Dynamics of droplets in an agitated dispersion with multiple breakage. Part II: uniqueness and global existence

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Abstract. In Part I (see [2]) a new model for the evolution of a system of droplets dispersed in an agitated liquid was presented. Our aim was to extend a previous version (see [4]) in order to describe the influence of each breakage mode. Here we complete the mathematical analysis to ensure the well posedness (in the sense of Hadamard) of the Cauchy problem for the main evolution equation.

1. Introduction

In Part I of this paper (see [2] this same volume) we presented a very general model for spatially homogeneous liquid dispersions in which any possible rupture mode is considered and the corresponding breakage frequencies are allowed to blow up as the droplet volume approaches a critical finite upper bound. Namely we allow a parent droplet to break in at most N pieces where N can be any finite positive integer greater or equal than two. The breakage frequency α_k of the $k-$th mode is allowed to tend to infinity as v tends to v_m (as in [1]). The main purpose of Part I was to show how to deal with the probability functions for each breakage mode and to verify the physical consistency of the whole model. Such a model generalized the one proposed in [4] (limited to binary breakage and with bounded breakage frequency), where the so–called *volume scattering operator* was introduced, preventing the appearance of droplets beyond a critical size (depending on the agitation speed).

Here we prove the well posedness (in the sense of Hadamard) of the Cauchy problem for the evolution equation derived in Part I.

In the final section of the paper we work out the specific example in which $N=3$, in order to show that it is indeed possible to construct a probability density function which meets all the requirements that the mathematics and the physics suggest.

2. Mathematical model

The reader should refer to Part I of this paper (see [2]) for all symbols and functions. For the reader's convenience we rewrite the main operators and the evolution equation: these are

$$\begin{aligned} L_c f(v,t) = & \int_0^{v/2} \tau_c(w, v-w) f(w,t) f(v-w,t) \, \mathrm{d}w \\ & - f(v,t) \int_0^{v_m - v} \tau_c(w,v) f(w,t) \, \mathrm{d}w, \end{aligned} \tag{2.1}$$

$$\begin{aligned} L_b f(v,t) = & \int_v^{v_m} \alpha_2(s) \beta_2(s,v) f(s,t) \, \mathrm{d}s \\ & + \sum_{k=3}^{N} \int_v^{v_m} \alpha_k(s) f(s,t) \, \mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s-v-U_{k-2}\right) \, \mathrm{d}\sigma_{k-2} \\ & - \sum_{k=2}^{N} \alpha_k(v) f(v,t), \end{aligned} \tag{2.2}$$

$$\begin{aligned} L_s f(v,t) = & \int_{v_m}^{v_m+v} \lambda_2(s) \beta_2(s, s-v) \, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w, w) f(w,t) f(s-w,t) \, \mathrm{d}w \\ & + \sum_{k=3}^{N} \left[\int_{v_m}^{v_m+v} \lambda_k(s) \, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w) f(w,t) f(s-w,t) \, \mathrm{d}w \right. \\ & \left. \times \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s-v-U_{k-2}\right) \, \mathrm{d}\sigma_{k-2} \right] \\ & - f(v,t) \int_{v_m}^{v_m+v} \tau_c(v, s-v) f(s-v,t) \, \mathrm{d}s, \end{aligned} \tag{2.3}$$

and

$$\frac{\partial f}{\partial t} = \phi(t) \left(L_c f + L_b f + L_s f \right) . \tag{2.4}$$

As in [4] we look for a solution– in a suitable class of regular functions f to be specified later – to both the *original Cauchy problem*

$$\begin{cases} \dfrac{\partial f}{\partial t} & = \phi(t)(L_c f + L_b f + L_s f), \\ f(v,0) = f_o(v), \end{cases} \tag{2.5}$$

and the so–called *modified Cauchy problem*

$$\begin{cases} \dfrac{\partial \psi}{\partial t} & = \phi(t)(L_c^+ \psi + L_b^+ \psi + L_s^+ \psi), \\ \psi(v,0) = f_o(v), \end{cases} \tag{2.6}$$

where the L^+–operators have been defined in Part I.

The reader should now bear in mind hypotheses (H1) to (H5) we stated in Part I.

