A rain water infiltration model with unilateral boundary condition

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Abstract

We present a new model for describing rain water infiltration through the soil above the aquifer assuming runoff of the excess water. The main feature of the model lies in the formulation of the boundary condition on the ground surface in the form of a unilateral constraint. The latter allows to estimate, after saturation, the real amount of water that penetrates the soil and the one running off. We present also the results of a numerical study comparing the proposed model with other models which approach in a different way the rain water infiltration problem.

Key words: Rain water infiltration; Unilateral boundary conditions; Unsaturated flow.

1 Introduction

This paper deals with the modelling of the rain water infiltration through the soil above an aquifer, i.e. through the so called vadose zone. One of the difficulties of such a problem is the formulation of the boundary condition on the inflow surface, namely on the ground surface. This aspect is particularly important to evaluate the quantity of water that is drained by the soil and the quantity that runs off.

Although modelling real situations often requires taking into account the full 3–D problem, possibly including inhomogeneities and anisotropy (the reader is referred to [1], [2], [3], [4], [5], [6], [7] and [8] and the literature quoted therein),

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we decide to confine the presentation to a one-dimensional model in planar symmetry in order to make more clear the discussion on boundary conditions.

We consider the case in which at time $t = 0$ (the rain starting time) the vadose zone is not saturated and we model percolation in time interval $(0, T)$ during which the rainfall event occurs. In particular we assume that $T$ is much smaller than the characteristic evolution time of the aquifer, so that its depth remain constant.

Let $N(t)$ represents the rate at which the rain falls on the unit area of the ground. Initially all rain water infiltrates into the soil, but if the rainfall event is particularly intense (as in the examples we will consider) a time $t_s \in (0, T)$ exists such that the maximum draining capacity of the soil is exceeded. For $t > t_s$ the model should assume:

(i) Either storage of the excess water above the surface until it can penetrate the soil.
(ii) Or runoff, so that all the unabsorbed water is instantaneously removed.
(iii) Or, finally, an intermediate situation between (i) and (ii).

We will mainly focus our attention on (ii) and we will present a new model in which the boundary condition on the ground surface is such that:

- As long as the ground surface is unsaturated, the water influx is equal to $N(t)$ (i.e. all rain water impinging on the ground surface penetrates the soil).
- If the above condition is violated, we prescribe that the pressure equals the saturation value.

The mathematical model emerging from this scheme is a unilateral boundary condition (UBC) or a Signorini-type boundary condition.

Leaving to a forthcoming paper the mathematical analysis of the problem (existence, uniqueness and properties of the solution) we confine ourselves to numerical simulations, discussion of results and comparison with other frequently used models. We also put in evidence that our model provides the correct determination of $t_s$. Moreover, we show that inaccurate models under this respect may lead to the violation of the obvious constraint that the surface flux cannot exceed that which is really available, i.e. the rainfall rate.

The paper develops as follows. In Section 2 we briefly recall the basic equation of hydrology (Richards’ equation) and formulate the physical assumptions. Section 3 is devoted to presentation and discussion of the most widely used models and of (UBC). Section 4 contains the numerical simulation we did for some model cases as well as the discussion of the results. In Section 5 we present, for sake of completeness, other models in which only partial or no
runoff is assumed to occur. Section 6 is finally devoted to some information on the numerical method we used.

2 Basic equations, physical assumptions and selection of the hydraulic functions

We consider a 1–D Darcian flow driven by gravity in an unsaturated porous medium. We call $z$ the vertical coordinate, setting $z = 0$ on the water table or phreatic surface and $z = Z$ on the ground surface. We assume that the aquifer depth remains constant and we suppose that the soil above the aquifer is homogeneous.

The flow is described by the Richards’ equation which, in the chosen coordinates, reads as (see [1], [2], [3] for more details)

$$\frac{\partial \theta(\psi)}{\partial t} = \frac{\partial}{\partial z} \left[ K(\psi) \left( \frac{\partial \psi}{\partial z} + 1 \right) \right],$$

(2.1)

where:

- $\psi$ is the suction or soil water pressure head.
- $\theta$ is the volumetric water content, or moisture content, and $\theta(\psi)$ is a smooth function (retention curve) such that:
  1. $\theta(0) = \theta_s$ and $\lim_{\psi \to -\infty} \theta(\psi) = \theta_r$ (notice that $\psi = 0$ represents the saturation pressure head), where $\theta_s$ denotes the soil porosity (saturated water content) and $\theta_r$ the residual water content.
  2. $\frac{d\theta}{d\psi} > 0$, for $-\infty < \psi < 0$.

