Dynamics of droplets in an agitated dispersion with multiple breakage. Part I: formulation of the model and physical consistency

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Abstract. In [7] a new model for the evolution of a system of droplets dispersed in an agitated liquid was presented, with the inclusion of the so–called *volume scattering effect* (a combination of coalescence and breakage). In that paper droplets breakage was considered to be binary, in order to simplify exposition. Here we remove that limitation, investigating the effect of each breakage mode and of scattering with multiple exits. We also allow the breakage kernel, at each mode, to become singular when droplets approach their *finite* maximum admissible size.

1. Introduction

A system of two immiscible liquids agitated in a batch under the action of impellers gives rise to a set of droplets of one phase dispersed in the other phase. The resulting system is called a *dispersion* (finer dispersions are called *emulsions*) and its evolution is caused by the fact that droplets during their motion may break up in two or more smaller droplets or they may coalesce (an essentially binary process), producing larger elements. Dispersions are commonly encountered food industry, cosmetics, pharmacology, photography and many others industrial processes . This justifies the large amount of scientific papers devoted to this subject during the last century (see, for example, [1,2,4] for the main relevant literature). However, many basic questions are still pending so that research is still very active in this area (see, for example, [5,8,12,14,15,10]).

Dealing with the specific case of the batch reactor, it is commonly assumed that spatial homogeneity is achieved, so that the droplet system is described by a volume distribution function f, so that $f(v,t)\ \mathrm{d}v$ represents the number of droplets having volume in the interval

$(v, v + \mathrm{d}v)$ at time t, per unit volume of dispersion. When we come to the question of describing the evolution of f, there are essentially two kinds of difficulties somehow related to each other: the first is that the main processes influencing the evolution of f, namely *coalescence* and *breakage*, are indeed complex phenomena, not completely understood (particularly at high rotational speeds) and for which various descriptions have been proposed in the experimental literature. Experiments and observations are also very delicate so possible insights have to be taken very carefully. However – and this is mainly the second basic difficulty – the proposed mathematical models seem to require further refinements. Indeed a typical feature usually adopted in the construction of the mathematical model is that v is allowed to take any positive value: this is clearly meaningless from the physical point of view, but writing an evolution model is generally much simpler if v ranges over $(0, +\infty)$. On the other hand it is true that coalescence tends naturally to produce *large* droplets. However, it is not a controversial point that (for a given agitation speed of the mixture) the maximum observable size of droplets v_m is *finite* (see *e.g.* [17]). In the mathematical literature this aspect of the problem is generally underestimated (if not ignored) or at most by–passed in some artificial way, allowing v to go to infinity.

Some years ago we proposed a model (see [6,7]) for the dynamics of droplets including a constraint on the droplet size in a consistent way and pointing out that this requires the presence of a third physical mechanism in the evolution of the system, called *volume scattering*. This effect consists in the immediate decay by rupture of a parent droplet resulting from coalescence and exceeding the threshold value v_m, so that all daughters remain in the allowed size range. Scattering is represented by a specific operator (with gain and loss terms) in the balance equation which adds to classical coalescence and breakage operators. The main advantage of this approach is that it is based on natural assumptions, reflecting the real physics. Also the mathematics appears to be simpler since subtle questions regarding summability in unbounded domains are automatically eliminated. The resulting model consists in an initial value problem for a Boltzmann–like equation for the function $f(v, t)$.

Our first contribution dealt with the somehow artificial case of *binary* rupture events: this means that any parent droplet involved within breakage or scattering produces exactly *two* daughters.

In [6,7] we proved that the problem is well posed under rather general hypotheses and with a *bounded* fragmentation kernel. The extension to the unbounded case was subsequently worked out in [3].

Here we show a possible approach to remove the main limitation that we put in [6,7], namely the hypothesis of binary ruptures with the aim of determining the influence of each breakage mode. Such a "fine structure" of breakage has not been incorporated in any previous model (at our knowledge), possibly because of its mostly theoretical interest and its high degree of complexity. We notice that, since the scattering operator involves a breakage event, it must be modified accordingly in order to allow volume scattering with *multiple exits*.

Indeed multiple breakage is usually considered (see e. g. [1,2,9,4,11,13,20] and many other references therein) but with the philosophy of capturing a global information about breakage, in view of the difficulty of analyzing the single modes. Such a global approach has also been introduced in the model with volume scattering, along with a new breakage mechanism, the

so–called *shattering* (see [16,20]). The goal we are pursuing here is instead to emphasize the contribution of each breakage class to the rate of change of the distribution function. Accordingly, the probability functions describing each rupture mode appear explicitly in the model. In such a way it becomes possible, at least in principle, to investigate the contribution to f due to each mode *individually*. In the present paper we show how to deal with the probability functions and we also prove the physical consistency of the model. We note explicitly that the breakage frequency α_k of the $k-$th mode is allowed to tend to infinity as v tends to v_m (as in [3]).

The proof of the well posedness of the Cauchy problem together with explicit examples will be presented subsequently (see Part II in this same volume).

2. Mathematical model

We assume droplets to be uniformly distributed in the reactor so that f does not depend on spatial coordinates. We also assume that the system is isolated, so that there is no heat or mass exchange. We assume that there exits a finite upper bound of the breakage mode, that is that drops may break at most in N pieces. Nevertheless N can be as large as we want. Of course we expect that the probability density β_N that a parent drop would produce N daughters, as the result of a single rupture event, goes uniformly to zero as N goes to infinity. However, in our analysis N will be a given parameter somehow related to the effective operating conditions (say, the agitation speed, the nature of liquids being mixed, the geometry of the apparatus and so on).

As in [6,7] we write the evolution equation for f as follows

$$\frac{\partial f}{\partial t} = \phi(t)\left(L_c f + L_b f + L_s f\right) \ . \tag{2.1}$$

Here $\phi(t) = \Psi\left[\mathcal{N}(t), \mathcal{S}(t)\right]$ with

$$\mathcal{N}(t) = \int_0^{v_m} f(v,t)\ \mathrm{d}v \ , \qquad \mathcal{S}(t) = \int_0^{v_m} v^{(2/3)} f(v,t)\ \mathrm{d}v \ ,$$

represents what we called an *efficiency factor*, $\mathcal{N}$ and $\mathcal{S}$ having, respectively, the meaning of the instantaneous total *number of droplet* and *inter–facial area* per unit volume of dispersion. The role of ϕ has been described in the quoted papers.

The operators at the r.h.s. of (2.1) have a rather complex structure: L_c is the *coalescence* operator and depends on a coalescence kernel τ_c which is a known function of the size of the two colliding droplets; L_b is the *breakage* operator summing up the contributions of the various rupture modes (binary, ternary, etc.), having defined, for each breakage mode, its *frequency* α_k and the *probability density* β_k of its outcome. Finally L_s is the *scattering* operator and the kernel of the k-th mode is just the product of β_k and τ_c. We recall that L_s has represented the main novelty of our model in the current literature about drops dynamics since we first proposed it in [6,7]. Its role is to justify the instability of droplets resulting from coalescence

and with volume above the threshold value v_m without invoking any extra condition besides the physics involved: indeed L_s is nothing but a suitable combination of the two main factors driving the dynamics of droplets, namely coalescence and breakage.

Natural size limitations among droplets impose particular care when the integration domains of the various terms on the r.h.s. of (2.1) are specified. To be precise we put

$$\begin{aligned} L_c f(v,t) = & \int_0^{v/2} \tau_c(w, v-w) f(w,t) f(v-w,t)\, \mathrm{d}w \\ & - f(v,t) \int_0^{v_m - v} \tau_c(w,v) f(w,t)\, \mathrm{d}w, \end{aligned} \tag{2.2}$$

$$\begin{aligned} L_b f(v,t) = & \int_v^{v_m} \alpha_2(s)\beta_2(s,v) f(s,t)\, \mathrm{d}s \\ & + \sum_{k=3}^{N} \int_v^{v_m} \alpha_k(s) f(s,t)\, \mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right)\, \mathrm{d}\sigma_{k-2} \\ & - \sum_{k=2}^{N} \alpha_k(v) f(v,t), \end{aligned} \tag{2.3}$$

$$\begin{aligned} L_s f(v,t) = & \int_{v_m}^{v_m+v} \lambda_2(s)\beta_2(s, s-v)\, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w) f(w,t) f(s-w,t)\, \mathrm{d}w \\ & + \sum_{k=3}^{N} \left[\int_{v_m}^{v_m+v} \lambda_k(s)\, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w) f(w,t) f(s-w,t)\, \mathrm{d}w \right. \\ & \left. \times \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right)\, \mathrm{d}\sigma_{k-2} \right] \\ & - f(v,t) \int_{v_m}^{v_m+v} \tau_c(v, s-v) f(s-v,t)\, \mathrm{d}s, \end{aligned} \tag{2.4}$$

where τ_c is a symmetric kernel and $\mathrm{d}\sigma_{k-2} = \mathrm{d}u_1 \cdots \mathrm{d}u_{k-2}$ denotes the measure element in $\mathbb{R}^{k-2}$. We will list soon the meaning and the assumptions on all the coefficients appearing in (2.2)–(2.4).

Remark 1. Notice that

$$\int_{v_m}^{v_m+v} \tau_c(v, s-v) f(s-v,t)\, \mathrm{d}s = \int_{v_m-v}^{v_m} \tau_c(w,v) f(w,t)\, \mathrm{d}w, \tag{2.5}$$

so that the last term in (2.4) is nothing but the continuation of the last term in (2.2). This can be easily interpreted bearing in mind the physical meaning of both operators: the negative term in L_c takes into account the loss of a drop with volume v due to coalescence with a drop with volume w *when the the resulting drop has volume* $v+w<v_m$. The role of the negative term in L_s is similar: *in this case the volume of the resulting drop* $v+w$ *is greater than* v_m because of the very meaning of L_s. The upper bound of the integral is equal to v_m since in any case w cannot exceed this limit.