Lemma 2.1. *Under assumptions from* (H1) *to* (H5)*, all bounded solutions to problem (2.6) satisfy the following conditions*

$$\psi(v_m, t) = \lim_{v \to v_m^-} \psi(v,t) = 0, \tag{2.7}$$

$$\psi(0, t) = \lim_{v \to 0^+} \psi(v,t) = 0. \tag{2.8}$$

Proof. Let us write problem (2.6) as follows

$$\frac{\partial \psi}{\partial t}(v,t) + \phi(t) \sum_{k=2}^{N} \alpha_k(v) \psi(v,t) = A(v,t) \tag{2.9}$$

where, because of Theorem 6.1 in Part I,

$$A(v,t) = \phi(t)\Bigg\{\int_0^{v/2} \tau_c(w,v-w)\psi_+(w,t)\psi_+(v-w,t)\ \mathrm{d}w - \psi(v,t)\int_0^{v_m-v} \tau_c(w,v)|\psi(w,t)|\ \mathrm{d}w$$

$$+\int_v^{v_m} \alpha_2(s)\beta_2(s,v)\psi(s,t)\ \mathrm{d}s$$

$$+\sum_{k=3}^{N}\int_v^{v_m} \alpha_k(s)\psi_+(s,t)\ \mathrm{d}s\int_{D_k(s,v)} \beta_k\left(s,u_1,\ldots,u_{k-2},s-v-U_{k-2}\right)\ \mathrm{d}\sigma_{k-2}$$

$$+\int_{v_m}^{v_m+v} \lambda_2(s)\beta_2(s,s-v)\ \mathrm{d}s\int_{s-v_m}^{s/2} \tau_c(s-w,w)\psi_+(w,t)\psi_+(s-w,t)\ \mathrm{d}w$$

$$+\sum_{k=3}^{N}\Bigg[\int_{v_m}^{v_m+v} \lambda_k(s)\ \mathrm{d}s\int_{s-v_m}^{s/2} \tau_c(s-w,w)\psi_+(w,t)\psi_+(s-w,t)\ \mathrm{d}w$$

$$\times\int_{D_k(s,v)} \beta_k\left(s,u_1,\ldots,u_{k-2},s-v-U_{k-2}\right)\ \mathrm{d}\sigma_{k-2}\Bigg] - \psi(v,t)\int_{v_m-v}^{v_m} \tau_c(v,w)|\psi(w,t)|\ \mathrm{d}s\Bigg\}$$

If we put $F(t) = \displaystyle\int_0^t \phi(u)\ \mathrm{d}u$, then (2.9) implies

$$\psi(v,t) = f_o(v)\exp\left(-\sum_{k=2}^{N}\alpha_k(v)F(t)\right) + \int_0^t A(v,\tau)\exp\left[-\sum_{k=2}^{N}\alpha_k(v)(F(t)-F(\tau))\right]\ \mathrm{d}\tau, \qquad v\in[0,v_m). \tag{2.10}$$

If C is an upper bound for $\phi\cdot\tau_c\cdot\psi^2$, then

$$
\begin{aligned}
\lim_{v\to v_m^-} A(v,t) = \phi(t) \Bigg\{ & \int_0^{v_m/2} \tau_c(w, v_m - w)\psi_+(w,t)\psi_+(v_m - w, t)\ \mathrm{d}w \\
& + \int_{v_m}^{2v_m} \lambda_2(s)\beta_2(s, s-v)\ \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w, w)\psi_+(w,t)\psi_+(s-w,t)\ \mathrm{d}w \\
& + \sum_{k=3}^{N} \Bigg[\int_{v_m}^{2v_m} \lambda_k(s)\ \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w)\psi_+(w,t)\psi_+(s-w,t)\ \mathrm{d}w \\
& \times \int_{D_k(s,v_m)} \beta_k\left(s, u_1, \dots, u_{k-2}, s - v_m - U_{k-2}\right)\ \mathrm{d}\sigma_{k-2} \Bigg] \\
& - \psi(v_m, t) \int_0^{v_m} \tau_c(v,w)|\psi(w,t)|\ \mathrm{d}s \Bigg\} \\
& \leq C \left(\frac{v_m}{2} + v_m \int_{v_m}^{2v_m} \lambda_2(s)\beta_2(s,v)\ \mathrm{d}s \right. \\
& \left. + v_m \sum_{k=3}^{N} \int_{v_m}^{2v_m} \lambda_k(s)\ \mathrm{d}s \int_{D_k(s,v_m)} \beta_k\left(s, u_1, \dots, u_{k-2}, s - v_m - U_{k-2}\right)\ \mathrm{d}\sigma_{k-2} \right)
\end{aligned}
$$

Recalling hypotheses (ii)-(a) and (iii)-(a) of (H5), we have, after a redefinition of C, that

$$
\lim_{v\to v_m^-} A(v,t) \leq C \tag{2.11}
$$

Thus from(2.10), being also $f_o(v_m) = 0$, it follows

$$
\psi(v_m, t) \leq C \lim_{v\to v_m^-} \int_0^t \exp\left[-\sum_{k=2}^{N} \alpha_k(v)(F(t) - F(\tau)) \right]\ \mathrm{d}\tau. \tag{2.12}
$$

