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Fluid Mechanics — *A free boundary model for the evolution of a geothermal system*, by Luca Meacci, Angiolo Farina and Mario Primicerio, communicated on November 9, 2018.

Abstract. — The evolution of a geothermal system is studied. A mathematical model is proposed and the corresponding free boundary problem is formulated in a one-dimensional geometry. A situation corresponding to the geothermal field in Larderello, Tuscany (Italy) is considered, showing that the problem has two characteristic time scales, related to the motion of interface and diffusion of vapor. Since the former is much slower, a quasi-steady approximation can be introduced and solved, obtaining a qualitative description of the evolution of the Larderello basin from a liquid-dominated to a vapor-dominated situation. This is in agreement with the geological results.

Key words: Geothermal system, porous media fluid mechanics, free boundary problem

Mathematics Subject Classification: 76S05, 35R35

1. Introduction

A geothermal system consists of three essential components: the heat source, the reservoir and the fluid. The heat source can be a magmatic intrusion located at a relatively small depth (5 to 10 km) producing temperatures of 500–700 °C in the surrounding rocks [10]. The reservoir is a complex of fractured permeable rocks confined by two impermeable layers and is saturated by water, in liquid and/or in vapor phase. The lower boundary of the reservoir is called *K-horizon*, while the upper limit is perforated by wells through which the hot fluid is extracted and its energy is exploited in geothermal power plants, [10], [7], [22].

Of course, this is just a very schematic description, since in practice chemicals are carried by the fluid, the geometry of the reservoir can be very complicated, with several interconnected layers, recharge channels can be present through which meteoric water or artificially pumped water can feed the reservoir, etc.

Geothermal systems are usual classified as *water-dominated* or *vapor-dominated* [22]. In the first case the fluid (often called "the energy vector", [7]) is mainly in liquid phase, while in the second case there is a significant fraction in vapor phase. The first experiment in electric energy generation from geothermal sources (an electric generator driven by geothermal steam put into operation in 1904 by the Prince Piero Ginori Conti) took place in Larderello (a site in Tuscany, Italy) where several power plants are still operating and the production of geothermal energy continues up to the present [23], [8].

We refer to the Larderello basin for the following reasons. Larderello is considered as an evolved geothermal basin. Indeed, geologically there is evidence that the Larderello basin was formed as a water dominated system. The transformation into a vapor dominated basin seems caused by superficial manifestations [21], [4].

In this paper we present a simplified one-dimensional mathematical model aimed at describing such evolution, starting from the may papers that have been devoted to empirical and more sophisticated studies of this phenomenon [20], [13], [15], [3], [17]. The main simplifications that we will introduce, besides of the 1-D geometry, are the following: (*i*) we assume that the fluid is pure water, (*ii*) we neglect recharge, (*iii*) we suppose that the motion of the fluid does not modify the temperature profile of the rocks in the reservoir, (*iv*) we neglect the width of the capillary fringe separating the zone of liquid water from the vapor zone. This means that, from the mathematical point of view, the problem is a typical *free boundary problem*, in which the movement of the interface is an unknown function since it is determined by evaporation/condensation dynamics.

We discuss the time scales of the two phenomena involved (movement of the free boundary and diffusion of vapor) and we give the analytical solution of the quasi-steady dynamics that provides an immediate interpretation of the historical evolution of the basin. The results are in good agreement with the papers that have studied the evolution of the Larderello basin with empirical or semiempirical models with a more sophisticated numerical modelling.

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2. Physical hypothesis and geometry

Let consider a geothermal fluid composed by water in a vapor and liquid state. We denote by x the vertical axis pointing upward (see Figure 1). The bottom and the top of the geothermal reservoir correspond to $x = L_i$, and to $x = L_s$, $L_i < L_s < 0$, respectively. The ground surface corresponds to $x = 0$, so that the domain of interest is $[L_i, L_s]$. We assume that the width of the capillary fringe is negligible with respect to the vertical dimensions of the reservoir and thus that the liquid and the vapor phase are separated by a sharp interface, $x = s(t)$, that may vary in time. So, the lower part of the domain $L_i \le x \le s(t)$, is filled by liquid, while in the upper part of the domain, $s(t) \le x \le L_s$, there is only vapor. At $x = L_s$ the vapor pressure is known and we denote it as P_s. The latter is measured essentially depends on the exploitation of the basin.