- $K$ is the hydraulic conductivity. It is assumed that $K(\psi)$ is a smooth function such that:
  1. $\lim_{\psi \to -\infty} K(\psi) = 0$.
  2. $\frac{dK}{d\psi} > 0$.
  3. $K(\psi) = K_{sat}$ for $\psi \geq 0$, where $K_{sat} = K(0)$ is the saturated hydraulic conductivity.

As well-known, $\psi$ is an experimentally determined function of the moisture content $\theta$ (for instance we refer the reader to [9], [10], [11], [12], [13], [14]).
We stipulate that $\psi$ is a single-valued invertible function of $\theta$, namely

$$\psi = \psi(\theta) \iff \theta = \theta(\psi).$$  \hfill (2.2)

Different forms of the retention curve $\theta(\psi)$ can be chosen (depending on numerical and experimental convenience, see [12], [13] and [14] for instance). However, defining the water capacity

$$C(\psi) = \frac{d\theta}{d\psi}(\psi),$$

the sign of $C(\psi)$ for $\psi = 0$ assumes particular relevance, since it modifies the mathematical properties of Richards’ equation. Indeed, there are two distinct possibilities: $C(0) > 0$ and $C(0) = 0$ (i.e. $C$ continuous for $\psi = 0$, since $C(\psi) \equiv 0$ for $\psi > 0$). In the first case, Richards’ equation is uniformly parabolic, whereas in the second case the equation degenerates when $\psi = 0$. In the sequel we shall refer to degenerate case and degenerate curve if $C(0) = 0$ and to non-degenerate case and non-degenerate curve if $C(0) > 0$.

In Section 4 we will make numerical simulations to compare different forms of the boundary conditions on $z = Z$. For a more complete analysis we performed computations in both cases $C(0) > 0$ and $C(0) = 0$.

The degenerate $\theta(\psi)$ which we consider is the so-called Van Genuchten curve [15]

$$\theta(\psi) = (\theta_s - \theta_r) S(\psi) + \theta_r,$$  \hfill (2.3)

with

$$S(\psi) = \left\{ \frac{1}{1 + (-\mu \psi)^n} \right\}^m,$$  \hfill (2.4)

where $n$ ($n > 1$) and $\mu$ are experimental shape parameters and $m = 1 - 1/n$. $S(\psi)$ is also called effective saturation.

To simulate a non-degenerate case we take

$$\theta(\psi) = \frac{\theta_s}{1 - \tanh(ab) + \varepsilon} \left[ 1 + \tanh(a(\psi - b)) + \varepsilon \right],$$  \hfill (2.5)

where $\varepsilon$ is chosen so that $\lim_{\psi \to -\infty} \theta(\psi) = \theta_r$, i.e.

$$\varepsilon = \frac{\theta_r (1 - \tanh(ab))}{\theta_s - \theta_r},$$  \hfill (2.6)

and $a$, $b$ are numerical parameters that were taken as in Tab. 1, so that forms (2.3) and (2.5) do not differ significantly (see Fig.1-B ) in the range $-10 \leq \psi \leq 0$, which is relevant in our simulations.

\footnote{More generally hysteresis in wetting and drying cycles produces an extremely complex relationship between $\psi$ and $\theta$.}
Concerning the hydraulic conductivity, the shape of $K(\psi)$ used in the simulations has been derived on the basis of Mualem’s theory [16], i.e.

$$K(\psi) = K(S(\psi)) = K_{sat}\sqrt{S(\psi)} [1 - (1 - S(\psi)^{1/m})^2]^{m/2},$$

where $m$ is the coefficient introduced in (2.4).

In Tab.1 we have reported the soil data used in the simulations we have performed (see [12] for an accurate description of the soil texture parameters). We remark that we have chosen a soil with a pretty low hydraulic conductivity and a rather high porosity, in order to outline the differences between the models we will compare.