Remark 2. The apparently complicated structure of equation (2.1) is essentially due to the decomposition into single modes of all breakage and scattering events. Indeed if we define

$$\begin{aligned}
\mathfrak{O}_{\mathrm{b}}^{+}(s,v) &= \alpha_2(s)\beta_2(s,v) + \sum_{k=3}^{N} \alpha_k(s) \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s-v-U_{k-2}\right)\, \mathrm{d}\sigma_{k-2}, \\
\mathfrak{O}_{\mathrm{b}}^{-}(v) &= \sum_{k=2}^{N} \alpha_k(v), \\
\mathfrak{O}_{\mathrm{s}}^{+}(s,v,w) &= \left(\lambda_2(s)\beta_2(s,s-v) + \sum_{k=3}^{N} \lambda_k(s) \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s-v-U_{k-2}\right)\, \mathrm{d}\sigma_{k-2} \right) \\
&\quad \times \tau_c(s-w,w)
\end{aligned}$$

then the three operator at the r.h.s. of (2.1) rewrite as

$$L_c f(v,t) = \int_0^{v/2} \tau_c(w, v-w) f(w,t) f(v-w,t)\, \mathrm{d}w - f(v,t) \int_0^{v_m} \tau_c(w,v) f(w,t)\, \mathrm{d}w, \tag{2.6}$$

$$L_b f(v,t) = \int_v^{v_m} \mathfrak{O}_{\mathrm{b}}^{+}(s,v) f(s,t)\, \mathrm{d}s - \mathfrak{O}_{\mathrm{b}}^{-}(v) f(v,t), \tag{2.7}$$

$$L_s f(v,t) = \int_{v_m}^{v_m+v} \int_{s-v_m}^{s/2} \mathfrak{O}_{\mathrm{s}}^{+}(s,v,w) f(w,t) f(s-w,t)\, \mathrm{d}w\, \mathrm{d}s, \tag{2.8}$$

where we made use of (2.5). Apart from the scattering term, this form appears much closer to the traditional structure of equation (2.1).

The symbols appearing above have the following meaning:

- $\alpha_k(s)$ is the *breakage rate* of droplets with volume $s \in (v_{\rm crit}^{(1)}, v_m)$ in k pieces, $v_{\rm crit}^{(1)}$ being a lower non–negative threshold.
- $\lambda_k(s)$ is a suitable *weight* (to be chosen conveniently) measuring the chance of the parent droplet $s \in (v_m, 2v_m]$ to break exactly in k pieces within the scattering process; of course

$$\sum_{k=2}^{N} \lambda_k(s) = 1. \tag{2.9}$$

- $U_n = \sum_{h=1}^{n} u_h$.
- $\beta_k(s, u_1, \ldots, u_{k-1})$ is the probability density of drops with volume $s \in (0, 2v_m)$ to generate by breakage k fragments with prescribed volumes u_j ($j = 1, \ldots, k-1$) in increasing order, $v_{\rm crit}^{(2)} \le u_1 \le u_2 \le \ldots \le u_{k-1}$ (the volume of the remaining drop is the complement to s and may occupy any position in the size order), $v_{\rm crit}^{(2)}$ being a lower non–negative threshold. The inequality $v_{\rm crit}^{(2)} \le u_1$ is replaced with $v_{\rm crit}^{(2)} < u_1$ in the case $v_{\rm crit}^{(2)}$ vanishes.
- $\tau_c(v, w)$ is the *coalescence kernel*, that is proportional to the probability that two colliding droplets of respective volumes v and w coalesce to form a single droplet of volume $v + w$.

We remark that α_k in (2.3) and λ_k in (2.4) have different roles. Indeed the rates α_k include the k–th mode breakage frequency, while in the operator L_s the breakage has probability one and all we need to know is the probability of each breakage mode. A way of slightly simplifying the model could be to define the breakage frequency α irrespectively of each mode, hence setting $\alpha_k(s) = \alpha(s)\lambda_k(s)$.

The precise definition of the functions $\beta_k(s, u_1, \ldots, u_{k-1})$ and of the domains $D_k(s, v)$ need several preliminaries: we devote the following Section just to this topic.

As to the regularity properties of the functions appearing in the kernels of L_c, L_b, L_s and the efficiency factor Ψ, we start by assuming the following

(H1) Ψ is strictly positive, Lipschitz continuous and bounded in $\mathbb{R}^2$. We also assume $\inf_{\mathbb{R}^2} \Psi = \widehat{\Psi} > 0$.

(H2) τ_c is non–negative, symmetric and continuously differentiable in $[v_{\rm crit}^{(3)}, v_m] \times [v_{\rm crit}^{(3)}, v_m]$, $v_{\rm crit}^{(3)}$ being a lower non–negative threshold.

(H3) For $k = 2, \ldots, N$, α_k is non–negative, continuously differentiable and non–decreasing in $[v_{\rm crit}^{(1)}, v_m)$, unbounded as v tends to v_m. We also assume $\sum_{k=2}^{N} \alpha_k > 0$ for all $v \in (v_{\rm crit}^{(1)}, v_m)$ and

  a) $\alpha_k \simeq (v_m - v)_+^{-\mu_k}$ with $\mu_k \in (0, 1)$ in a left neighbourhood of $v = v_m$,

  b) $\alpha_k \simeq (v - v_{\rm crit}^{(1)})_+^{\delta_k}$ with $\delta_k > 0$ in a right neighbourhood of $v = v_{\rm crit}^{(1)}$, being $(\bullet)_+ := \max\{\bullet, 0\}$.

(H4) For $k = 2, \ldots, N$, functions λ_k are continuous in $[v_m, 2v_m]$.

To investigate the well-posedness of the Cauchy problem for equation (2.1) extra assumptions are needed. These will be presented in Section 6 (see hypothesis (H5) therein).

Remark 3. The thresholds $v_{\rm crit}^{(1)}$ (for breakable drops), $v_{\rm crit}^{(2)}$ (for breakage generated drops), and $v_{\rm crit}^{(3)}$ (for coalescence) are not necessarily related. A widely used empirical law, called *Weber relation* (see, for example, [18]), assumes

$$v_{\rm crit}^{(1)} = 10^{-4}\pi\,(\sigma/\varrho)^{9/5}\left(\omega^2 D^{4/3}\right)^{-9/5},$$

where σ and ϱ are, respectively, the surface tension and the density of the dispersed phase, ω is the angular velocity of the impeller and D is the impeller diameter. Therefore it is quite reasonable to think of $v_{\rm crit}^{(1)}$ as a very small but not vanishing value. Indeed $v_{\rm crit}^{(1)} \to 0$ only if either $\omega \to +\infty$ or $\sigma \to 0$, both unphysical situation. As far as we know, something similar to the *Weber relation* is not available for $v_{\rm crit}^{(2)}$ and $v_{\rm crit}^{(3)}$. However experiments show clearly that $v_{\rm crit}^{(2)}$ and $v_{\rm crit}^{(3)}$ are very unlike to be zero. This means that droplets with size below $v_{\rm crit}^{*} = \min\{v_{\rm crit}^{(1)}, v_{\rm crit}^{(2)}, v_{\rm crit}^{(3)}\}$ (supposed to be strictly positive) are only those pre–existing the agitation process. These drops are *stable* against both breakage and coalescence. It is worth noticing that – while the local existence in time of the unique solution to the Cauchy problem for equation (2.1) can be achieved regardless of being $v_{\rm crit}^{*}$ equal to zero or not – to prove the global existence *we are forced to assume* $v_{\rm crit}^{*} > 0$ (see Part II of the present paper in this same volume). Thus the mathematical consistency of the model appear to be strictly related to the very physics of the problem. However, in order to simplify the exposition, we put $v_{\rm crit}^{*} = 0$ throughout except in the last section of Part II of this paper (in this same volume), where the role of this parameter will be better emphasized.

Remark 4. If $v_{\rm crit}^{*} > 0$, the highest number of allowable rupture modes N can be roughly estimated from above through the ratio $2v_m/v_{\rm crit}^{*}$.

Remark 5. It is worth noticing that in the exceptional case of binary ruptures only (that is $\alpha_k = \lambda_k = 0$ for all $k \geq 3$) the model we propose coincides with the one presented in [6,7].

3. The functions β_k, their domains and the sets D_k

The function β_2 is such that

$$\beta_2(s,u) = \beta_2(s, s-u), \tag{3.1}$$

and

$$\beta_2(s,u) = 0, \quad \text{if } s \leq u. \tag{3.2}$$

In other words, for each $s \in (0, v_m]$, we only need to define β_2 in $[0, s/2]$. If $s \in (v_m, 2v_m]$ being $u = s - (s-u) > s - v_m$, the function is defined in $(s - v_m, s/2)$. We set

$$T_{2,1}(s) = \{u_1 \mid 0 < u_1 \leq s - u_1 \leq v_m\} = \left(\max\{0, s - v_m\}, \frac{s}{2}\right), \tag{3.3}$$

$$T_{2,2}(s) = \{u_1 \mid 0 < s - u_1 < u_1 \leq v_m\} = \left(\frac{s}{2}, \min\{s, v_m\}\right), \tag{3.4}$$

Notice that the map $C_0 : s - u \mapsto u$, transforms $T_{2,2}(s)$ one-to-one onto $T_{2,1}(s)$. Therefore, for any value of $s \in (0, 2v_m)$, we assign β_2 on $T_{2,1}(s)$ in such a way that

$$\int_{T_{2,1}(s)} \beta_2(s, u)\, \mathrm{d}u = 1, \tag{3.5}$$

and think of $\beta_2 \circ C_0$ as its extension on $T_{2,2}(s)$. Also notice that $C_0 = C_0^{-1}$; moreover C_0 is measure–conserving, so that

$$\int_{T_{2,2}(s)} \beta_2(s, s-u)\, \mathrm{d}u = \int_{T_{2,1}(s)} \beta_2(s, u)\, \mathrm{d}u = 1. \tag{3.6}$$

Now, for a given $k \geq 3$, let us first consider the case $s \in (0, v_m]$ and define the set of $\mathbb{R}^{k-1}$

$$T_{k,1}(s) = \{(u_1, \ldots, u_{k-1}) \mid 0 < u_1 \leq \ldots \leq u_{k-1} \leq s - U_{k-1} \leq v_m\}. \tag{3.7}$$

Clearly $\widetilde{u} = s - U_{k-1}$ identifies one of the k daughters and $T_{k,1}$ is characterized by the circumstance of $\widetilde{u}$ being the volume of the *largest* daughter(s). Function β_k is assigned on $T_{k,1}$ in such a way that

$$\int_{T_{k,1}(s)} \beta_k\left(s, u_1, \ldots, u_{k-1}\right)\, \mathrm{d}\sigma_{k-1} = 1. \tag{3.8}$$

We then introduce, for a fixed $s \in (0, v_m]$, the following domains in $\mathbb{R}^{k-1}$

$$\begin{aligned} &T_{k,j}(s) = \{(u_1, \ldots, u_{k-1}) \mid 0 < u_1 \leq \ldots \leq u_{k-j} \leq s - U_{k-1} \leq u_{k-j+1} \leq \ldots \leq u_{k-1} \leq v_m\}, \\ &j = 2, \ldots, k, \end{aligned} \tag{3.9}$$

where, by definition, $u_0 = 0$ (*i. e.* in $T_{k,k}(s)$, $\widetilde{u}$ is the volume of the smallest drop(s)).