According to the experimental data [2], the temperature within the reservoir is linear

$$T(x) = T_s + \gamma(L_s - x), \quad L_i \le x \le L_s, \tag{1}$$

where $\gamma > 0$, constant in time, is the known thermal gradient.

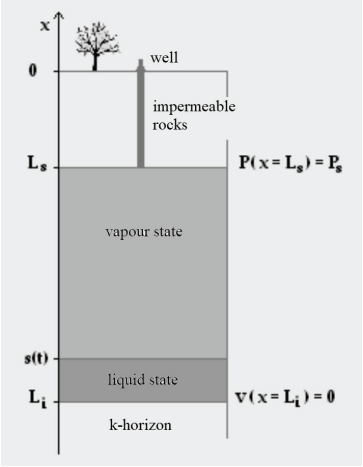

Figure 1. The one-dimensional domain is the interval $[L_i, L_s]$ that is splitted into two parts: the upper part is saturated by vapor while the bottom by the liquid phase. The temperature decreases linearly from T_s to T_i. At $x = L_s$ the vapor pressure is P_s (known) while the bottom of the reservoir, $x = L_i$, is impermeable.

3. The mathematical model

We denote by P_l the liquid pressure. We have

$$P_l(x) = P_l(s(t)) + \rho_l g(s(t) - x), \quad L_i < x < s(t) \leq L_s < 0, \tag{2}$$

where ρ_l is the density of the liquid and g the gravity acceleration.

The domain $s(t) < x < L_s$, is saturated by vapor whose continuity equation is

$$\frac{\partial}{\partial t}(\phi\rho) + \frac{\partial}{\partial x}(\phi\rho v) = 0, \quad s(t) < x \leq L_s, \tag{3}$$

where ϕ is the porosity of the medium assumed uniform and constant, v the velocity of the vapor and ρ its density. The vapor is assumed to behave as a perfect gas, i.e.

$$\rho = \frac{P}{rT}, \tag{4}$$

where P the vapor pressure and r the molar vapor constant. We recall Darcy's law

$$v = -\frac{K}{\phi\mu}\left(\frac{\partial P}{\partial x} + \rho g\right), \tag{5}$$

where μ is the vapor viscosity and K the permeability of the medium. Thus we rewrite (3) as

$$\frac{1}{T}\frac{\partial P}{\partial t} - \frac{K}{\phi\mu}\frac{\partial}{\partial x}\left[\frac{P}{T}\left(\frac{\partial P}{\partial x} + \frac{g}{r}\frac{P}{T}\right)\right] = 0, \quad s(t) < x \le L_s < 0, \tag{6}$$

where T is given by (1).

3.1. Free boundary conditions

The continuity of the mass flux across $s(t)$ entails (see [5])

$$\rho(v - \dot{s}) = -\rho_l \dot{s}, \tag{7}$$

and, with a reasonable approximation, $\rho v = -\rho_l \dot{s}$, so that (5) yields

$$\begin{aligned} \dot{s} &= \frac{\rho}{\rho_l}\frac{K}{\phi\mu}\left(\frac{\partial P}{\partial x} + \rho g\right)\Big|_{x=s(t)} \\ &\underset{(4)}{=} \frac{P}{rT\rho_l}\frac{K}{\phi\mu}\left(\frac{\partial P}{\partial x} + \frac{P}{rT}g\right)\Big|_{x=s(t)}. \end{aligned} \tag{8}$$

A second free boundary condition is provided by the Clapeyron's law that gives the relationship between P and T at the liquid-vapor interface. In the range of temperature relevant to our problem this law takes the empirical form

$$P^*(T) = a\exp\left[b\frac{T - 273}{T}\right], \tag{9}$$

where T is °K, P in Pascal and a and b have values $a = 961,7$ Pa, $b = 17.35$. Consequently the second condition on the free boundary is

$$P(s(t), t) = P^*(T(s(t))), \tag{10}$$

and (8) rewrites as

$$\dot{s} = \frac{P^*(T(s(t)))}{rT(s(t))\rho_l}\frac{K}{\phi\mu}\left(\frac{\partial P}{\partial x}\Big|_{s(t)} + \frac{P^*(T(s(t)))}{rT(s(t))}g\right). \tag{11}$$

We remark that (10) requires that the following conditions have to be fulfilled

$$P(x,t) < P^*(T(x)), \quad \text{in the vapor domain, i.e. } s(t) < x \le L_s, \tag{12}$$

$$P(x,t) > P^*(T(x)), \quad \text{in the liquid domain, i.e. } L_i \le x < s(t), \tag{13}$$

in order to have a physically consistent solution. We easily realize that (2) satisfies (13). We shall analyze (12) in Section 3.3.