Recall now that, because of (H1),

$$
F(t) - F(\tau) = \int_\tau^t \phi(u)\ \mathrm{d}u > \widehat{\Psi}(t-\tau),
$$

and so

$$
\psi(v_m, t) \leq C \lim_{v\to v_m^-} \int_0^t \exp\left[-\widehat{\Psi} \sum_{k=2}^{N} \alpha_k(v)(t-\tau) \right]\ \mathrm{d}\tau. \tag{2.13}
$$

If we put $\Lambda(v) = \widehat{\Psi} \sum\limits_{k=2}^{N} \alpha_k(v)$, we have

$$\psi(v_m,t) \le C \lim_{v\to v_m^-} \frac{1-\exp\left[-\Lambda(v)t\right]}{\Lambda(v)} = 0. \tag{2.14}$$

since (recall hypothesis (H3)-(a)) $\Lambda(v)$ goes to infinity as $v \to v_m^-$.

To prove (2.8) we proceed similarly: notice that

$$\begin{aligned}\lim_{v\to 0^+} A(v,t) = \phi(t)\Bigg[-\psi(0,t)\int_0^{v_m} \tau_c(0,w)|\psi(w,t)|\ \mathrm{d}s + \int_0^{v_m} \alpha_2(s)\beta_2(s,0)\ \mathrm{d}s \\ + \sum_{k=3}^{N}\int_0^{v_m} \alpha_k(s)\psi_+(s,t)\ \mathrm{d}s \int_{D_k(s,0)} \beta_k\left(s,u_1,\ldots,u_{k-2},s-U_{k-2}\right)\ \mathrm{d}\sigma_{k-2}\Bigg].\end{aligned} \tag{2.15}$$

Hypothesis (i) of (H5) and Theorem 6.1in Part I imply that $\lim_{v\to 0^+} A(v,t) \le 0$; consequently, recalling once more Theorem 6.1, relation (2.10) and that $f_o(0)=0$, we get (2.8).

□

3. Well posedness of the Cauchy problem

In this section we prove that the Cauchy problem (2.5) is well posed and that the unique solution exists at any time.

Theorem 3.1. (UNIQUENESS) *Under assumptions from (H1) to (H5), problem (2.5) has at most one bounded solution.*

Proof. Let f_1, f_2 be two bounded solutions to (2.5) such that $f_1(v,0) = f_2(v,0) = f_o(v)$ and let us define

$$\gamma(v,t) = f_1(v,t) - f_2(v,t)$$

$$\gamma_M(t) = ||\gamma(v,t)|| \qquad \text{(where} \quad ||\bullet|| \equiv \sup_{(0,v_m)} |\bullet|)$$

$$\phi_i \quad = \Psi\left[\int_0^{v_m} f_i(v,t)\ \mathrm{d}v, \int_0^{v_m} v^{(2/3)} f_i(v,t)\ \mathrm{d}v\right], \qquad i=1,2.$$

The difference function γ obeys the equation

$$\frac{\partial\gamma}{\partial t} = H_c(v,t) + H_s(v,t) + \phi_1(t)L_b f_1(v,t) - \phi_2(t)L_b f_2(v,t) \tag{3.1}$$

where H_c and H_s can be written as

$$H_\xi(v,t) = \phi_1(t)\,[L_\xi f_1(v,t) - L_\xi f_2(v,t)] + [\phi_1(t) - \phi_2(t)]\,L_\xi f_2(v,t), \qquad \xi = c, s.$$

The last two terms on the right hand side of (3.1) rewrite as follows

$$\phi_1(t) L_b f_1(v,t) - \phi_2(t) L_b f_2(v,t) = -\phi_2(t) \sum_{k=2}^{N} \alpha_k(v)\gamma(v,t) + H_b(v,t),$$

where

$$\begin{aligned}
H_b(v,t) = &- \sum_{k=2}^{N} \alpha_k(v)\,[\phi_1(t) - \phi_2(t)]\,f_1(v,t) \\
&+ \phi_1(t) \int_v^{v_m} \alpha_2(s)\beta_2(s,v) f_1(s,t)\,\mathrm{d}s - \phi_2(t) \int_v^{v_m} \alpha_2(s)\beta_2(s,v) f_2(s,t)\,\mathrm{d}s \\
&+ \phi_1(t) \sum_{k=3}^{N} \int_v^{v_m} \alpha_k(s) f_1(s,t)\,\mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right)\,\mathrm{d}\sigma_{k-2} \\
&- \phi_2(t) \sum_{k=3}^{N} \int_v^{v_m} \alpha_k(s) f_2(s,t)\,\mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}.
\end{aligned}$$