Remark 1. If $s \leq v_m$, the last inequality in (3.7) and (3.9) is obviously redundant.

We now consider, again for a fixed s, the maps

$$\begin{aligned} &C_j : (\xi_1, \ldots, \xi_{k-1}) \mapsto (u_1, \ldots, u_{k-1}), \\ &j = 1, \ldots, k-1, \end{aligned} \tag{3.10}$$

defined by

$$\begin{cases} u_1 = \xi_1, \ldots, u_{k-j-1} = \xi_{k-j-1}, \\ \\ u_{k-j} = s - \sum_{i=1}^{k-1} \xi_i, \\ \\ u_{k-j+1} = \xi_{k-j+1}, \ldots, u_{k-1} = \xi_{k-1}. \end{cases} \tag{3.11}$$

The purpose of maps (3.11) is to "re–locate" the *residual drop* $\widetilde{u}$ with respect to the ordered set of the other daughters. Indeed $U_{k-1} = s - \xi_{k-j}$ with $\widetilde{u}$ taking the place of ξ_{k-j}. It is easy to see that the Jacobian of each map C_j is equal to one and that

$$\begin{cases} C_j\left(T_{k,j}(s)\right) = T_{k,j+1}(s), \\ \\ C_j\left(T_{k,j+1}(s)\right) = T_{k,j}(s) \end{cases} \tag{3.12}$$

so that $C_j = C_j^{-1}$. Recalling the definition of the sets $T_{k,j}$, the role of the maps C_j is rather evident. Since the residual drop is the largest one in $T_{k,1}$ and the smallest one in $T_{k,k}$, we can complete the definition of the maps (3.10)–(3.11) also for $j = k$, by taking

$$C_k\left(T_{k,k}(s)\right) = T_{k,1}(s), \tag{3.13}$$

which makes the family of maps C_j *cyclic* among the domains $T_{k,j}(s)$. Because of (3.12) it turns out that

$$C_k = C_1 \circ C_2 \circ \ldots \circ C_{k-1}. \tag{3.14}$$

The main reason for introducing the maps C_j is to extend the probability density over all domains $T_{k,j}$. The procedure is the following. First we prove that, for all $k \geq 3$ all open domains $\overset{\circ}{T}_{k,j}$ are mutually disjoint and that $\bigcap_{j=1}^{k} T_{k,j}$ reduces to a single point which can be identified with the event

$$u_1 = u_2 = \ldots = u_{k-1} = \frac{s}{k}, \tag{3.15}$$

that is "all droplets have the same volume".

The first statement above is obvious: for $i \neq j$, say for example $i = j + 1$, we have that

$$u_{k-i} < s - U_{k-1} < u_{k-i+1} \quad \Leftrightarrow \quad u_{k-j-1} < s - U_{k-1} < u_{k-j}$$

holds true in $\overset{\circ}{T}_{k,i}$ and this is incompatible with

$$u_{k-j} < s - U_{k-1} < u_{k-j+1}$$

which holds true in $\overset{\circ}{T}_{k,j}$. Moreover a point $(u_1, \dots, u_{k-1})$ belongs to $\bigcap_{j=1}^{k} T_{k,j}$ if and only if

$$\begin{cases} s - U_{k-1} = u_{k-j}, \\ \\ j = 1, \dots, k-1 \end{cases} \tag{3.16}$$

This is a linear system $A \cdot X = B$ with coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 & 2 \\ 1 & 1 & \dots & 2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & 1 & 1 \\ 2 & 1 & \dots & 1 & 1 \end{pmatrix} \tag{3.17}$$

and vector

$$B = \begin{pmatrix} s \\ s \\ \vdots \\ s \end{pmatrix}$$

Lemma 3.1. *Any matrix of type*

$$A = \begin{pmatrix} a & a & \dots & a & b \\ a & a & \dots & b & a \\ \dots & \dots & \dots & \dots & \dots \\ a & b & \dots & a & a \\ b & a & \dots & a & a \end{pmatrix}, \qquad (a \neq 0, b \neq 0),$$

with dimension $n \times n$ ($n > 1$) is singular if and only if either $a = b$ or $b = (1-n)a$.

Proof. The case $a = b$ is elementary. As far as the other case is concerned, we notice that, up to the sign, A has the same determinant as

$$\widetilde{A} = \begin{pmatrix} b & a & \dots & a & a \\ a & b & \dots & a & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & \dots & b & a \\ a & a & \dots & a & b \end{pmatrix}.$$

More precisely $\det \widetilde{A} = (-1)^{n-1} \det A$. Both A and $\widetilde{A}$ are symmetric and therefore admit a diagonal form. Since

$$\widetilde{A} - (b-a)I = \begin{pmatrix} a & a & \dots & a & a \\ a & a & \dots & a & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & \dots & a & a \\ a & a & \dots & a & a \end{pmatrix} = A^*,$$

being $\det A^* = 0$, $b-a$ is an eigenvalue of $\widetilde{A}$. Moreover, since $\operatorname{rank} A^* = 1$, this eigenvalue turns out to have its geometric and algebraic multiplicity both equal to $n-1$. The remaining eigenvalue λ of $\widetilde{A}$ can be evaluated explicitly recalling that, being the trace of $\widetilde{A}$ an invariant, we have

$$\operatorname{trace} \widetilde{A} = nb = \operatorname{trace} \begin{pmatrix} b-a & 0 & \dots & 0 & 0 \\ 0 & b-a & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b-a & 0 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix} = \lambda + (n-1)(b-a).$$

Thus $\lambda = b + (n-1)a$ and consequently

$$\det \widetilde{A} = \det \begin{pmatrix} b-a & 0 & \dots & 0 & 0 \\ 0 & b-a & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b-a & 0 \\ 0 & 0 & \dots & 0 & b+(n-1)a \end{pmatrix} = (b-a)^{n-1}(b+(n-1)a).$$

We conclude that $\widetilde{A}$ (and so A) is singular if and only either $b = a$ or $b = (1-n)a$.

□

In our case $b = 2$, $a = 1$ and $n \geq 2$ so that matrix (3.17) is not singular and the unique solution of system $A \cdot X = B$ is just (3.15).

Now, by means of the maps C_j we extend β_k from $T_{k,1}$ to $T_{k,2}$, from $T_{k,2}$ to $T_{k,3}$ and so on, up to $T_{k,k}$. In other words we put

$$\widetilde{\beta}_k(s, u_1, \dots, u_{k-1}) = \begin{cases} \beta_k(s, u_1, \dots, u_{k-1}), & \text{if } (u_1, \dots, u_{k-1}) \in T_{k,1}(s), \\ \beta_k \circ C_1(u_1, \dots, u_{k-1}), & \text{if } (u_1, \dots, u_{k-1}) \in T_{k,2}(s), \\ \vdots & \vdots \\ \beta_k \circ C_1 \circ C_2 \circ \dots \circ C_{k-1}(u_1, \dots, u_{k-1}), & \text{if } (u_1, \dots, u_{k-1}) \in T_{k,k}(s). \end{cases} \tag{3.18}$$

Because of the properties of the maps C_j, we have

$$\int_{T_{k,1}(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = \int_{T_{k,2}(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = \dots = \int_{T_{k,k}(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = 1; \tag{3.19}$$

if we define $T_k = \bigcup_{j=1}^{k} T_{k,j}$ and recall that $\overset{\circ}{T}_{k,j} \cap \overset{\circ}{T}_{k,i} = \varnothing$ for $i \neq j$, we also have

$$\int_{T_k(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = k. \tag{3.20}$$

We now put

$$D_k(s,v) = T_k(s) \cap \{U_{k-1} = s - v\}. \tag{3.21}$$

Thus in all the $T_{k,j}$ contributing to $D_k(s,v)$, the volume v is the one of the "residual drop". Notice that $T_k(s)$ is $(k-1)-$dimensional polytope, so that $D_k(s,v)$ is nothing but a finite portion of an hyperplane in $\mathbb{R}^{k-2}$. From now on we drop the "tilde" above β_k in (3.18), i.e. we identify β_k with its extension over $T_k(s)$.

Since we allow s in the interval $(0, 2v_m]$, function β_k is defined in a $k-$dimensional polytope

$$\mathcal{T}_k = \left\{ (s, u_1, \ldots, u_{k-1}) \in \mathbb{R}^k \mid s \in (0, 2v_m], 0 < u_1 \leq u_2 \leq \ldots \leq u_{k-1} \leq u_k, U_k = s \right\}. \tag{3.22}$$

The domain $T_k(s)$ is nothing but the intersection of $\mathcal{T}_k$ with the plane $s =$constant.

We now extend the definitions (3.7) and (3.9) of the domains $T_{k,j}(s)$ when $s \in (v_m, 2v_m]$. In this case the last inequality appearing in the definitions (3.7) and (3.9) (which, when $s \in (0, v_m]$, is automatically satisfied) plays an effective role.

We also extend the assumption (3.8): we put

$$\int_{T_{k,1}(s)} \beta_k \left(s, u_1, \ldots, u_{k-1}\right) \, \mathrm{d}\sigma_{k-1} = 1, \tag{3.23}$$

regardless of the size of s in $(0, 2v_m]$. The maps C_j then allow to extend β_k over the whole set $T_k(s)$ also for $s \in (v_m, 2v_m)$. Of course also (3.19) and (3.20) extend to this case.

Remark 2. We notice explicitly that (3.19) is the natural extension to the case $k \geq 3$ of (3.6) and that, being $T_{2,1}(s) \cap T_{2,2}(s) = \{s/2\}$, $\overset{\circ}{T}_{2,1} \cap \overset{\circ}{T}_{2,2} = \varnothing$ and $\operatorname{meas} T_{2,1}(s) = \operatorname{meas} T_{2,2}(s)$, relation (3.20) also holds true for $k = 2$. The same conclusion concerns (3.21): for $k = 2$ this set reduces the single point of abscissa $s - v$ in the interval $(\max\{0, s - v_m\}, \min\{s, v_m\})$.

Figure 3.1 shows how $T_2(s) = T_{2,1}(s) \cup T_{2,2}(s)$ changes with s: $T_{2,1}(s)$ and $T_{2,2}(s)$ are just a pair of equal length one-dimensional intervals with s-dependent endpoints.