Of course (2.1) has to be supplemented with initial and boundary conditions. As an initial condition we give $\theta(z,0)$ for $z \in (0, Z)$. We can suppose, for instance, to assign the moisture content corresponding to a stationary solution in which capillary and hydraulic pressure balance each other (see Fig. 1-A)

$$\psi(z,0) = -z.$$  

Concerning the boundary condition on $z = 0$, we will have

$$\psi(0,t) = 0,$$

since it is the location of the phreatic surface. The more delicate point is the boundary condition on the soil surface $z = Z$ that will be discussed in the next section.

3 Discussing boundary condition at $z = Z$

We are assuming to know the rainfall rate $N(t)$ and we also suppose that no overpressure is applied on the soil surface (in Section 5 we briefly discuss cases in which this assumption is no longer made). Consequently, the following two conditions are to be satisfied

$$K(\psi(Z,t)) [\psi_z(Z,t) + 1] \leq N(t), \ t \geq 0,$$

where

<table>
<thead>
<tr>
<th>Texture</th>
<th>$\theta_s$</th>
<th>$\theta_r$</th>
<th>$K_{sat}$ (m/s)</th>
<th>$\mu$ (m$^{-1}$)</th>
<th>$n$</th>
<th>$a$ (m$^{-1}$)</th>
<th>$b$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sandy clay loam</td>
<td>0.390</td>
<td>0.100</td>
<td>3.64x10$^{-6}$</td>
<td>0.059</td>
<td>1.48</td>
<td>0.083</td>
<td>-20.063</td>
</tr>
</tbody>
</table>

Table 1
Soil sample data from [12].
The first one simply states that water drained by the soil cannot exceed the rainfall, while the second one guarantees the absence of overpressure; moreover, recalling the maximum principle for parabolic equations, we have

\[
\psi(z, t) \leq 0, \quad \text{for } 0 \leq z \leq Z \text{ and } t \geq 0.
\] (3.3)

From a physical point of view the conditions above mean that all excess water runs off.

**Remark 1** Assume that \( N(t) < K_{\text{sat}} \) in some interval \([0, T)\). It easy to check (for a proof see Appendix A) that imposing

\[
K(\psi(Z, t)) \left[ \psi_z(Z, t) + 1 \right] = N(t),
\] (3.4)

automatically yields \( \psi(z, t) < 0 \) in \([0, T)\). Thus, if \( N(t) < K_{\text{sat}} \) \( (3.1) \) implies \( (3.2) \).

**Remark 2** Any stationary solution \( \tilde{\psi}(z) \) of \((2.1)\) has to satisfy

\[
K(\tilde{\psi}) \left[ \tilde{\psi}'(z) + 1 \right] = \Phi_0,
\]

where \( \Phi_0 \) is the constant flux flowing through the vadose zone. Imposing \( \tilde{\psi}(0) = 0 \), we have

\[
K_{\text{sat}} \left[ \tilde{\psi}'(0) + 1 \right] = \Phi_0,
\]

and since \( \tilde{\psi}'(0) \leq 0 \), the maximum stationary flux is \( K_{\text{sat}} \) that corresponds to a fully saturated vadose zone \( (\tilde{\psi}(z) \equiv 0) \).

Starting from the facts illustrated in the two remarks above and from the experimental evidence that a given soil cannot drain an arbitrary rainfall rate \( N(t) \), it is generally assumed that the boundary condition on \( z = Z \) is given by \( (3.4) \) only if \( N(t) \leq K_{\text{sat}} \) and the following formulations have been proposed:

(i) **Hedged Boundary Condition** (HBC)

\[
K(\psi) \left[ \psi_z(Z, t) + 1 \right] = \min \{ N(t), K_{\text{sat}} \}.
\] (3.5)

(ii) **Switched Boundary Condition** (SBC)

\[
\begin{cases}
K(\psi) \left[ \psi_z(Z, t) + 1 \right] = N(t), & \text{if } N(t) \leq K_{\text{sat}}, \\
\psi(Z, t) = 0, & \text{if } N(t) > K_{\text{sat}}.
\end{cases}
\] (3.6)

Concerning (i), according to Remark 1, it is clear that the two constraints \( (3.1), (3.2) \) are satisfied, but the choice of cutting \( N(t) \) at the level \( K_{\text{sat}} \) cannot
be motivated by the need of satisfying (3.2). Indeed, assume (2.8) - (2.9) and take \( N(t) = N_0 > K_{sat} \). For any \( N_0 \) there exists a time interval \([0, T_0]\) of a finite width (that can be estimated from below, see Appendix B) such that \( \psi(z, t) < 0 \) in \((0, Z) \times [0, T_0]\). Therefore choice (3.5) seems arbitrarily too conservative.