Thus equation (3.1) rewrites

$$\frac{\partial \gamma}{\partial t} + \phi_2(t) \sum_{k=2}^{N} \alpha_k(v)\gamma(v,t) = H_c(v,t) + H_b(v,t) + H_s(v,t). \tag{3.2}$$

Because of our hypotheses, the following estimates can immediately be proved

$$|H_c(v,t)| \le A_1 \gamma_M(t) + B_1 |\phi_1(t) - \phi_2(t)|,$$

$$|H_s(v,t)| \le A_3 \gamma_M(t) + B_3 |\phi_1(t) - \phi_2(t)|,$$

Because of (H1) we have

$$|\Psi(\xi_1,\eta_1) - \Psi(\xi_2,\eta_2)| \le L\,|||(\xi_1,\eta_1) - (\xi_2,\eta_2)|||$$

where $|||\bullet|||$ denotes the Euclidean length in $\mathbb{R}^2$ and $L > 0$ is a suitable constant. Consequently we have

$$
\begin{aligned}
&|\phi_2 - \phi_1| \\
&\le L \left|\left|\left| \left(\int_0^{v_m} f_1(v,t)\, \mathrm{d}v, \int_0^{v_m} v^{2/3} f_1(v,t)\, \mathrm{d}v \right) - \left(\int_0^{v_m} f_2(v,t)\, \mathrm{d}v, \int_0^{v_m} v^{2/3} f_2(v,t)\, \mathrm{d}v \right) \right|\right|\right| \\
&\le L \left|\left|\left| \left(\int_0^{v_m} (f_1(v,t) - f_2(v,t))\, \mathrm{d}v, \int_0^{v_m} v^{2/3} (f_1(v,t) - f_2(v,t))\, \mathrm{d}v \right) \right|\right|\right| \\
&\le \widehat{L} \gamma_M(t)
\end{aligned}
\tag{3.3}
$$

Recalling Lemma 3.1 and hypotheses (H5), we have

$$
|H_b(v,t)| \le A_2 \gamma_M(t) + B_2 |\phi_1(t) - \phi_2(t)|
$$

and, consequently, the estimate (3.3) finally implies

$$
|\mathcal{H}(v,t)| \equiv |H_c(v,t) + H_b(v,t) + H_s(v,t)| \le C \gamma_M(t)
$$

for a suitable positive constant C (which evidently depends also upon $\sup_{[0,v_m]} f$). Now, by integrating (3.2) and recalling that $\gamma(v,0) = 0$, we get

$$
\gamma(v,t) = \int_0^t \mathcal{H}(v,\tau) \exp\left(-\sum_{k=2}^{N} \alpha_k(v) \int_\tau^t \phi_2(u)\, \mathrm{d}u \right) \mathrm{d}\tau
$$

which in turn implies

$$
|\gamma(v,t)| \le \int_0^t |\mathcal{H}(v,\tau)|\, \mathrm{d}\tau \le C \int_0^t \gamma_M(\tau)\, \mathrm{d}\tau, \qquad \forall v \in [0, v_m].
$$

Thus

$$
0 \le \gamma_M(t) \le C \int_0^t \gamma_M(\tau)\, \mathrm{d}\tau
$$

which, by Gronwall's Lemma, implies $\gamma_M(t) \equiv 0$.

□

We now prove that problem (2.6) has a local bounded solution provided that the initial data go to zero sufficiently fast as v goes to v_m. Because of positivity (see Theorem 6.1 in Part I), all bounded solutions to problem (2.6) with initial data $f_o(v)$ also satisfy problem (2.5) with the same data. Moreover, because of the uniqueness theorem, to achieve the existence of solutions to problem (2.5), it suffices to prove it for problem (2.6).

Theorem 3.2. (LOCAL EXISTENCE) *Assume that hypotheses from (H1) to (H5) be satisfied and that $f_o(v)$ obeys both conditions*

$$\begin{aligned} &f_o(v) \textit{ is piecewise continuously differentiable in } [0, v_m], \\ &f_o(v) \textit{ is non–negative in } [0, v_m], \\ &f_o(0) = f_o(v_m) = 0. \end{aligned} \tag{3.4}$$

and

$$||\alpha_k'(v) f_o(v)|| < +\infty, \qquad \forall k = 2, \ldots, N. \tag{3.5}$$

Then problem (2.6) has at least one Lipschitz continuous solution in $[0, v_m] \times [0, T)$ for a suitable finite $T > 0$.