3.2. The problem

To complete the information, we need to give the pressure at the reservoir top, $P(L_s,t)=P_s$ (already introduced in Section 2) and the initial conditions $s(0)=s_o$, $L_i<s_o<L_s$, and $P(x,0)=P_o(x)$, $s_o<x<L_s$, fulfilling (12) and (10), i.e. $P_o(x)<P^*(T(x))$ and $P_o(s_o)=P^*(T(s_o))$.

Thus the mathematical problem to be solved is the following: find a time $\tau>0$, a function $s(t)\in C[0,\tau]\cap C^1(0,\tau)$ such that $s(t)\in(L_i,L_s)$ and a function $P(x,t)$ continuous in $\overline{D}_\tau=\{(x,t): s(t)\le x\le L_s,\ 0\le t\le\tau\}$, $C^{2,1}(D_\tau)$ having P_x continuous up to $x=s(t)$, $t\in(0,\tau)$, so that, for $T(x)$ given by (1), the system is fulfilled

$$
\begin{cases}
\dfrac{\partial P}{\partial t}-\dfrac{KT}{\phi\mu}\dfrac{\partial}{\partial x}\left[\dfrac{P}{T}\left(\dfrac{\partial P}{\partial x}+\dfrac{g}{r}\dfrac{P}{T}\right)\right]=0, & (x,t)\in D_\tau,\\
P(L_s,t)=P_s, & t\in[0,\tau],\\
P(s(t),t)=P^*(T(s(t))), & t\in[0,\tau],\\
\dot{s}=\dfrac{P^*(T(s))}{rT(s)\rho_l}\dfrac{K}{\phi\mu}\left(\dfrac{\partial P}{\partial x}\Big|_s+\dfrac{P^*(T(s))}{rT(s)}g\right), & t\in(0,\tau),\\
P(x,0)=P_o(x), & x\in[s_o,L_s],\\
s(0)=s_o, & s_o\in(L_i,L_s).
\end{cases}
\tag{14}
$$

According to the results [9] the problem has one unique solution provided that $P_o\in C[s_o,L_s]$ and $P_o(L_s)=P_s$, $P_o(s_o)=P^*(T(s_o))$. We remark again that the solution is physically consistent only if (12) holds true.

3.3. Discussing time scales

To analyze the time scales that characterize the problem, we normalize the variables (excluding time) in the following way

$$
\tilde{x}=\frac{x-L_s}{L},\quad\Rightarrow\quad\tilde{x}\in[-1,0],
\tag{15}
$$

$$
\tilde{s}=\frac{s-L_s}{L},\quad\Rightarrow\quad\tilde{s}\in[-1,0],
\tag{16}
$$

$$
\tilde{T}=\frac{T}{T_s},
\tag{17}
$$

$$
\tilde{P}=\frac{P}{P_s},
\tag{18}
$$

$$
\tilde{P}^*(\tilde{T})=\frac{a}{P_s}\exp\left[b\frac{\tilde{T}-273/T_s}{\tilde{T}}\right],
\tag{19}
$$

Symbol	Description	Value	Units
L_s	Reservoir top depth	-1300	m
L_i	Reservoir bottom depth	-2500	m
T_s	Temperature at the top of the reservoir	650	°K
γ	Thermal gradient	0.04	°K/m
P_s	Vapor pressure at the top of the reservoir	$3,1 \times 10^6$	Pa
r	Universal gas constant over H_2O molar mass	$4,6 \times 10^2$	J/Kg°K
ρ_l	Liquid water density	10^3	Kg/m^3
ϕ	Medium porosity	10^{-2}	–
g	Standard gravity	9,8	m/s^2
μ	Vapor viscosity	2×10^{-5}	Pa · s
K	Medium permeability	3×10^{-16}	m^2

Table 1. Table of parameters referring to the reservoir of Larderello.