Concerning SBC, in the example above the switch would start from \( t = 0 \) so that \( \psi_z(z, t) \) would be unbounded and in any case there is no control on its value guaranteeing that (3.1) is fulfilled. In other words, imposing SBC implies that the inflow at \( z = Z \) can exceed the rainfall rate. This has been proved analytically in [17] for arbitrary \( N(t) \) and in the non-degenerate case. Indeed, it is shown that in that case both \( \psi(Z, t) \) and \( \psi_z(Z, t) \) are allowed to be discontinuous and the boundary point principle is contradicted.

3.1 The UBC Model

We stipulate that the correct boundary condition has to be such that not only (3.1) and (3.2) have to be satisfied, but also that (at least) one of the two has to hold as an equality, i.e.

\[
\begin{align*}
K(\psi(Z, t)) \{ \psi_z(Z, t) + 1 \} & \leq N(t), \psi(Z, t) \leq 0, \\
\psi(Z, t) \{ K(\psi(Z, t)) \{ \psi_z(Z, t) + 1 \} - N(t) \} &= 0.
\end{align*}
\]

(3.7)

This is a typical unilateral boundary condition (UBC) also known as Signorini-type condition, which is frequently encountered in continuum mechanics in the framework of the unilateral constraints (see e.g. [18]).

As suggested in [19], it is possible to give (3.7) in a mathematically compact form introducing the Heaviside graph

\[
H(\psi) = \begin{cases} 
0 & \text{if } \psi < 0, \\
[0, 1] & \text{if } \psi = 0, \\
1 & \text{if } \psi > 0,
\end{cases}
\]

(3.8)

and writing (3.7) as an inclusion, namely

\[
K(\psi(Z, t)) \{ \psi_z(Z, t) + 1 \} \in N(t)H(-\psi(Z, t)).
\]

(3.9)

It might be conjectured that since (3.5) and (3.6) look much simpler than (3.9), they are so frequently used, although from the point of view of numerical computation the complexity of (3.9) is more apparent than real.
Now, we present the results of the numerical simulations we have performed. We took $Z = 10 \text{ m}$ and we imposed conditions (2.8) and (2.9).

We considered three examples corresponding to three rainfall situations and we computed the numerical solutions corresponding to (SBC), (HBC) and (UBC).

**Example 1.a.** We start by comparing the solutions of the three models in the case

$$N(t) = K_{\text{sat}}(1 + \alpha), \quad \alpha = 0.1.$$  \hspace{1cm} (4.1)

There is a meaningful difference between modelling the evolution of the water content in the vadose zone by SBC and UBC (see Fig.2). In particular, since in SBC the top boundary condition is controlled by the rainfall rate $N(t)$, it is evident that here the value of $\psi$ is always forced to 0, i.e. $\psi(Z, t) \equiv 0 \quad \forall t \geq 0$. Conversely, by UBC we see that the saturation of the ground surface is reached at $t = 8\text{h}50\text{m}$. This considerable gap strongly affects the shape of the solution.

In Fig. 3 we have compared the infiltration rates at $z = Z$ computed by SBC and UBC with the constant rate $K_{\text{sat}}$ predicted by HBC and the rainfall rate (4.1). It is interesting to notice that according to UBC the influx is equal to $N(t)$ until the top surface saturates. Thus, introducing the runoff rate

$$R(t) = N(t) - K(\psi(Z, t))[\psi(Z, t) + 1],$$  \hspace{1cm} (4.2)

we get three different evaluations of this quantity (in SBC the runoff is “negative”!).