Proof. Let $0 = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_n$ be a monotone increasing finite sequence of time steps. In each interval $(\vartheta_i, \vartheta_{i+1})$ we consider the problem

$$\begin{cases} \dfrac{\partial \psi^{(n)}}{\partial t}(v,t) = -\phi_i \psi^{(n)}(v,t) \left\{ \displaystyle\sum_{k=2}^{N} \alpha_k(v) + \int_0^{v_m} \tau_c(w,v) \left| \psi_{+,i}^{(n)}(w) \right| \, \mathrm{d}w \right\} \\ \qquad + \phi_i \left\{ \displaystyle\int_0^{v/2} \tau_c(w, v-w) \psi_{+,i}^{(n)}(w) \psi_{+,i}^{(n)}(v-w) \, \mathrm{d}w \right. \\ \qquad + \displaystyle\int_v^{v_m} \alpha_2(s) \beta_2(s,v) \psi_{+,i}^{(n)}(s) \, \mathrm{d}s \\ \qquad + \displaystyle\sum_{k=3}^{N} \int_v^{v_m} \alpha_k(s) \psi_{+,i}^{(n)}(s) \, \mathrm{d}s \int_{D_k(s,v)} \beta_k \left(s, u_1, \ldots, u_{k-2}, s-v-U_{k-2} \right) \, \mathrm{d}\sigma_{k-2} \\ \qquad + \displaystyle\int_{v_m}^{v_m+v} \lambda_2(s) \beta_2(s, s-v) \, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w, w) \psi_{+,i}^{(n)}(w) \psi_{+,i}^{(n)}(s-w) \, \mathrm{d}w \\ \qquad + \displaystyle\sum_{k=3}^{N} \int_{v_m}^{v_m+v} \lambda_k(s) \, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w, w) \psi_{+,i}^{(n)}(w) \psi_{+,i}^{(n)}(s-w) \, \mathrm{d}w \\ \qquad \left. \times \displaystyle\int_{D_k(s,v)} \beta_k \left(s, u_1, \ldots, u_{k-2}, s-v-U_{k-2} \right) \, \mathrm{d}\sigma_{k-2} \right\}, \\ \psi^{(n)}(v, \vartheta_i) = \psi_{+,i}^{(n)}(v), \end{cases} \tag{3.6}$$

where the subscript "$_{,i}$" means that the corresponding function is evaluated in the limit $\vartheta \to \vartheta_i^-$. For $i = 0$, functions are evaluated in $t = 0$. At this stage of the proof we suppose that $\psi_i^{(n)}$ is bounded.

Equation (3.6)–1 can be written as

$$\frac{\partial \psi^{(n)}}{\partial t}(v,t) = -\mathcal{A}_i(v)\psi^{(n)}(v,t) + \mathcal{B}_i(v) \tag{3.7}$$

where $\mathcal{A}_i(v) = \phi_i \widetilde{\mathcal{A}}_i(v)$, $\mathcal{B}_i(v) = \phi_i \widetilde{\mathcal{B}}_i(v)$, $\widetilde{\mathcal{A}}_i$ and $\widetilde{\mathcal{B}}_i$ being evident from the context.

Hypotheses from (H1) to (H5) imply, for a suitable positive constant C, that

$$|\mathcal{B}_i(v)| \leq C\left(\left|\left|\psi_{+,i}^{(n)}\right|\right|^2 + \left|\left|\psi_{+,i}^{(n)}\right|\right|\right), \tag{3.8}$$

being $||\bullet|| \equiv \sup\limits_{(0,v_m)} (\bullet)$. By integrating (3.7) we get

$$\psi^{(n)}(v,t) = \psi_{+,i}^{(n)}(v) \exp\left[-\mathcal{A}_i(v)(t-\vartheta_i)\right] + \int_{\vartheta_i}^{t} \mathcal{B}_i(v) \exp\left[-\mathcal{A}_i(v)(t-\tau)\right]\, \mathrm{d}\tau. \tag{3.9}$$

Since $\mathcal{A}_i > 0$ we have

$$\left|\psi^{(n)}(v,t)\right| \leq \left|\left|\psi_{+,i}^{(n)}\right|\right| + |\mathcal{B}_i(v)|\,(t-\vartheta_i). \tag{3.10}$$

From (3.8) and (3.10) we get

$$\left|\psi^{(n)}(v,t)\right| \leq \left|\left|\psi_{+,i}^{(n)}\right|\right| \left\{1 + (\vartheta_{i+1}-\vartheta_i)\left(1 + C\left|\left|\psi_{+,i}^{(n)}\right|\right|\right)\right\}, \qquad t \in (\vartheta_{i+1}-\vartheta_i)\,. \tag{3.11}$$