where $L = L_s - L_i = 1200$ m (see Table 1). Equation $(14)_1$ rewrites as

$$t_1 \frac{\partial \tilde{P}}{\partial t} - \tilde{T} \frac{\partial}{\partial \tilde{x}} \left[\frac{\tilde{P}}{\tilde{T}} \left(\frac{\partial \tilde{P}}{\partial \tilde{x}} + \alpha \frac{\tilde{P}}{\tilde{T}} \right) \right] = 0, \tag{20}$$

where

$$t_1 = \frac{\phi \mu L^2}{K P_s}, \quad \text{and} \quad \alpha = \frac{gL}{rT_s}. \tag{21}$$

Referring to the parameters of Table 1 we have $t_1 \approx 50$ years, while $\alpha \approx 4 \cdot 10^{-2}$. We remark that the estimate of t_1 is close to the one often appearing in the literature concerning the Larderello reservoir, [1].

In turn equation $(14)_4$ rewrites as

$$\frac{d\tilde{s}}{dt} = t_2 \frac{\tilde{P}^*(\tilde{T}(\tilde{s}))}{\tilde{T}(\tilde{s})} \left(\frac{\partial \tilde{P}}{\partial \tilde{x}} \bigg|_{\tilde{s}} + \alpha \frac{\tilde{P}^*(T(\tilde{s}))}{\tilde{T}(\tilde{s})} \right), \tag{22}$$

where $\tilde{P}^*(\tilde{T})$ is given by (19) and where

$$t_2 = t_1 \frac{rT_s \rho_l}{P_s} \approx 9000 \text{ years}.$$

This datum is confirmed by the geological researches, which show that the evolution of the Larderello basin towards a ''vapor dominated'' system took place on temporal scales $\sim 10^3$ years.

In Table 1 we list the values of the constants involved in the model. The parameters refer to the geothermal reservoir of Larderello, Tuscany (see also [11]).

The problem has clearly two different time scales. If our analysis is confined to ''short'' periods, the free boundary is practically immobile and the dimensionless

vapor pressure $\tilde{P}$ evolves from the initial value $P_o(x)/P_s$ to the stationary profile. Such a problem is not particularly interesting from a mathematical point of view (a standard parabolic problem in a fixed domain) nor for applications (indeed, the "initial" pressure $P_o(x)$ is not easily estimated).

On the contrary, the motion of the interface over long periods can be much more interesting to understand the trend of the geothermal reservoir in terms of the condition of exploitation (the known pressure P_s). Thus, in the time scale of the order of centuries the vapor pressure is in its stationary configuration, while the interface is moving. Thus, selecting t_2 as characteristic time and introducing

$$\tilde{t} = \frac{t}{t_2},$$

the problem to be studied has the form

$$\begin{cases} \dfrac{\partial}{\partial \tilde{x}}\left[\dfrac{\tilde{P}}{\tilde{T}}\left(\dfrac{\partial \tilde{P}}{\partial \tilde{x}} + \alpha \dfrac{\tilde{P}}{\tilde{T}}\right)\right] = 0, & \tilde{x} \in (\tilde{s}(\tilde{t}), 0),\ \tilde{t} > 0, \\ \tilde{P}(0, \tilde{t}) = 1, & \tilde{t} > 0, \\ \tilde{P}(\tilde{s}, \tilde{t}) = \tilde{P}^*(T(\tilde{s})), & \tilde{t} > 0, \\ \dfrac{d\tilde{s}}{d\tilde{t}} = \dfrac{\tilde{P}^*(\tilde{T}(\tilde{s}))}{\tilde{T}(\tilde{s})}\left(\left.\dfrac{\partial \tilde{P}}{\partial \tilde{x}}\right|_{\tilde{x}=\tilde{s}} + \alpha \dfrac{\tilde{P}^*(\tilde{T}(\tilde{s}))}{\tilde{T}(\tilde{s})}\right), & \tilde{t} > 0, \\ \tilde{s}(0) = \tilde{s}_o. \end{cases} \tag{23}$$

Recalling now (1), we can easily rewrite the problem (23) in the "temperature-time domain". We remark that $T = 1$ corresponds to the top of the basin while $1 + \frac{\gamma L}{T_s}$ to the bottom. Indeed, we have

$$\tilde{T} = 1 - \frac{\gamma L}{T_s}\tilde{x}, \quad \Rightarrow \quad \tilde{T} \in \left(1, 1 + \frac{\gamma L}{T_s}\right) \approx (1, 1.07), \tag{24}$$

and define $\tilde{\sigma}(t) = \tilde{T}(\tilde{s})$ as the temperature at which the change of state occurs, i.e.