**Example 1.b.** The same computation has been carried out for the rainfall rate

$$N(t) = K_{\text{sat}}(1 + \alpha), \quad \alpha = 0.5.$$  \hspace{1cm} (4.3)

In this case, UBC predicts that the saturation of the top surface is reached at $t = 4\text{h}43\text{m}$. Although the contradictions above mentioned still occur, we can observe that in this example SBC is a better approximation than HBC to the solution UBC (see Fig. 4 and 5).

**Example 2.** Next we show the results of the same simulations for a model rainfall event, whose rate is shown in Fig. 6. The results are shown in Fig.s
We can conclude this section by noting that as far as the rainfall rate $N(t)$ exceeds $K_{sat}$ for small time intervals, the differences among the three models are smoothed out over reasonable time scales, so that assuming SBC or HBC is an acceptable approximation.

As a final remark, we add that considering the degenerate case instead of the non-degenerate one produces essentially the same results (see Fig 10).

5 Models with reduced runoff

So far we have assumed that if the impinging rain exceeds the water that can be drained by the soil, the excess quantity runs off.

Conversely, if we assume that all the excess water accumulates on the surface and forms a pond, we will clearly have that its thickness is given by

$$\Omega(t) = \max \left\{ 0, \int_0^t R(\tau) d\tau \right\}, \quad (5.1)$$

with $R(t)$ defined by (4.2).

For those $t$ for which $\Omega(t) > 0$ the pressure at $z = Z$ will be the hydrostatic pressure due to the pond

$$\psi(Z, t) = \Omega(t), \text{ whenever } \Omega(t) > 0. \quad (5.2)$$

Now, constraint (3.1) becomes

$$\psi(Z, t) \leq \Omega(t), \quad (5.3)$$

and constraint (3.2) is active only when $\Omega(t) = 0$. Consequently, the boundary condition can be written as

$$K(\psi) [\psi_2(Z, t) + 1] \in N(t) \cdot H(-\psi) + \left[ N(t) - \dot{\Omega}(t) \right] \cdot \Gamma(\psi), \quad (5.4)$$

where $\Gamma$ is the following function

$$\Gamma(\psi) = \begin{cases} 
1, & \text{if } \psi > 0, \\
0, & \text{if } \psi \leq 0,
\end{cases}$$
and $H$ is the Heaviside graph defined in (3.8).

A problem of this kind (with $C(0) > 0$) has been considered in [20] in the case where $\Omega(t) > 0 \forall t$, reducing it to an elliptic-parabolic free boundary problem. Of course, intermediate cases between total ponding and total runoff can be considered introducing an appropriate partition ratio in front of the integral defining $\Omega$.

In [4], in the framework of a very general model, which is fully analyzed also from a pure mathematical point of view, ponding is incorporated in the boundary condition neglecting the hydrostatic pressure caused by the pond. Since the latter is very low in practical cases, this model is definitely the closest to our approach.

For sake of completeness we also quote the model [17] in which the transfer of momentum from rain drops to the water saturating the soil surface is considered. According to this picture the constraint on $\psi$ becomes

$$\psi(Z, t) \leq \Psi_{\text{rain}}(t) = \frac{v}{g}N(t),$$

where $v$ is the averaged rain drops velocity and $g$ is the gravity acceleration (see [17] for more details).

The mathematical formulation of this case follows the same pattern as in Section 3. In particular, such a model was studied considering the non-degenerate case with the aim of analyzing the evolution of the free boundary $z = s(t)$ separating the saturated region $s(t) < z < Z$ where $\psi$ is positive (and linear), from the unsaturated region $0 < z < s(t)$ where $\psi < 0$. Computations show that the thickness of this region is negligible in most practical cases.

6 Numerical methods for solving UBC model

The numerical treatment of the Richards’ equation has been extensively investigated (see [21] and [22] for instance). Nevertheless, to deal with the unilateral boundary condition in UBC needs some modifications of the usual numerical scheme.