We now choose an arbitrary $\varepsilon > 0$ and take

$$\vartheta_{i+1} - \vartheta_i = \frac{\varepsilon}{1 + C\,||f_o||\,(1+\varepsilon)^i} \tag{3.12}$$

so that, (3.11) gives

$$\begin{aligned} \left|\left|\psi_{i+1}^{(n)}\right|\right| &\leq \left|\left|\psi_{+,i}^{(n)}\right|\right| \left\{1 + \frac{\varepsilon}{1 + C\,||f_o||\,(1+\varepsilon)^i}\left(1 + C\left|\left|\psi_{+,i}^{(n)}\right|\right|\right)\right\}, \qquad t \in (\vartheta_{i+1}, \vartheta_i) \\ &\leq \left|\left|\psi_{i}^{(n)}\right|\right| \left\{1 + \frac{\varepsilon}{1 + C\,||f_o||\,(1+\varepsilon)^i}\left(1 + C\left|\left|\psi_{i}^{(n)}\right|\right|\right)\right\}, \qquad t \in (\vartheta_{i+1}, \vartheta_i) \end{aligned} \tag{3.13}$$

Recalling that $\psi_0^{(n)} = f_o$ for any n, it is immediate to check, by induction from (3.13), that

$$\left|\left|\psi_i^{(n)}\right|\right| \leq ||f_o||\,(1+\varepsilon)^i \qquad \forall i = 0,\ldots,n, \qquad \forall n \in \mathbb{N}. \tag{3.14}$$

Assume now $\varepsilon = 1/n$ and notice that

$$(1+\varepsilon)^i = (1+1/n)^i \leq (1+1/n)^n \leq e, \qquad \forall i \leq n, \quad \forall n \in \mathbb{N}.$$

Therefore, from (3.14) and (3.11), it follows that

$$\left|\psi^{(n)}(v,t)\right| \leq e\,||f_o||\,. \tag{3.15}$$

As far as the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ is concerned, it is immediately seen that

$$\vartheta_n = \frac{1}{C\,||f_o||}\varepsilon\sum_{h=0}^{n-1}(1+\varepsilon)^{-h} = \frac{1}{C\,||f_o||\,(1+\varepsilon)}\left(1-\frac{1}{(1+\varepsilon)^n}\right);$$

Thus, with the choice made for ε,

$$\lim_{n\to+\infty}\vartheta_n = \frac{1}{C\,||f_o||}\left(\frac{e-1}{e}\right) \equiv T. \tag{3.16}$$

In conclusion the sequence $\left|\psi^{(n)}(v,t)\right|$ is uniformly bounded in $[0,v_m]\times[0,T]$.

We prove that, in the same region of $\mathbb{R}^2$, the two partial derivatives of $\psi^{(n)}$ are uniformly bounded too. To this aim we first multiply (3.9) by $\alpha_k(v)$ to obtain

$$\alpha_k(v)\left|\psi^{(n)}(v,t)\right| \leq \alpha_k(v)\psi_{+,i}^{(n)}(v)\exp[-\mathcal{A}_i(v)(t-\vartheta_i)] + \alpha_k(v)\int_{\vartheta_i}^{t}\mathcal{B}_i(v)\exp[-\mathcal{A}_i(v)(t-\tau)]\,\mathrm{d}\tau. \tag{3.17}$$

Because of (3.15), estimate (3.8) can also be written as

$$|\mathcal{B}_i(v)| \leq \widetilde{C}\left|\left|\psi_{+,i}^{(n)}\right|\right| \leq \widetilde{C}e\,||f_o||\,. \tag{3.18}$$

Let us now define

$$d_{n,j}^{(k)} = \left|\left|\alpha_k\left|\psi_j^{(n)}\right|\right|\right| \tag{3.19}$$

Notice that $d_{n,0}^{(k)}$ is bounded for all k. Indeed it suffices to recall that $\psi_0^{(n)} = f_o$ and to proceed as in Lemma 3.1: evaluating (3.17) in ϑ_{i+1} we get

$$
\begin{aligned}
d_{n,i+1}^{(k)} &\le d_{n,i}^{(k)}(1+\widetilde{C}(t-\vartheta_i)) \\
&\le d_{n,i}^{(k)}(1+\widetilde{C}(\vartheta_{i+1}-\vartheta_i)) \le d_{n,i}^{(k)}(1+\overline{C}\frac{\varepsilon}{(1+\varepsilon)^i}) \\
&\le d_{n,i}^{(k)}(1+\overline{C}\varepsilon).
\end{aligned}
$$

Thus

$$
d_{n,i}^{(k)} \le d_{n,0}^{(k)}(1+\overline{C}\varepsilon)^i.
$$

For $\varepsilon = 1/n$, the sequence $\left\{d_{n,i}^{(k)}\right\}_i$ with $i = 0, \dots, n-1$ turns out to be bounded for all $n \in \mathbb{N}$. If v goes to v_m^-, being in that case α_k unbounded for all k, (3.9) also implies $\lim_{v\to v_m^-} \psi^{(n)}(v,t) = \psi^{(n)}(v_m,t) = 0$. From (3.7) and the boundedness of $\psi^{(n)}(v,t)$ it also follows that $\left|\frac{\partial\psi^{(n)}}{\partial t}\right|$ is uniformly bounded in the same region of $\mathbb{R}^2$.