For $k = 3$ the regions $T_{k,j}(s)$ are two–dimensional. Figure 3.2 shows the two-dimensional polytopes $T_{3,j}(s)$ for $j = 1, 2, 3$ when $s \in (0, v_m)$ and $s \in (v_m, 2v_m)$ respectively. The case $k = 4$ is also shown (see figures 3.3 and 3.4).

In all these figures the subset $D_k(s,v)$ for a given value of v is also shown.

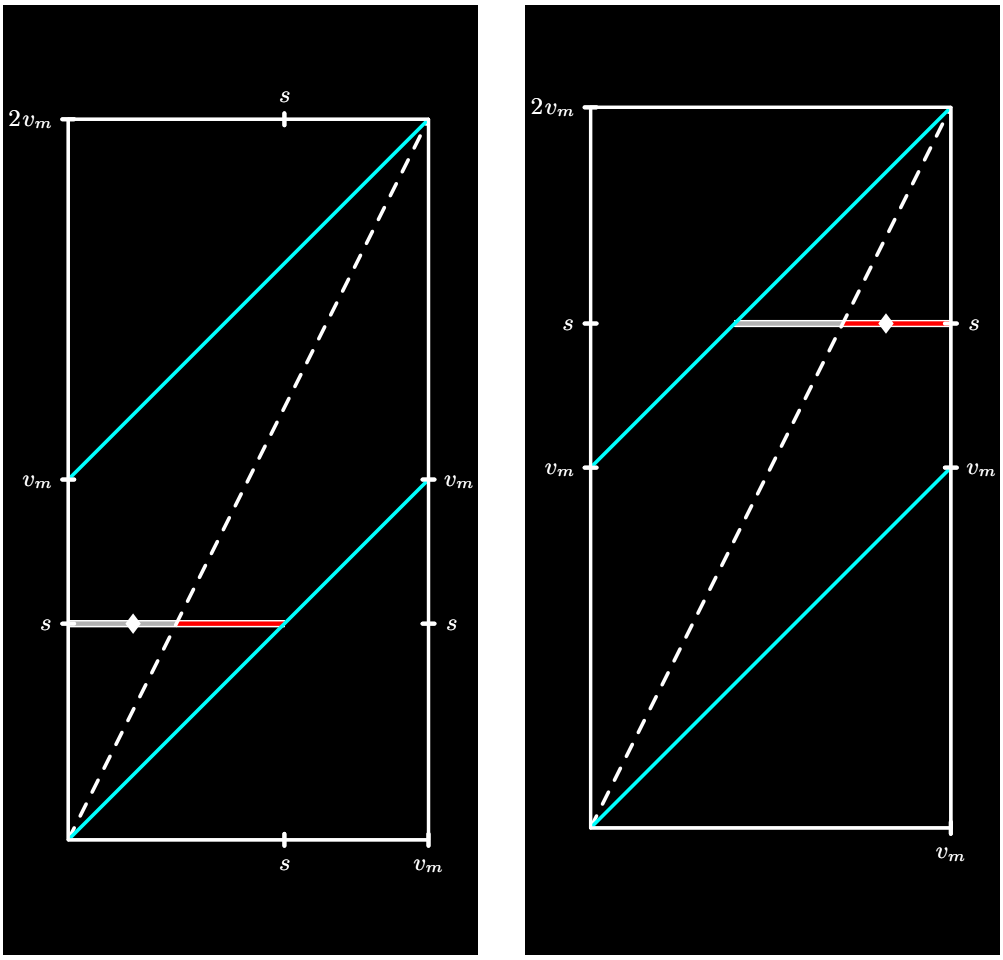

Fig. 3.1. The domains $T_{2,1}(s)$ and $T_{2,2}(s)$ when s is smaller than v_m (left) and when s is larger than v_m (right). For a given $v \in (\max\{0, s - v_m\}, \min\{s, v_m\})$, the domain $D_2(s, v)$ reduces to a single point (denoted by the white diamond) along these intervals. In both figures $T_{2,1}$ is gray-colored, $T_{2,2}$ is red-colored. The left endpoint of $T_{2,1}$ is on the line $u = \max\{s - v_m, 0\}$, the right endpoint of $T_{2,2}$ is on the line $u = \min\{s, v_m\}$, the common endpoint being on the line $u = (\max\{s - v_m, 0\} + \min\{s, v_m\})/2$

4. Drops production rate

The drops production rate N_c due to coalescence only does not differ from the one presented in [6,7] and it will not be re–discussed here.

As far as the contributions due to breakage and scattering are concerned, let us distinguish between their "gain" and "loss" parts and consider, in each case, the $k-$th mode only. Accord-

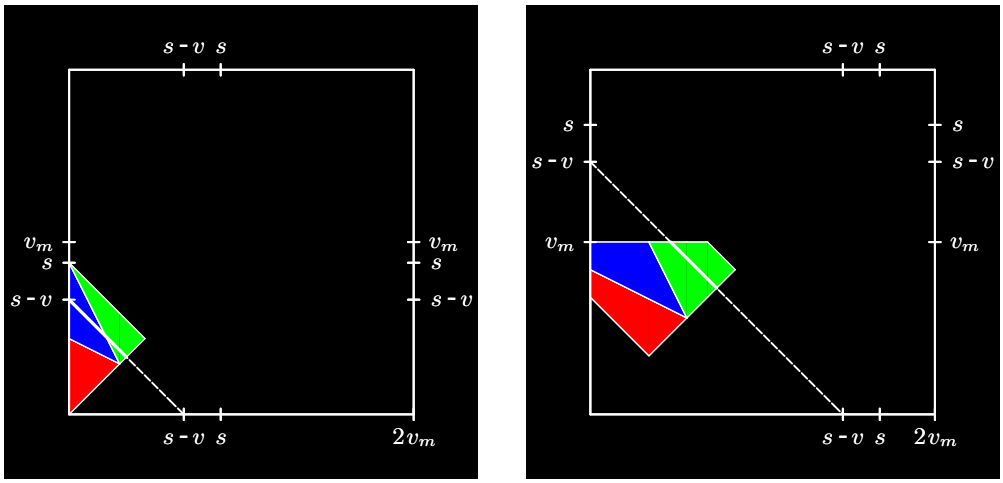

Fig. 3.2. The domains $T_{3,j}(s)$ and $D_3(s,v)$ when s is smaller than v_m (left) and when s is larger than v_m (right). For a fixed $v \in (0, \min\{s, v_m\}$, the domain $D_3(s,v)$ is represented by the intersection of the white dashed line with the $T_3(s)$ domain. In both figures $T_{3,1}$ is red-colored, $T_{3,2}$ is blue-colored, $T_{3,3}$ is green-colored

ingly we introduce *the $k-$th net drop number production rate $\mathcal{N}_b^{(k)}$, at time t, due to breakage* as the difference between the rates

$$\begin{aligned} \mathcal{N}_{b,g}^{(k)} &= \int_0^{v_m} L_{b,g}^{(k)}(v) f \, \mathrm{d}v, \\ \mathcal{N}_{b,\ell}^{(k)} &= \int_0^{v_m} L_{b,\ell}^{(k)}(v) f \, \mathrm{d}v. \end{aligned} \tag{4.1}$$

where, $L_{b,g}^{(k)}$, $L_{b,\ell}^{(k)}$ are the *gain rate breakage operator* and *loss rate breakage operator* respectively, namely

$$L_{b,g}^{(k)}(v) f = \begin{cases} \displaystyle\int_v^{v_m} \alpha_2(s) f(s,t) \beta_2(s,v) \, \mathrm{d}s, \\ \qquad\qquad\qquad\qquad \text{for } k = 2, \\ \displaystyle\int_v^{v_m} \alpha_k(s) f(s,t) \, \mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right) \, \mathrm{d}\sigma_{k-2}, \\ \qquad\qquad\qquad\qquad \text{for } k > 2, \end{cases} \tag{4.2}$$

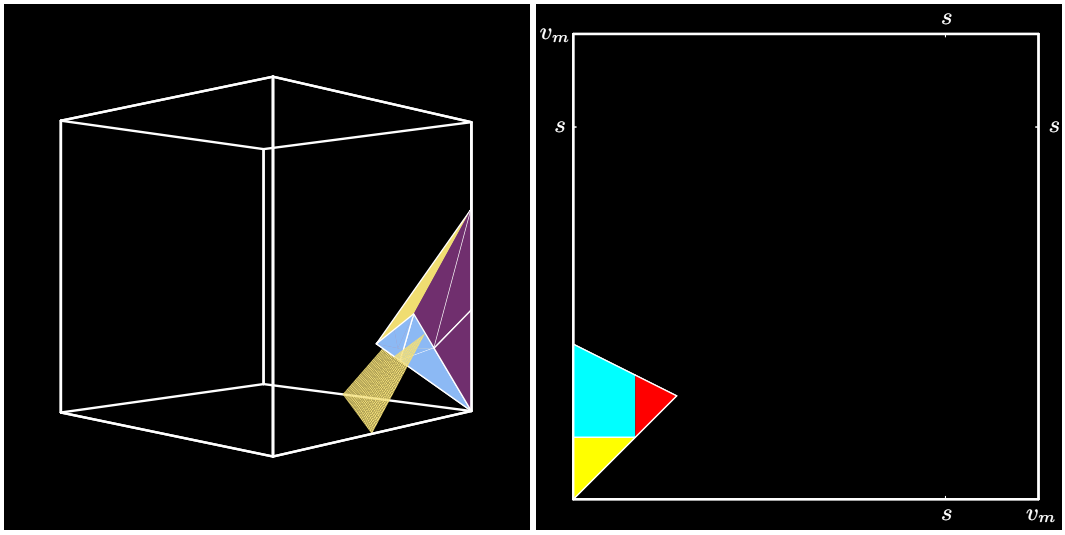

Fig. 3.3. A three–dimensional view of the domain $T_4(s)$ intersected by the plane $u_1 + u_2 + u_3 = s - v$ (at left) and the projection of the intersection $D_4(s, v)$ over u_1, u_2 (at right) when s is smaller than v_m (in this case $s = 0.7\, v_m$ and $v = 0.13\, v_m$). Different colors distinguish among the intersections of the plane with the various disjoint connected components of $T_4(s)$: gray is that with $T_{4,1}(s)$, yellow that with $T_{4,2}(s)$, cyan that with $T_{4,3}(s)$, and red that with $T_{4,4}(s)$

and

$$L^{(k)}_{b,\ell}(v) f = \alpha_k(v) f(v,t), \qquad \text{for } k \geq 2. \tag{4.3}$$

The term $\mathcal{N}^{(k)}_{b,g}$ represents the number of droplets with volume v produced per unit time by breakage events in which exactly k fragments are generated, while $\mathcal{N}^{(k)}_{b,\ell}$ those that are lost in the same process.