For describing this procedure, it is useful to rewrite UBC as

$$K(\psi)[\psi_z(Z, t) + 1] = N(t), \text{ if } \psi(Z, t) < 0,$$  \hspace{1cm} (6.1)

$$K(\psi)[\psi_z(Z, t) + 1] \leq N(t), \text{ if } \psi(Z, t) = 0.$$  \hspace{1cm} (6.2)
To solve the Richards’ equation we first use a standard finite difference discretization in space (see [23] for instance). We divide $[0, Z]$ in $(N-1)$ intervals $\{[z_i, z_{i+1}]\}_{i=1}^{N}$ of length $\Delta z$. Approximating the r.h.s. of (2.1) gives the nonlinear system of ODEs

$$
G(\Psi) \frac{d\Psi}{dt} = A(\Psi) \Psi + d(\Psi),
$$

(6.3)

where

- $\Psi = \{\psi(z_i, t)\}_{i=1,...,N}$. The $i^{th}$ component of the vector $\Psi$ is denoted by $(\Psi)_i$.
- $G(\Psi)$ is a diagonal matrix defined as $[G(\Psi)]_{i,i} = C(\psi(z_i, t))$.
- $A(\Psi)$ is a $N \times N$ tridiagonal matrix.
- $d(\Psi)$ is a vector of length $N$.

To solve system (6.3) we apply a fully implicit scheme in time, namely

$$
G(\Psi^{j+1}) \frac{\Psi^{j+1} - \Psi^j}{\Delta t} = A(\Psi^{j+1}) \Psi^{j+1} + d(\Psi^{j+1}),
$$

(6.4)

obtaining for the unknown vector

$$\Psi^{j+1} = [\psi(z_1, t_{j+1}), \psi(z_2, t_{j+1}), \ldots, \psi(z_N, t_{j+1})],$$

the following nonlinear system

$$
\left[ G(\Psi^{j+1}) \frac{1}{\Delta t} - A(\Psi^{j+1}) \right] \Psi^{j+1} - d(\Psi^{j+1}) - G(\Psi^{j+1}) \frac{\Psi^j}{\Delta t} = 0,
$$

(6.5)

which in turn is solved by a Newton iteration scheme.

Notice that 1\textsuperscript{st} and N\textsuperscript{th} equations of the system (6.5) correspond to the boundary conditions on $z = 0$ and $z = Z$, respectively.

The Dirichlet boundary condition on $z = 0$ presents no difficulty. Handling the unilateral boundary condition on $z = Z$ is more involved. We proceed as follows (see [24]):

1. Since we assumed $\psi_0(Z) < 0$, \( \Psi \) (i.e. $\Psi$ at the second time step) is computed using the flux boundary condition (6.1).
2. For $j \geq 2$, $\Psi^{j+1}$ is determined analyzing the sign of $(\Psi^j)_N$. We have:
   2.1 If $(\Psi^j)_N \leq 0$, then $\Psi^{j+1}$ is computed considering the flux condition (6.1).
   2.2 Else, we compute the flux

$$
F = K \left( (\Psi^j)_N \right) \left[ \frac{(\Psi^j)_N - (\Psi^j)_{N-1}}{\Delta z} + 1 \right].
$$
2.2.1 If $F \leq N(t_j)$ then we choose condition (6.2).

2.2.2 Else (i.e. if the constraint in (6.2) is violated) we use the flux condition (6.1).

3 Once $\Psi^{j+1}$ is computed we check that the constraints in (6.1)-(6.2) are satisfied by $(\Psi^{j+1})_N$. If one of those is violated then we repeat the procedure of point 2 choosing the previously not used boundary condition.

Such an algorithm presents the advantage that, requiring a double check on the boundary conditions set, avoids possible violations of the constraints.

7 Conclusions

We presented a new model (UBC) for the rain water percolation through the vadose zone when a rainfall event of particular high intensity occurs, in the sense that the ground surface comes to saturation in a finite time $t_s$. So, assuming runoff, for $t > t_s$ the boundary condition imposing that the entering water flux equals the rain water intensity is in general no more valid.

Our model is focused on a unilateral boundary condition on $z = Z$, guaranteeing an exact evaluation of $t_s$ and satisfying the obvious constraint that entering flux cannot exceed the one which is really available (rainfall rate flux). Indeed, in other models commonly used, incorrect estimation of $t_s$ (assumed to coincide with the time such that $N(t) = K_{sat}$) lead to a possible physical inconsistency.

We reported the results of numerical simulations in some model cases and compared UBC with other models. The differences are shown to be independent of the analytical properties of the retention curve (the results coincide in the degenerate and non-degenerate cases) but are of course strongly influenced by the soil texture.