To estimate $\left|\frac{\partial\psi^{(n)}}{\partial v}\right|$ we first proceed formally, the correctness of this approach being justifiable *a posteriori*. Let us define

$$
\begin{aligned}
\mathfrak{S}_1^{(i,n)}(v) &= \phi_i \int_0^{v_m} \tau_c(v,w) \left|\psi_i^{(n)}(w)\right| \, \mathrm{d}w, \\
\mathfrak{S}_2^{(i,n)}(v) &= \phi_i \left\{ \int_0^{v/2} \tau_c(w, v-w)\psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(v-w)\, \mathrm{d}w \right. \\
&\quad + \int_{v_m}^{v_m+v} \lambda_2(s)\beta_2(s,v)\, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w)\psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(s-w)\, \mathrm{d}w \\
&\quad + \sum_{k=3}^{N} \int_{v_m}^{v_m+v} \lambda_k(s)\, \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w)\psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(s-w)\, \mathrm{d}w \\
&\quad \left. \times \int_{D_k(s,v)} \beta_k\left(s, u_1, \dots, u_{k-2}, s-v-U_{k-2}\right)\, \mathrm{d}\sigma_{k-2} \right\}, \\
\mathfrak{S}_3^{(i,n)}(v) &= \phi_i \left\{ \int_v^{v_m} \alpha_2(s)\beta_2(s,v)\psi_{+,i}^{(n)}(s)\, \mathrm{d}s \right. \\
&\quad \left. + \sum_{k=3}^{N} \int_v^{v_m} \alpha_k(s)\psi_{+,i}^{(n)}(s)\, \mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \dots, u_{k-2}, s-v-U_{k-2}\right)\, \mathrm{d}\sigma_{k-2} \right\},
\end{aligned}
$$

so that the first of (3.6) rewrites

$$\frac{\partial \psi^{(n)}}{\partial t} = -\phi_i \psi^{(n)}(v,t) \sum_{k=2}^{N} \alpha_k(v) - \mathfrak{S}_1^{(i,n)}(v)\psi^{(n)}(v,t) + \left(\mathfrak{S}_2^{(i,n)}(v) + \mathfrak{S}_3^{(i,n)}(v)\right), \tag{3.20}$$

for each t in the interval $(\vartheta_i, \vartheta_{i+1})$. Because of the second of (3.6) we also have

$$\frac{\mathrm{d}\psi^{(n)}(v,\vartheta_i)}{\mathrm{d}v} = \frac{\mathrm{d}\psi_{+,i}^{(n)}(v)}{\mathrm{d}v}. \tag{3.21}$$

Differentiate now (3.20) with respect to v to get

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \psi^{(n)}(v,t)}{\partial v} &= -\phi_i \sum_{k=2}^{N} \alpha_k'(v)\psi^{(n)}(v,t) - \phi_i \sum_{k=2}^{N} \alpha_k(v)\frac{\partial \psi^{(n)}(v,t)}{\partial v} \\ &-\frac{\mathrm{d}\mathfrak{S}_1^{(i,n)}}{\mathrm{d}v}(v)\psi^{(n)}(v,t) - \mathfrak{S}_1^{(i,n)}(v)\frac{\partial \psi^{(n)}(v,t)}{\partial v} \\ &+\frac{\mathrm{d}\mathfrak{S}_2^{(i,n)}}{\mathrm{d}v}(v) + \frac{\mathrm{d}\mathfrak{S}_3^{(i,n)}}{\mathrm{d}v}(v), \\ &t \in (\vartheta_i, \vartheta_{i+1}), \qquad i = 0, \dots, n. \end{aligned} \tag{3.22}$$

Because of the hypotheses from (H1) to (H5) and of the boundedness of $\alpha_k \psi^{(n)}$ (proved above) it turns out that all terms $\mathfrak{S}_h^{(i,n)}$ are uniformly bounded. Also notice that

$$\frac{\mathrm{d}\mathfrak{S}_1^{(i,n)}}{\mathrm{d}v}(v) = \int_0^{v_m} \frac{\partial \tau_c}{\partial v}\left|\psi_i^{(n)}(w)\right| \, \mathrm{d}w, \tag{3.23}$$