$$\tilde{\sigma} = 1 - \frac{\gamma L}{T_s}\tilde{s}, \tag{25}$$

problem $(23)_1$–$(23)_3$ can be rewritten as (we omit the "~" for simplicity)

$$\begin{cases} \dfrac{\partial}{\partial T}\left[\dfrac{P}{T}\left(\dfrac{\partial P}{\partial T} - \delta \dfrac{P}{T}\right)\right] = 0, \quad T \in (1, \sigma(t)) \\ P(1) = 1, \\ P(\sigma) = P^*(\sigma), \end{cases} \tag{26}$$

where $\delta = \dfrac{\alpha T_s}{\gamma L} \approx 0.75$, and $P^*(\sigma)$ is given by (19) with $\tilde{T} = \sigma$.

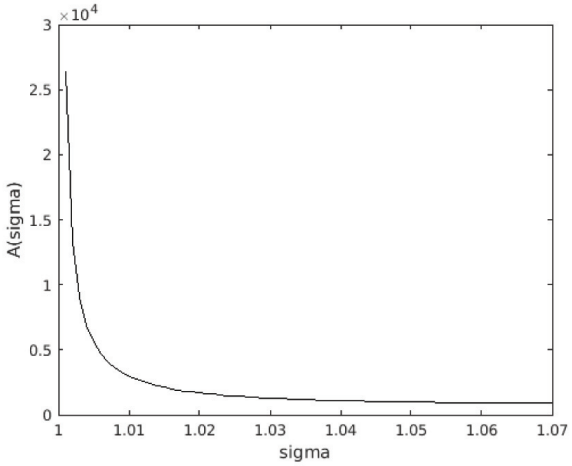

Figure 2. Plot of $A(\sigma)$. The function diverges as $\sigma \to 1$.

The solution to (26) is

$$P(T,\sigma) = \sqrt{\frac{A(\sigma)}{1-\delta}(T^2 - T^{2\delta}) + T^{2\delta}}, \quad 1 \le T \le \sigma, \tag{27}$$

where

$$A(\sigma) = (1-\delta)\frac{P^*(\sigma)^2 - \sigma^{2\delta}}{\sigma^2 - \sigma^{2\delta}}. \tag{28}$$

The behavior of $A(\sigma)$, when $\sigma \in (1, 1.07]$, is shown in Fig. 2.

We note that, for any $\sigma > 1.01$ it is $P^*(T) > P(T,\sigma)$, $1 < T < \sigma$, as required by the compatibility condition i.e. (12). As an example, Fig. 3 displays the vapor pressure and the Clapeyron pressure for $0 < T < \sigma$, with $\sigma = 1.04$.

Taking (27) into account, the free boundary equation becomes

$$\begin{cases} \dot{\sigma} = \left(\dfrac{\gamma L}{T_s}\right)^2 \left[\dfrac{P}{T}\left(\dfrac{\partial P}{\partial T} - \delta\dfrac{P}{T}\right)\right]_{T=\sigma} = \left(\dfrac{\gamma L}{T_s}\right)^2 A(\sigma), \\ \sigma(0) = \sigma_o, \end{cases} \tag{29}$$

where $\sigma_o \in \left(1, 1 + \dfrac{\gamma L}{T_s}\right)$ is the initial temperature of the interface.

Looking at Figure 2 we realize that, if the interface is sufficiently far from the top of the reservoir (where $A(\sigma)$ diverges), we can assume A constant $A = A_0 = A(\sigma_0)$. So, we obtain a linear evolution of the free boundary

$$\sigma(t) = A_0\left(\frac{\gamma L}{T_s}\right)^2 t + \sigma_o, \tag{30}$$

where, for instance, $A_0\left(\dfrac{\gamma L}{T_s}\right)^2 \approx 4.9$ if $A_0 = 10^3$.