Proposition 4.1. *If f is a solution of (2.1) then*

$$\mathcal{N}^{(k)}_{b,g}(t) - \mathcal{N}^{(k)}_{b,\ell}(t) = (k-1) \int_0^{v_m} \alpha_k(s) f(s,t)\, \mathrm{d}s, \tag{4.4}$$

so that the net drop number rate due to breakage is

$$\mathcal{N}_{b,g}(t) - \mathcal{N}_{b,\ell}(t) = \sum_{k=2}^{N} (k-1) \int_0^{v_m} \alpha_k(s) f(s,t)\, \mathrm{d}s. \tag{4.5}$$

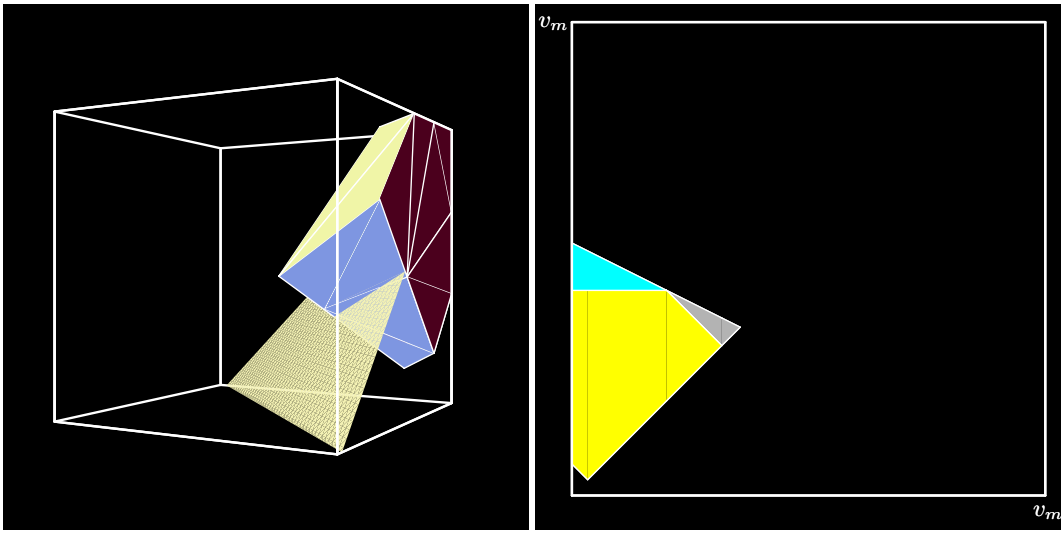

Fig. 3.4. A three–dimensional view of the domain $T_4(s)$ intersected by the plane $u_1+u_2+u_3 = s-v$ (at left) and the projection of the intersection $D_4(s,v)$ on the plane u_1, u_2 (at right) when s is greater than v_m (in this case $s = 1.4\, v_m$ and $v = 0.43\, v_m$). Colors are as in Figure 3.3

Proof. The case $N = 2$ has been discussed in [6,7] where, in particular, we proved (4.4) for $k = 2$. Thus let us take $k > 2$ and notice first that

$$\begin{aligned}
&\int_0^{v_m} L^{(k)}_{b,g}(v) f \, \mathrm{d}v \\
&= \int_0^{v_m} \left(\int_v^{v_m} \alpha_k(s) f(s,t) \, \mathrm{d}s \int_{D_k(s,v)} \beta_k \left(s, u_1, \dots, u_{k-2}, s - v - U_{k-2}\right) \, \mathrm{d}\sigma_{k-2} \right) \mathrm{d}v \qquad (4.6) \\
&= \int_0^{v_m} \alpha_k(s) f(s,t) \left(\int_0^{s} \mathrm{d}v \int_{D_k(s,v)} \beta_k \left(s, u_1, \dots, u_{k-2}, s - v - U_{k-2}\right) \, \mathrm{d}\sigma_{k-2} \right) \mathrm{d}s,
\end{aligned}$$

Applying to (4.6) the change of variable

$$\begin{cases} u_1 = \xi_1, \dots, u_{k-2} = \xi_{k-2}, \\ v = s - \xi_{k-1} - \sum_{h=1}^{k-2} \xi_h = s - \sum_{h=1}^{k-1} \xi_h \end{cases} \qquad (4.7)$$

we get

$$\begin{aligned}\mathcal{N}_{b,g}^{(k)}(t) &= \int_0^{v_m} \alpha_k(s) f(s,t) \left(\int_{T_{k,h}(s)} \beta_k\left(s, u_1, \ldots, u_{k-1}\right)\, \mathrm{d}\sigma_{k-1} \right) \mathrm{d}s \\ &= k \int_0^{v_m} \alpha_k(s) f(s,t)\, \mathrm{d}s \end{aligned} \tag{4.8}$$

Indeed the change of variable (4.7) maps $\bigcup_{v\in(0,s)} \overset{\circ}{D}_k(s,v) \times \{v\}$ one–to–one onto $\overset{\circ}{T}_k\ (s)$. To understand this point, recall that $\overset{\circ}{D}_k(s,v) = \overset{\circ}{T}_k\ (s) \cap \{U_{k-1} = s - v\}$ and that, when using (4.7), condition $U_{k-1} = s - v$ is automatically satisfied. Thus, if $v \in (0,s)$, we have $0 < U_{k-1} < s$ which reconstructs the whole set $\overset{\circ}{T}_k\ (s)$: indeed, if P is a point in $\overset{\circ}{T}_k\ (s)$, there exists one and only one index $j \in \{1, \ldots, k\}$ such that $P \in \overset{\circ}{T}_{k,j}\ (s)$, that is

$$u_{k-j} + U_{k-1} < s < u_{k-j+1} + U_{k-1},$$

which in turn implies $U_{k-1} < s$ for all $j \in \{1, \ldots, k\}$ and $U_{k-1} > 0$.
Finally, recalling (4.1) and (4.3), the proof of (4.5) is complete.

□

Let us now proceed similarly for the scattering operator and define *the $k-$th net drop number production rate $\mathcal{N}_s^{(k)}$, at time t, due to scattering* as the difference between the *gain rate scattering operator* $L_{s,g}^{(k)}$ and *loss rate scattering operator* $L_{s,\ell}$, given respectively by

$$\begin{aligned}\mathcal{N}_{s,g}^{(k)} &= \int_0^{v_m} L_{s,g}^{(k)}(v) f\, \mathrm{d}v, \\ \mathcal{N}_{s,\ell} &= \int_0^{v_m} L_{s,\ell}(v) f\, \mathrm{d}v, \end{aligned} \tag{4.9}$$

being

$$L_{s,g}^{(k)}(v)f = \begin{cases} \displaystyle\int_{v_m}^{v_m+v} \lambda_2(s)\beta_2(s,s-v)\,\mathrm{d}s \\ \quad \displaystyle\times \int_{s-v_m}^{s/2} \tau_c(s-w,w)f(w,t)f(s-w,t)\,\mathrm{d}w, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } k=2, \\ \\ \displaystyle\int_{v_m}^{v_m+v} \lambda_k(s)\,\mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w)f(w,t)f(s-w,t)\,\mathrm{d}w \\ \quad \displaystyle\times \int_{D_k(s,v)} \beta_k\left(s,u_1,\ldots,u_{k-2},s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } k>2, \end{cases} \tag{4.10}$$

and

$$L_{s,\ell}(v)f = f(v,t)\int_{v_m-v}^{v_m} \tau_c(v,w)f(w,t)\,\mathrm{d}w \tag{4.11}$$

The term $\mathcal{N}_{s,g}^{(k)}$ represents the number of droplets with volume v produced per unit time by scattering events in which exactly k fragments are generated, while $\mathcal{N}_{s,\ell}$ those that are lost because of their coalescence with other droplets to form an unstable drop above the threshold value v_m.

Proposition 4.2. *If f is a solution of (2.1) then*

$$\begin{aligned} \mathcal{N}_{s,g}^{(k)}(t) &= k\iint_{M_1} \lambda_k(u+w)\tau_c(w,u)f(w,t)f(u,t)\,\mathrm{d}u\,\mathrm{d}w,\ \ \forall k\geq 2, \\ \mathcal{N}_{s,\ell}(t) &= 2\iint_{M_1} \tau_c(w,u)f(w,t)f(u,t)\,\mathrm{d}u\,\mathrm{d}w, \end{aligned} \tag{4.12}$$

where (see Figure 4.1)

$$M_1 := \left\{(u,w)\in \mathbb{R}_+^2 \mid u+w\geq v_m, u\leq w\leq v_m\right\}. \tag{4.13}$$

Consequently the net drop number rate due to scattering is

$$\mathcal{N}_{s,g}(t) - \mathcal{N}_{s,\ell}(t) = \iint_{M_1} \left(-2 + \sum_{k=2}^{N} k\lambda_k(u+w) \right) \tau_c(w,u) f(w,t) f(u,t) \, \mathrm{d}u \, \mathrm{d}w. \tag{4.14}$$

Proof. The case $N = 2$ has been examined in [6,7] where we proved, in particular, that, in that case the net drop number rate due to scattering is equal to zero. If the scattering process involves only binary ruptures, then necessarily $\lambda_2(s) \equiv 1$ and the right hand side of (4.14) vanishes. In the more general case of multiple ruptures, $\lambda_2(s)$ is strictly less than one. However the same proof proposed in [7] applies to show that

$$\begin{aligned} \mathcal{N}_{s,g}^{(2)}(t) &= 2 \iint_{M_1} \lambda_2(u+w) \tau_c(w,u) f(w,t) f(u,t) \, \mathrm{d}u \, \mathrm{d}w, \\ \mathcal{N}_{s,\ell}(t) &= 2 \iint_{M_1} \tau_c(w,u) f(w,t) f(u,t) \, \mathrm{d}u \, \mathrm{d}w. \end{aligned} \tag{4.15}$$

To examine the case $k > 2$, first notice that we can write

$$\mathcal{N}_{s,g}^{(k)}(t) = \int_0^{v_m} \mathrm{d}v \int_{v_m}^{v_m+v} I_k(s,v) J_k(s) \, \mathrm{d}s, \tag{4.16}$$

where

$$I_k(s,v) = \int_{D_k(s,v)} \beta_k \left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2} \right) \, \mathrm{d}\sigma_{k-2}, \tag{4.17}$$

$$J_k(s) = \lambda_k(s) \left(\int_{s-v_m}^{v_m} \tau_c(u, s-u) f(u,t) f(s-u,t) \, \mathrm{d}u \right). \tag{4.18}$$