Although for simplicity we have worked in planar geometry (the extension to 3-D can be done with minimal changes) we believe that we have focused on an important aspect of rain water percolation, proposing a rational methodology for describing how rain water infiltrates through the soil.
A Analytical properties

As mentioned in Remark 1, we consider the following problem for the Richards’
equation

\[ C(\psi) \psi_t = [K(\psi)(\psi_z + 1)]_z, \quad 0 < z < Z, \ t > 0, \]  
\[ \psi(z, 0) = \psi_0(z), \quad 0 \leq z \leq Z, \]  
\[ \psi(0, t) = 0, \quad t \geq 0, \]  
\[ [K(\psi)(\psi_z + 1)]_z = N(t), \quad t \geq 0, \]  

(A.1) (A.2) (A.3) (A.4)

with \( N(t) < K_{sat} \), and assume the existence of a classical solution \( \psi(z, t) \) of
the above problem.

Let \((z^*, t^*)\) such that \( \psi(z^*, t^*) > 0 \), the maximum principle entails

\[ z^* = Z \text{ and } \psi_z(Z, t^*) \geq 0. \]  

(A.5)

But, recalling that \( N(t) < K_{sat} \), we have

\[ \psi_z(Z, t^*) = \frac{N(t^*)}{K(\psi(Z, t^*))} - 1 = \frac{N(t^*)}{K_{sat}} - 1 < 0, \]

which contradicts (A.5). It follows that such a point \((z^*, t^*)\) cannot exist and so

\[ \psi(z, t) \leq 0, \quad 0 \leq z \leq Z, t \geq 0. \]

We also note that if in (A.4) the “=” sign is replaced by “\(\leq\)” the result holds
a fortiori.

\[ \square \]

B An estimate for the saturation time

Let us consider problem (A.1)-(A.4) with \( N(t) = N_0 > K_{sat} \) and assume the
following properties:

(i) \( C(0) > 0 \), so that \( C(\psi) > 0 \) for all \( \psi \).

(iii) \( \psi_0(z) = -z \).

(iii) \( N_0 = K_{sat}(1 + \alpha) \), where \( 0 < \alpha < 1 \).

Notice that we introduce (iii) in order to deal with a simpler case (i.e. the
constant rainfall rate \( N_0 > K_{sat} \) is not arbitrary, as mentioned in Remark 2).
Nevertheless, with simple additional work the restriction \( 0 < \alpha < 1 \) can be avoided.
Now, we prove the following

**Proposition 1** Suppose that assumptions (i)-(iii) are fulfilled. Then, there exists a time \( T_0 > 0 \) such that the (classical) solution of problem (A.1)-(A.4) satisfies

\[
\psi(z, t) < 0, \quad \forall (z, t) \in (0, Z] \times [0, T_0].
\]

**Proof.** We confine ourselves to the case

\[
K(\psi) = K_{sat},
\]

although the proof can be worked out also in the general case (see [17] and [25]). Therefore, (A.1)-(A.4) rewrite as the following problem

\[
\begin{align*}
C (\psi) \psi_t &= K_{sat} \psi_{zz}, & 0 < z < Z, \ t > 0, \\
\psi(z, 0) &= -z, & 0 \leq z \leq Z, \\
\psi(0, t) &= 0, & t \geq 0, \\
\psi_z(Z, t) &= \alpha, & t \geq 0,
\end{align*}
\]

which has a unique classical solution \( \psi \) (see [26]). Thus, we can consider it as a linear problem by reading \( C(\psi) \) as a function of \((z, t)\), i.e. \( C(z, t) = C(\psi(z, t)) \).

Now, we are looking for a function \( \omega(z, t) \) such that both following properties hold true

\[
\begin{align*}
\psi(z, t) &\leq \omega(z, t), \quad \forall z \in [0, Z], \ t \geq 0, \\
\omega(Z, t) &< 0, \quad \forall t \in [0, T_0],
\end{align*}
\]

with \( T_0 \) to be determined later.

We define \( \omega(z, t) \) as

\[
\omega(z, t) = az^2 - z + \lambda t,
\]

with

\[
a = \frac{\alpha + 1}{2Z} \quad \text{and} \quad \lambda = \frac{K_{sat} (\alpha + 1)}{ZC_{\min}},
\]

being \( C_{\min} = \min_{z \in (0, Z), \ t > 0} C(\psi(z, t)) \), which is surely positive because of (ii).

