)$ in the class of all convex bodies is attained if $K$ is a triangle.*

*Proof.* By (11), $m_p$ is affine invariant. If $T$ is a regular triangle, then, owing to the simmetries of $T$, $m_p(T) = C_p(T)$. By the affine invariance of both functionals, the same equality holds for an arbitrary triangle. Therefore, by Theorem 4.2, for an arbitrary two-dimensional convex body $K$, we have that

$$m_p(K) \le C_p(K) \le C_p(T) = m_p(T)\,,$$

where $T$ is any triangle. $\square$

In [BB] it is shown that parallelograms are the only maximizers of $C_1$ and $C_2$ among all centrally symmetric plane convex figures. For an arbitrary $p \ge 1$ the following holds true.

**Theorem 4.4.** *For $d = 2$, the maximum of $C_p(K)$ (or $m_p(K)$) in the class of all centrally symmetric convex bodies is attained if $K$ is a parallelogram.*

*Proof.* One can easily adapt the proof of Theorem 3.3 to the functional $C_p$. Precisely, by the continuity of $C_p$, it is sufficient to show that parallelograms are maximizers in the class of all origin symmetric polygons. Let $P$ belong to this class and assume that $P$ is not a parallelogram. Label three consecutive vertices of $P$ by $v_1$, $v_2$ and $v_3$. The shadow system $\{P_t : t \in [t_0, t_1]\}$ along a direction parallel to $v_1 - v_3$, with speed 1 at $v_2$, $-1$ at $-v_2$ and 0 at the other vertices is a parallel chord movement for $t_0$, $t_1$ sufficiently close to 0. If $[t_0, t_1]$ is the largest interval with such a property, then $P_{t_0}$ and $P_{t_1}$ have exactly two vertices less than $P$ and, by Theorem 4.1,

$$C_p(P) < max\{C_p(P_{t_0}), C_p(P_{t_1})\}\,.$$

Possible iterations of this argument give the conclusion. $\square$

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