Also notice that

$$\int_0^{v_m} \mathrm{d}v \int_{v_m}^{v_m+v} I_k(s,v) J_k(s) \, \mathrm{d}s = \int_{v_m}^{2v_m} J_k(s) \, \mathrm{d}s \int_{s-v_m}^{v_m} I_k(s,v) \, \mathrm{d}v \tag{4.19}$$

and that, by means of transformation (4.7) again and recalling (3.20), we have

$$\begin{aligned} \int_{s-v_m}^{v_m} I_k(s,v) \, \mathrm{d}v &= \int_{s-v_m}^{v_m} \mathrm{d}v \int_{D_k(s,v)} \beta_k \left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2} \right) \, \mathrm{d}\sigma_{k-2} \\ &= \int_{T_k(s)} \beta_k \left(s, u_1, \ldots, u_{k-1} \right) \, \mathrm{d}\sigma_{k-1} = k. \end{aligned} \tag{4.20}$$

Moreover

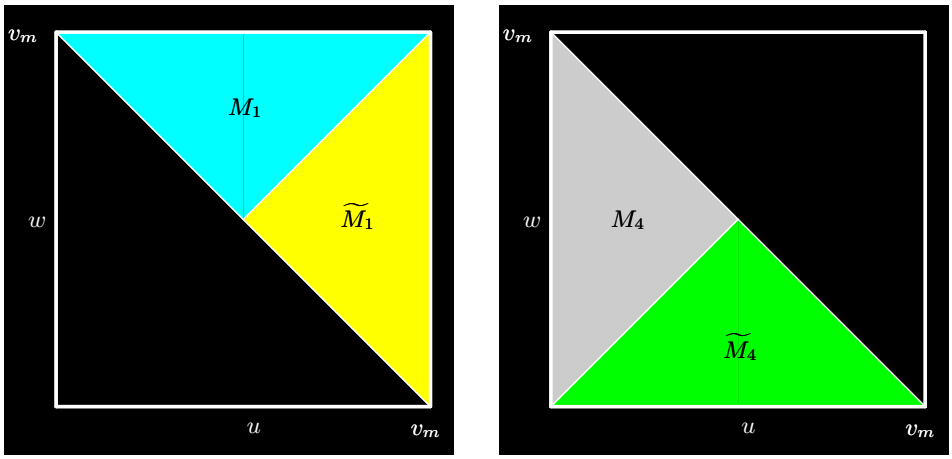

Fig. 4.1. Triangular regions M_1 and M_4 and symmetric ones with respect to $u = w$

$$\begin{aligned}\mathcal{N}_{s,\ell}(t) &= \int_0^{v_m} f(v,t)\,\mathrm{d}v \int_{v_m - v}^{v_m} \tau_c(v,w) f(w,t)\,\mathrm{d}w \\ &= \iint_{M_2} \tau_c(u,w) f(w,t) f(u,t)\,\mathrm{d}u\,\mathrm{d}w,\end{aligned} \tag{4.21}$$

where (see Figure 4.1)

$$M_2 := \{(u,w) \mid 0 \le u \le v_m, v_m - u \le w \le v_m\}\,. \tag{4.22}$$

Notice that if $\widetilde{M}_1$ is the set symmetric to M_1 with respect to the line $u = w$ in $\mathbb{R}^2$, then $M_2 = M_1 \cup \widetilde{M}_1$ being $\overset{\circ}{M}_1 \cap \overset{\circ}{\widetilde{M}}_1 = \varnothing$. Thus, because of the symmetry of τ_c,

$$\mathcal{N}_{s,\ell}(t) = 2 \iint_{M_1} \tau_c(u,w) f(w,t) f(u,t)\,\mathrm{d}\sigma_2. \tag{4.23}$$

.

On the other hand, because of (4.20),

$$\begin{aligned}\mathcal{N}^{(k)}_{s,g}(t) &= k \int_{v_m}^{2v_m} \lambda_k(s)\,\mathrm{d}s \int_{s - v_m}^{s/2} \tau_c(w, s-w) f(w,t) f(s-w,t)\,\mathrm{d}w \\ &= k \iint_{M_1} \lambda_k(u+w) \tau_c(w,u) f(w,t) f(u,t)\,\mathrm{d}u\,\mathrm{d}w,\end{aligned} \tag{4.24}$$

which just implies (4.12).

□

We now recall the following Proposition (proved in [7]):

Proposition 4.3. *The net rate of drop production due to coalescence only is given by*

$$\begin{aligned}\mathcal{N}_{c,g}(t) - \mathcal{N}_{c,\ell}(t) &= \frac{1}{2}\iint_{M_3} \tau_c(w,u) f(w,t) f(u,t)\, \mathrm{d}\sigma_2 \\ &= \iint_{M_4} \tau_c(w,u) f(w,t) f(u,t)\, \mathrm{d}\sigma_2,\end{aligned} \tag{4.25}$$

where (see figure 4.1)

$$M_3 := M_4 \cup \widetilde{M_4} = \{(u,w) \mid 0 \le w \le v_m, 0 \le u \le v_m - w\}\,, \tag{4.26}$$

being

$$M_4 := \{(u,w) \mid 0 \le w \le v_m, 0 \le u \le v_m - v, u \le w\}\,. \tag{4.27}$$

Propositions 4.1, 4.2, and 4.3 imply the following

Proposition 4.4. *At any time, the total net rate of drops production (per unit volume of dispersion) is given by*

$$\begin{aligned}\dot{\mathcal{N}}(t) \equiv \int_0^{v_m} \frac{\partial f}{\partial t}(v,t) = \phi(t) \Bigg[&- \iint_{M_4} \tau_c(u,w) f(u,t) f(w,t)\, \mathrm{d}u\, \mathrm{d}w \\ &+ \sum_{k=2}^{N} (k-1) \int_0^{v_m} \alpha_k(s) f(s,t)\, \mathrm{d}s \\ &+ \iint_{M_1} \left(-2 + \sum_{k=2}^{N} k \lambda_k(u+w) \right) \tau_c(w,u) f(u,t) f(w,t)\, \mathrm{d}u\, \mathrm{d}w \Bigg]\,.\end{aligned} \tag{4.28}$$

5. Volume conservation

In this section we prove that, in our model, the total volume of droplets is conserved. We notice explicitly that since drops are not allowed to have arbitrarily large size, counterexamples like that found by Simons in [19] do not apply to our case.

As in the previous section, it suffices to consider just the $k-$th mode of rupture to meet our goal. Also, since L_c is the same as in [7], we do not repeat the proof that L_c is *volume conserving* and we address the interested reader to that paper for all details about this point.

Instead, consider first the contribution to the rate of volume change due to L_b: the case $k = 2$ has been analyzed in [7] so that we assume $k > 2$. The change of variable (4.7) still applies to show that the integral

$$\begin{aligned}
&\int_0^{v_m} v L_{b,g}^{(k)} f(v,t)\,\mathrm{d}v \\
&\quad = \int_0^{v_m} \mathrm{d}v \int_v^{v_m} \mathrm{d}s \int_{D_k(s,v)} v\beta_k\left(s,u_1,\dots,u_{k-2},s-v-U_{k-2}\right)\alpha_k(s)f(s,t)\,\mathrm{d}\sigma_{k-2} \qquad (5.1)\\
&\quad = \int_0^{v_m} \alpha_k(s)f(s,t)\left[\int_0^s \int_{D_k(s,v)} v\beta_k\left(s,u_1,\dots,u_{k-2},s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}\,\mathrm{d}v\right]\,\mathrm{d}s
\end{aligned}$$

can be written as

$$\int_0^{v_m} \alpha_k(s)f(s,t)\left[\int_{T_k(s)} \left(s-U_{k-1}\right)\beta_k\left(s,u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}\right]\,\mathrm{d}s, \qquad (5.2)$$

that is, recalling (3.20),

$$\begin{aligned}
&\int_0^{v_m} s\alpha_k(s)f(s,t)\left[\int_{T_k(s)} \beta_k\left(s,u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}\right]\,\mathrm{d}s \\
&-\int_0^{v_m} \alpha_k(s)f(s,t)\left[\int_{T_k(s)} U_{k-1}\beta_k\left(s,u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}\right]\,\mathrm{d}s \qquad (5.3)\\
&= k\int_0^{v_m} s\alpha_k(s)f(s,t)\,\mathrm{d}s - \int_0^{v_m} \alpha_k(s)f(s,t)\int_{T_k(s)} U_{k-1}\beta_k\left(s,u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}\,\mathrm{d}s.
\end{aligned}$$

Here once more we made use of the fact that (4.7) maps $\bigcup_{v\in(0,s)} \overset{\circ}{D}_k(s,v)\times\{v\}$ one–to–one onto $\overset{\circ}{T}_k\,(s)$.

We now prove the following

Lemma 5.1. *For all* $h = 2,\dots,k$

$$\int_{T_{k,h}(s)} U_{k-1}\beta_k\left(s,u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1} = \int_{T_{k,1}(s)} \left(s-u_{h-1}\right)\beta_k\left(s,u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}. \qquad (5.4)$$

Proof. We recall that we denoted with β_k both the function assigned on $T_{k,1}(s)$ and its extension over $T_k(s)$. To prove (5.4) this may be source of confusion and this abuse will be momentarily abandoned.

Let us then recall (3.11), (3.12), (3.18) and that, when C_j is applied, we have

$$u_{k-j} = s - \sum_{h=1}^{k-1} \xi_h \quad \Leftrightarrow \quad \xi_j = s - \sum_{h=1}^{k-1} u_h = s - U_{k-1}.$$

Then, applying C_j repeatedly to the left hand side of (5.4) (written as it should be), we obtain

$$\begin{aligned}
&\int_{T_{k,h}(s)} U_{k-1}\left(\beta_k \circ C_1 \circ \ldots \circ C_{h-1}\right)(u_1, \ldots, u_{k-1}) \ \mathrm{d}\sigma_{k-1} \\
&= \int_{T_{k,h-1}(s)} (s - \xi_{h-1})\left(\beta_k \circ C_1 \circ \ldots \circ C_{h-2}\right)(\xi_1, \ldots, \underbrace{s - \sum_{m=1}^{k-1} \xi_m}_{(k-h+1)-\text{th position}}, \ldots, \xi_{k-1}) \ \mathrm{d}\sigma_{k-1} \\
&= \int_{T_{k,h-2}(s)} (s - u_{h-1})\left(\beta_k \circ C_1 \circ \ldots \circ C_{h-3}\right)(u_1, \ldots, u_{k-h+1}, \underbrace{s - \sum_{m=1}^{k-1} \xi_m}_{(k-h+2)-\text{th position}}, \ldots, \xi_{k-1}) \ \mathrm{d}\sigma_{k-1} \\
&\ \ \vdots \quad \vdots \quad \vdots \\
&= \int_{T_{k,2}(s)} (s - \xi_{h-1})\left(\beta_k \circ C_1\right)(\xi_1, \xi_2, \ldots, \underbrace{s - \sum_{m=1}^{k-1} \xi_m}_{(k-1)-\text{th position}}) \ \mathrm{d}\sigma_{k-1} \\
&= \int_{T_{k,1}(s)} (s - u_{h-1})\, \beta_k(u_1, u_2, \ldots, u_{k-1}) \ \mathrm{d}\sigma_{k-1},
\end{aligned}$$

which completes the proof.