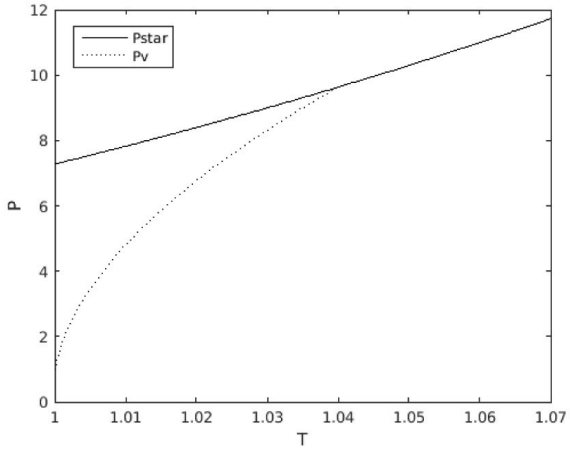

Figure 3. Plot of $P^*(T)$, $P(T,\sigma)$ given by (27), for $1 < T < \sigma$, with $\sigma = 1.04$.

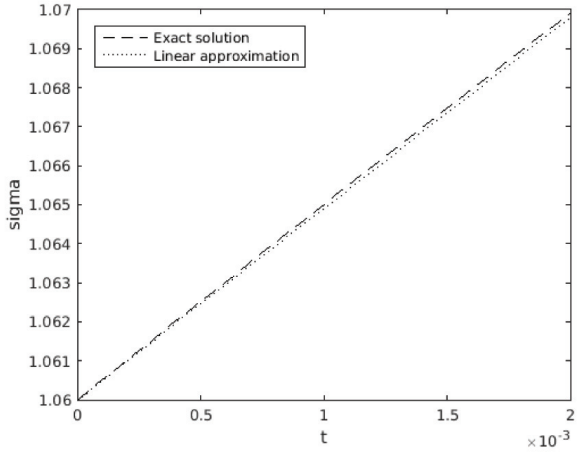

Figure 4. $\sigma(t)$ given by (30) in dotted line considering $\sigma_0 = 1.06$, i.e. $s_0 = -0.8$, and $A_0 = 10^3$. The dashed line corresponds to the solution of (29).

Figure 4 shows the linear approximation (dashed line) given by (30) and the exact solution to problem (29) (dotted line) when $s_0 = -0.8$, namely $\sigma_0 = 1.06$.

Figure 5 corresponds to, $s_0 = -0.13$ (i.e. the interface is close to the top of the basin) namely $\sigma_0 = 1.01$ and $A_0 = 1.25 \times 10^3$. Even in such a configuration, the linear approximation (30) is sufficiently close to the exact solution to (29). The above results highlight that the complete model can be considerably simplified without substantially losing its physical consistency. We refer the readers to [12] where the full problem is numerical solved by means a moving mesh finite elements method.

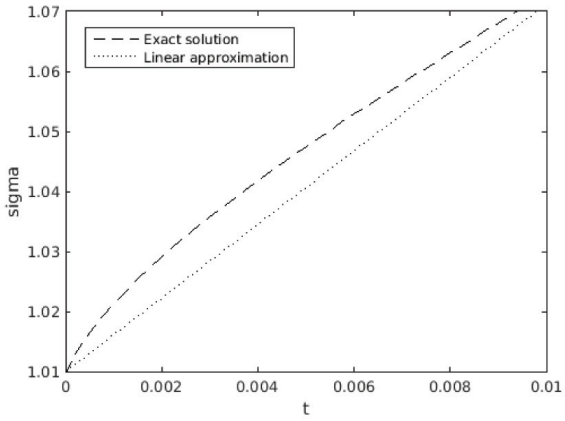

Figure 5. $\sigma(t)$ given by (30) in dashed line when $\sigma_0 = 1.01$, i.e. $s_0 = -0.13$, and $A_0 = 1.25 \times 10^3$. The dotted line corresponds to the solution of (29).

4. Conclusions: Historical evolution of Larderello basin

The results obtained describes qualitatively the historical evolution of the Larderello basin from the water-dominated to a vapor-dominated system. Of course our solution cannot be presented as a quantitatively exact description of the phenomenon that depends on the unknown "initial" conditions.

An important characteristic of the model is the fact that the system evolution, i.e. the motion of the interface away from the top of the reservoir, takes place over a time scale of millennia, whereas the effects of diffusion of the vapor have characteristic times of the order of decades. Such a conclusion agrees with the results of geological studies, despite the drastic simplifications introduced in our model: in particular, 1-D geometry, the absence of chemicals dissolved in water and the absence of recharge.

We note that recharge could be simulated by modifying the boundary condition at the reservoir. Of course, recharge will slow down the motion of the interface, thus enhancing the difference between the two time scales.

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