Introducing

\[
\phi(z, t) = \omega(z, t) - \psi(z, t),
\]

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we have that $\phi$ satisfies the problem

$$\begin{aligned}
K_{\text{sat}} \phi_{zz} - C(z,t) \phi_t &= 2aK_{\text{sat}} - C(z,t) \lambda = K_{\text{sat}} \frac{\alpha + 1}{2Z} \left( 1 - \frac{C(z,t)}{C_{\min}} \right) < 0, \\
\phi(z,0) &> 0, \\
\phi(0,t) &\geq 0, \\
\phi_z(Z,t) &= 0.
\end{aligned}$$

Applying the maximum principle and the parabolic version of Hopf’s Lemma (see [26] for instance), implies $\phi(z,t) \geq 0$, i.e. (B.7). On the other hand

$$\omega(Z,t) = Z \left( \frac{\alpha - 1}{2} \right) + \left[ K_{\text{sat}}(\alpha + 1) \right] t,$$

and so for

$$t < T_0 = \frac{Z^2 (1 - \alpha) C_{\min}}{2K_{\text{sat}}(\alpha + 1)},$$

we have $\omega(Z,t) < 0$, i.e. (B.8). But, recalling that $\psi$ can take its maximum only on $z = Z$, it is easy to check that property (B.1) follows by (B.7) and (B.8) (notice that assumption $0 < \alpha < 1$ guarantees $T_0 > 0$).

□
References


Fig. 1. (A) Initial suction $\psi_0$ used in the simulations: $\psi_0(z) = -z$. Notice that $-10 \text{ m} \leq \psi \leq 0$. (B) Initial moisture content $\theta_0(z) = \theta(\psi_0(z))$ according to (2.3) and (2.5).
Fig. 2. Example 1.a: evolution of the water content $\theta(z, t)$ versus the soil depth (in m). The assumed rainfall rate is $N(t) = K_{\text{sat}}(1 + \alpha)$, with $\alpha = 0.1$, i.e. $N(t) = 14.3$ mm/h. Concerning $K_{\text{sat}}$ see Tab. 1.
Fig. 3. Example 1.a: comparative plot between different infiltration rates $I(t) = K(\psi(Z,t)) [\psi_z(Z,t) + 1]$. The rainfall rate is given by (4.1) with $\alpha = 0.1$, i.e. $N(t) = 14.3$ mm/h. Recall $K_{sat} = 13$ mm/h.
Fig. 4. Example 1.b: evolution of the water content $\theta(z,t)$ versus the soil depth (in m). The assumed rainfall rate is $N(t) = K_{sat}(1+\alpha)$, with $\alpha = 0.5$, i.e. $N(t) = 19.5$ mm/h.
Fig. 5. Example 1.b: comparative plot between different infiltration rates $I(t) = K(\psi(Z,t))|\psi_z(Z,t) + 1|$. The rainfall rate is given by (4.3) with $\alpha = 0.5$, i.e. $N(t) = 19.5 \text{ mm/h}$. Recall $K_{sat} = 13 \text{ mm/h}$.
Fig. 6. Rainfall Rate $N(t)$ in mm/h of Example 2. The maximum value of $N(t)$ is 50 mm/h.
Fig. 7. Example 2: evolution of the water content $\theta(z, t)$ versus the soil depth (in m). The assumed rainfall rate is reported in Fig. 6.
Fig. 8. Example 2: comparative plot between the rainfall rate $N(t)$ and the infiltration rate $I(t) = K(\psi(Z,t)) [\psi_z(Z,t) + 1]$. Notice that water flux penetrating through the soil evaluated by UBC is always less or equal than $N(t)$. 

Infiltration Rate (mm/h)

Time (h)
Fig. 9. Example 2: runoff rate $R(t) = N(t) - K(\psi(Z,t))[\psi_z(Z,t) + 1]$. $N(t)$ is the one reported in Fig. 6. Notice that HBC (diamonds) overestimates $R(t)$ with respect to UBC (continuous line).
Fig. 10. Evolution of the water content $\theta(z, t)$ versus the soil depth (in m): comparison between degenerate and non-degenerate case.