□

Lemma 5.1 implies that

$$\int_{T_k(s)} U_{k-1}\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1} = \sum_{h=1}^{k}\int_{T_{k,h}(s)} U_{k-1}\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}$$

$$\begin{aligned}
&= \int_{T_{k,1}(s)} U_{k-1}\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1} + \sum_{h=2}^{k}\int_{T_{k,h}(s)} U_{k-1}\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}\\
&= \int_{T_{k,1}(s)} U_{k-1}\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1} + \sum_{h=2}^{k}\int_{T_{k,1}(s)} \left(s-u_{h-1}\right)\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}\\
&= (k-1)s + \int_{T_{k,1}(s)} U_{k-1}\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1} - \int_{T_{k,1}(s)} U_{k-1}\beta_k\left(u_1,\dots,u_{k-1}\right)\,\mathrm{d}\sigma_{k-1}\\
&= (k-1)s.
\end{aligned} \tag{5.5}$$

From (5.1), (5.3) and (5.5) we get

$$\int_0^{v_m} vL_{b,g}^{(k)}f(v,t)\,\mathrm{d}v = \int_0^{v_m} s\alpha_k(s)f(s,t)\,\mathrm{d}s.$$

Since

$$\int_0^{v_m} vL_{b,\ell}^{(k)}f(v,t)\,\mathrm{d}v = \int_0^{v_m} v\alpha_k(v)f(s,t)\,\mathrm{d}v,$$

we conclude that

Proposition 5.1. *The operator L_b in (2.1) is volume conserving.*

We now analyze the scattering term. For $k \geq 3$ we have

$$\begin{aligned}
&\int_0^{v_m} vL_{s,g}^{(k)}f(v,t)\,\mathrm{d}v\\
&= \int_0^{v_m} v\,\mathrm{d}v \int_{v_m}^{v_m+v}\,\mathrm{d}s \int_{s-v_m}^{s/2}\,\mathrm{d}w \int_{D_k(s,v)} S_g^{(k)}\left(s,w,u_1,\dots,u_{k-1}\mid U_{k-1}=s-v\right)\,\mathrm{d}\sigma_{k-2},
\end{aligned}$$

where we put

$$S_g^{(k)}\left(s,w,u_1,\dots,u_{k-1}\right) = \lambda_k(s)\tau_c(w,s-w)\beta_k\left(s,u_1,\dots,u_{k-1}\right)f(w,t)f(s-w,t)$$

Lemma 5.2. *For all* $k = 3, \dots, N$

$$\int_{s-v_m}^{v_m} v \, \mathrm{d}v \int_{D_k(s,v)} \beta_k \left(s, u_1, \dots, u_{k-1} \mid U_{k-1} = s - v\right) \, \mathrm{d}\sigma_{k-2} = s. \tag{5.6}$$

Proof. First of all notice that, independently of the position of v with respect to the ordered sequence $u_1 \leq \ldots \leq u_{k-1}$, condition $U_{k-1} = s - v$ can be written as $u_{k-1} = s - v - U_{k-2}$, so that

$$\beta_k \left(s, u_1, \dots, u_{k-1} \mid U_{k-1} = s - v\right) = \beta_k \left(s, u_1, \dots, u_{k-2}, s - v - U_{k-2}\right).$$

Now the change of variable

$$\begin{aligned} &C_j^* : (u_1, \dots, u_{k-2}, v) \mapsto (\xi_1, \xi_2 \dots, \xi_{k-1}), \\ &j = 1, \dots, k-1, \end{aligned} \tag{5.7}$$

defined by

$$\begin{cases} \xi_j = u_j, & \text{for } j = 1, \dots, k-2, \\ \xi_{k-1} = s - v - U_{k-2}, \end{cases} \tag{5.8}$$

maps the domain $\bigcup\limits_{v \in (s-v_m, v_m)} \overset{\circ}{D}_k(s,v) \times \{v\}$ one–to–one onto $\overset{\circ}{T}_k(s)$. Indeed, by definition,

$$\overset{\circ}{D}_k(s,v) = \overset{\circ}{T}_k(s) \cap \{U_{k-1} = s - v\};$$

for $v \in (s - v_m, v_m)$ the hyperplane $U_{k-1} = s - v$ spans the whole region between the hyperplanes $U_{k-1} = s - v_m$ and $U_{k-1} = v_m$. Therefore $s - v_m < s - U_{k-1} < v_m$; bearing in mind the definition of $T_{k,j}(s)$ and that $T_k = \bigcup\limits_{j=1}^{k} T_{k,j}(s)$, we have that

$$\bigcup_{v \in (s-v_m, v_m)} \overset{\circ}{D}_k(s,v) \times \{v\} \subseteq \overset{\circ}{T}_k(s).$$

On the other hand, if $(s, u_1, \dots, u_{k-1}) \in \overset{\circ}{T}_k(s)$, then

$$s - v_m < s - U_{k-1} < v_m,$$

and so also

$$\overset{\circ}{T}_k(s) \subseteq \bigcup_{v \in (s-v_m, v_m)} \overset{\circ}{D}_k(s,v) \times \{v\}.$$

In conclusion

$$\int_{s-v_m}^{v_m} v \, \mathrm{d}v \int_{D_k(s,v)} \beta_k \left(s, u_1, \ldots, u_{k-1} \mid U_{k-1} = s - v\right) \, \mathrm{d}\sigma_{k-2}$$

$$= \int_{T_k(s)} \left(s - \sum_{m=1}^{k-1} \xi_m \right) \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1}.$$

Now we can proceed as we did for the breakage operator. Namely

$$\int_{T_k(s)} \sum_{m=1}^{k-1} \xi_m \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1} = \sum_{j=1}^{k} \int_{T_{k,j}(s)} \sum_{m=1}^{k-1} \xi_m \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1}$$

$$= \int_{T_{k,1}(s)} \sum_{m=1}^{k-1} \xi_m \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1} + \sum_{j=2}^{k} \int_{T_{k,j}(s)} \sum_{m=1}^{k-1} \xi_m \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1} \tag{5.9}$$

$$= \int_{T_{k,1}(s)} \sum_{m=1}^{k-1} \xi_m \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1} + \sum_{j=1}^{k-1} \int_{T_{k,1}(s)} (s - u_j) \beta_k \left(s, u_1, \ldots, u_{k-1}\right) \, \mathrm{d}\sigma_{k-1}$$

$$= (k-1)s.$$

Finally

$$\int_{s-v_m}^{v_m} v \, \mathrm{d}v \int_{D_k(s,v)} \beta_k \left(s, u_1, \ldots, u_{k-1} \mid U_{k-1} = s - v\right) \, \mathrm{d}\sigma_{k-2}$$

$$= \int_{T_k(s)} \left(s - \sum_{m=1}^{k-1} \xi_m \right) \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1}$$

$$= s \int_{T_k(s)} \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1} - \int_{T_k(s)} \sum_{m=1}^{k-1} \xi_m \beta_k \left(s, \xi_1, \ldots, \xi_{k-1}\right) \, \mathrm{d}\sigma_{k-1}$$

$$= ks - (k-1)s = s$$

and (5.6) is completely proved.

□

We now recall that (see (4.19))

$$\int_0^{v_m} \mathrm{d}v \int_{v_m}^{v_m+v} I_k(s,v) J_k(s) \, \mathrm{d}s = \int_{v_m}^{2v_m} J_k(s) \, \mathrm{d}s \int_{s-v_m}^{v_m} I_k(s,v) \, \mathrm{d}v. \tag{5.10}$$

Consequently, because of (5.6), we can write

$$\int_0^{v_m} vL_{s,g}^{(k)} f(v,t)\,\mathrm{d}v = \int_{v_m}^{2v_m} s\lambda_k(s)\,\mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(w,s-w)f(w,t)f(s-w,t)\,\mathrm{d}w$$

$$= \iint_{M_1} (u+w)\lambda_k(u+w)\tau_c(w,u)f(w,t)f(u,t)\,\mathrm{d}u\,\mathrm{d}w,$$

where M_1 is defined by (4.13). Again the symmetry of τ_c implies

$$\int_0^{v_m} vL_{s,g}^{(k)} f(v,t)\,\mathrm{d}v = 2\iint_{M_1} u\lambda_k(u+w)\tau_c(w,u)f(w,t)f(u,t)\,\mathrm{d}u\,\mathrm{d}w. \tag{5.11}$$

We notice explicitly that (5.11) (here proved for $k \geq 3$) has been established in [7] also for $k = 2$.

We now evaluate the contribution due to the loss term: we have

$$\int_0^{v_m} vL_{s,\ell} f(v,t)\,\mathrm{d}v = \int_0^{v_m} v\,\mathrm{d}v \int_{v_m}^{v_m+v} \tau_c(v,s-v)f(v,t)f(s-v,t)\,\mathrm{d}s$$

$$= \int_0^{v_m} v\,\mathrm{d}v \int_{v_m-v}^{v_m} \tau_c(v,w)f(v,t)f(w,t)\,\mathrm{d}w$$

$$= \iint_{M_2} v\tau_c(w,v)f(v,t)f(w,t)\,\mathrm{d}w\,\mathrm{d}v = 2\iint_{M_1} v\tau_c(w,v)f(v,t)f(w,t)\,\mathrm{d}w\,\mathrm{d}v.$$

Finally, since $\sum_{k=2}^{N} \lambda_k(s) = 1$,

$$\int_0^{v_m} \left(-L_{s,\ell} + \sum_{k=2}^{N} L_{s,g}^{(k)}\right) vf(v,t)\,\mathrm{d}v$$

$$= 2\iint_{M_1} \left(-1 + \sum_{k=2}^{N} \lambda_k\right) u\tau_c(w,u)f(u,t)f(w,t)\,\mathrm{d}u\,\mathrm{d}w$$

$$= 0.$$

In conclusion we have the following

Proposition 5.2. *The operator L_s in (2.1) is volume conserving.*

Recalling that also L_c is volume conserving (see [7]), Proposition 5.1 and 5.2 imply the following *conservation law*.

Theorem 5.1. *Let $f_o(v)$ be a (continuous) initial data for $f(v,t)$. Then, if $f(v,t)$ is a regular solution to equation (2.1), we have*

$$\int_0^{v_m} v f(v,t)\,\mathrm{d}v = \int_0^{v_m} v f_o(v)\,\mathrm{d}v. \tag{5.12}$$

Remark 1. As we said, usually no distinction is made among the various rupture modes; therefore $\beta(v,w)$ represents the distribution of products from a particle v breaking after collision, w being the *cumulative volume of all fragments*. To guarantee volume conservation (see [20]) *it is necessary to assume* in particular that

$$\int_0^{\min\{v,v_m\}} w\beta(v,w)\,\mathrm{d}w = v, \qquad \text{for a.a.} \quad 0 < v \le 2v_m. \tag{5.13}$$

We notice that (5.13) *is implicit in our approach*. Indeed relations (5.5) and (5.9) (that *we already proved*) imply that

$$\frac{1}{k-1}\int_{T_k(s)} \sum_{m=1}^{k-1} w_m \beta_k(v, w_1, \dots, w_{k-1})\,\mathrm{d}\sigma_{k-1} = v, \qquad \forall v \in (0, 2v_m], \tag{5.14}$$

and this is exactly the same as (5.13) distributed over the various breakage modes, defining β_k as we did. The factor $1/(k-1)$ is evidently justified by the fact that β_k is normalized on $T_{k,j}$, not on T_k.

6. Positivity theorem

In this section we prove that any solution of the Cauchy problem for equation (2.1) which originates from a non–negative initial data remains non-negative, a major qualitative requirement for the physical consitency of our model. As we shall see in Part II, the proof of the local existence theorem requires to handle several integral terms involving β_k. Since these functions are probability densities, extra care is needed to guarantee that the assumptions we need are all consistent. To this aim we need to complete the set of hypotheses (H) by adding the following

(H5) (*Regularity*).

(i) for all $k \ge 2$, β_k vanishes if the size of the smallest daughter goes to zero[1]; in particular

$$\lim_{v\to 0} \beta_k\Big|_{D_k(s,v)} = 0,$$

for all $k \ge 3$.

[1] We recall (see the remark 3 at page 7) that the lower bound for droplets volume should be taken strictly positive. However this assumption is not needed until we have to deal with the problem of global existence (see Part II). For this reason we maintain $v^* = 0$ until then.

(ii) $\beta_2(s,v)$ is piecewise continuously differentiable in $\overset{\circ}{\mathcal{T}}_2 = \bigcup_{s\in(0,2v_m)} \{s\}\times \overset{\circ}{T}_2\ (s)$ and, in addition, there exists a suitable positive constant C such that, for all $v \in (0, v_m)$,

(ii–a) $\lambda_2(v_m+v)\beta_2(v_m+v,v) \le C$,

(ii–b) $\left|\int_{v_m}^{v_m+v} \lambda_2(s)\dfrac{\partial\beta_2(s,v)}{\partial v}\ \mathrm{d}s\right| \le C$,

(ii–c) $\int_v^{v_m} \alpha_2(s)\beta_2(s,v)\ \mathrm{d}s \le C$,

(ii–d) $\left|\int_v^{v_m} \alpha_2(s)\dfrac{\partial\beta_2(s,v)}{\partial v}\ \mathrm{d}s\right| \le C$.

(iii) For $k = 3,\dots,N$, β_k is piecewise continuously differentiable in $\overset{\circ}{\mathcal{T}}_k = \bigcup_{s\in(0,2v_m)} \{s\}\times \overset{\circ}{T}_k\ (s)$ and, in addition, there exists a suitable positive constant C such that, for all $v \in (0, v_m)$,

(iii–a) $\left(\int_{D_k(s,v)} \beta_k\left(s,u_1,\dots,u_{k-2},s-v-U_{k-2}\right)\ \mathrm{d}\sigma_{k-2}\right)\Big|_{s=v_m+v} \le C$,

(iii–b) $\int_v^{v_m} \alpha_k(s)\left(\int_{D_k(s,v)} \beta_k\left(s,u_1,\dots,u_{k-2},s-v-U_{k-2}\right)\ \mathrm{d}\sigma_{k-2}\right)\ \mathrm{d}s \le C$,

(iii–c) $\int_v^{v_m} \alpha_k(s)\left(\int_{\partial D_k(s,v)} \beta_k\left(s,u_1,\dots,u_{k-2},s-v-U_{k-2}\right)\ \mathrm{d}\sigma_{k-3}\right)\ \mathrm{d}s \le C$,

(iii–d) $\left|\int_v^{v_m} \alpha_k(s)\int_{D_k(s,v)} \left[\dfrac{\partial\beta_k}{\partial u_{k-1}}\right]_{u_{k-1}=s-v-U_{k-2}} \mathrm{d}\sigma_{k-2}\ \mathrm{d}s\right| \le C$.

Since $D_k(s,v)$ is nothing but a finite portion of an hyperplane in $\mathbb{R}^{k-2}$, $\partial D_k(s,v)$ is evidently an orientable hypersurface in $\mathbb{R}^{k-3}$. In (iii–c) (and everywhere in this paper) $\partial D_k(s,v)$ is meant to be "positively oriented", that is such that the integral over it of a positive function is positive too.

Concerning the initial data we assume that

$$\begin{aligned} &f_o(v) \text{ is piecewise continuously differentiable in } [0,v_m],\\ &f_o(v) \text{ is non–negative in } [0,v_m],\\ &f_o(0) = f_o(v_m) = 0. \end{aligned} \tag{6.1}$$

As in [7] we look for a solution– in a suitable class of regular functions f to be specified later – to both the *original Cauchy problem*

$$\begin{cases} \dfrac{\partial f}{\partial t} & = \phi(t)(L_c f + L_b f + L_s f), \\ f(v,0) = f_o(v), \end{cases} \tag{6.2}$$

and the so–called *modified Cauchy problem*

$$\begin{cases} \dfrac{\partial \psi}{\partial t} & = \phi(t)(L_c^+ \psi + L_b^+ \psi + L_s^+ \psi), \\ \psi(v,0) = f_o(v), \end{cases} \tag{6.3}$$

where the L^+–operators are defined as follows

$$\begin{aligned} L_c^+\psi(v,t) = & \int_0^{v/2} \tau_c(w, v-w)\underset{+}{\psi}(w,t)\underset{+}{\psi}(v-w,t)\,\mathrm{d}w \\ & - \psi(v,t)\int_0^{v_m - v} \tau_c(w,v)|\psi(w,t)|\,\mathrm{d}w, \end{aligned} \tag{6.4}$$

$$\begin{aligned} L_b^+\psi(v,t) = & \int_v^{v_m} \alpha_2(s)\beta_2(s,v)\underset{+}{\psi}(s,t)\,\mathrm{d}s \\ & + \sum_{k=3}^{N}\int_v^{v_m} \alpha_k(s)\underset{+}{\psi}(s,t)\,\mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right)\,\mathrm{d}\sigma_{k-2} \\ & - \sum_{k=2}^{N} \alpha_k(v)\psi(v,t), \end{aligned} \tag{6.5}$$

$$\begin{aligned} L_s^+\psi(v,t) = & \int_{v_m}^{v_m+v} \lambda_2(s)\beta_2(s, s-v)\,\mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w)\underset{+}{\psi}(w,t)\underset{+}{\psi}(s-w,t)\,\mathrm{d}w \\ & + \sum_{k=3}^{N}\left[\int_{v_m}^{v_m+v} \lambda_k(s)\,\mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w)\underset{+}{\psi}(w,t)\underset{+}{\psi}(s-w,t)\,\mathrm{d}w \right. \\ & \left. \times \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}\right] \\ & - \psi(v,t)\int_{v_m - v}^{v_m} \tau_c(v,w)|\psi(w,t)|\,\mathrm{d}w, \end{aligned} \tag{6.6}$$

where, in writing L_s^+, we made use of (2.5).

Theorem 6.1. (POSITIVENESS) *Under assumptions from* (H1) *to* (H5)*, all bounded solutions to problem (6.3) are non–negative.*

Proof. Were $(\overline{v}, \overline{t})$ a negative minimum of ψ in the open region $[0, v_m] \times (0, t)$, then the inequality

$$\frac{\partial \psi}{\partial t}(\overline{v}, \overline{t}) \geq -P(\overline{v}, \overline{t})\psi(\overline{v}, \overline{t}) > 0,$$

where

$$\frac{P(v,t)}{\phi(t)} = \left\{ \int_0^{v_m} \tau_c(w,v)|\psi(w,t)| \, \mathrm{d}w + \sum_{k=2}^{N} \alpha_k(v) \right\} > 0$$

leads to a contradiction.

Remark 1. An alternative (but slightly longer) proof of Theorem 6.1 is the following. Because of the definition of the L^+–operators, we have

$$\frac{\partial \psi}{\partial t}(v,t) \geq -P(v,t)\psi(v,t),$$

where

$$\begin{aligned} P(v,t) &:= \phi(t) \left\{ \int_0^{v_m - v} \tau_c(w,v)|\psi(w,t)| \, \mathrm{d}w + \sum_{k=2}^{N} \alpha_k(v) + \int_{v_m - v}^{v_m} \tau_c(v,w)|\psi(w,t)| \, \mathrm{d}w \right\} \\ &= \phi(t) \left\{ \int_0^{v_m} \tau_c(w,v)|\psi(w,t)| \, \mathrm{d}w + \sum_{k=2}^{N} \alpha_k(v) \right\} \geq 0. \end{aligned}$$

Moreover

$$P(v,t) \leq C \left(1 + \sum_{k=2}^{N} \alpha_k(v) \right)$$

for a suitable constant $C > 0$. Thus

$$\frac{\partial \psi}{\partial t}(v,t) \geq -C \left(1 + \sum_{k=2}^{N} \alpha_k(v) \right) \psi(v,t),$$

so that

$$\psi(v,t) \exp \left[C \left(1 + \sum_{k=2}^{N} \alpha_k(v) \right) t \right] \geq \psi(v,0) = f_o(v) \geq 0,$$

and ψ turns out to be non–negative.

Corollary 6.1. *If ψ is a bounded solution of (6.3), then it is also a solution to problem (6.2).*

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