$$
\begin{aligned}
\frac{\mathrm{d}\mathfrak{S}_2^{(i,n)}}{\mathrm{d}v}(v) = \phi_i \Bigg\{ & \frac{1}{2}\tau_c\left(\frac{v}{2},\frac{v}{2}\right)\left[\psi_{+,i}^{(n)}\left(\frac{v}{2}\right)\right]^2 + \int_0^{v/2}\left(\frac{\partial \tau_c}{\partial u_2}\right)\Bigg|_{(u_1,u_2)=(w,v-w)} \psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(v-w)\,\mathrm{d}w \\
& + \int_0^{v/2}\tau_c(w,v-w)\psi_{+,i}^{(n)}(w)\left(\frac{\mathrm{d}\psi_{+,i}^{(n)}}{\mathrm{d}\xi}\right)\Bigg|_{\xi=v-w}\mathrm{d}w \\
& + \lambda_2(v_m+v)\beta_2(v_m+v,v)\int_v^{(v_m+v)/2}\tau_c(v_m+v-w,w)\psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(v_m+v-w)\,\mathrm{d}w \\
& - \int_{v_m}^{v_m+v}\lambda_2(s)\frac{\partial \beta_2(s,v)}{\partial v}\,\mathrm{d}s\int_{s-v_m}^{s/2}\tau_c(s-w,w)\psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(s-w)\,\mathrm{d}w \\
& + \sum_{k=3}^{N}\lambda_k(v_m+v)\int_v^{(v_m+v)/2}\tau_c(v_m+v-w,w)\psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(v_m+v-w)\,\mathrm{d}w \\
& \times \int_{D_k(v_m+v,v)}\beta_k\left(v_m+v,u_1,\ldots,u_{k-2},v_m-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2} \\
& + \sum_{k=3}^{N}\int_{v_m}^{v_m+v}\lambda_k(s)\,\mathrm{d}s\int_{s-v_m}^{s/2}\tau_c(s-w,w)\psi_{+,i}^{(n)}(w)\psi_{+,i}^{(n)}(s-w)\,\mathrm{d}w \\
& \times \frac{\partial}{\partial v}\left(\int_{D_k(s,v)}\beta_k\left(s,u_1,\ldots,u_{k-2},s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}\right)\Bigg\},
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\mathfrak{S}_3^{(i,n)}}{\mathrm{d}v}(v) = \phi_i \Bigg\{ & -\alpha_2(v)\beta_2(v,v)\psi_{+,i}^{(n)}(v) - \int_v^{v_m}\alpha_2(s)\frac{\partial \beta_2(s,v)}{\partial v}\psi_{+,i}^{(n)}(s)\,\mathrm{d}s \\
& - \sum_{k=3}^{N}\alpha_k(v)\psi_{+,i}^{(n)}(v)\int_{D_k(v,v)}\beta_k\left(v,0,0,\ldots,0\right)\,\mathrm{d}\sigma_{k-2} \\
& + \sum_{k=3}^{N}\int_v^{v_m}\alpha_k(s)\psi_{+,i}^{(n)}(s)\frac{\partial}{\partial v}\left(\int_{D_k(s,v)}\beta_k\left(s,u_1,\ldots,u_{k-2},s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}\right)\,\mathrm{d}s\Bigg\}
\end{aligned} \tag{3.25}
$$

Both (3.24) and (3.25) contain the term

$$
\frac{\partial}{\partial v}\left(\int_{D_k(s,v)}\beta_k\left(s,u_1,\ldots,u_{k-2},s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}\right)
$$

which, recalling the "transport theorem", can be written as

$$
-\int_{D_k(s,v)}\left(\frac{\partial}{\partial u_{k-1}}\beta_k\right)\Bigg|_{u_{k-1}=s-v-U_{k-2}}\mathrm{d}\sigma_{k-2} + \int_{\partial D_k(s,v)}\beta_k\Big|_{u_{k-1}=s-v-U_{k-2}}\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}\sigma_{k-3},
$$

where $\mathbf{V}$ is the rate of change of the boundary $\partial D_k(s,v)$ of $D_k(s,v)$ with respect to v and $\mathbf{n}$ is the outward unit normal to $\partial D_k(s,v)$. Notice that $\partial D_k(s,v)$ is the union of the intersections the hyperplanes $U_{k-1} = s - v$ with the boundary of $T_k(s)$. Since these hyperplanes depend linearly on v and the boundary of $T_k(s)$ is also made of hyperplanes, the vector $\mathbf{V}$ is independent of v itself. The unit vector $\mathbf{n}$, being normal to $\partial T_k(s)$, does not depend on v either. Therefore $\mathbf{V}\cdot\mathbf{n}$ is a piecewise continuous function on $\partial D_k(s,v)$ independent of v. Since $\partial D_k(s,v)$ is compact, $\mathbf{V}\cdot\mathbf{n}$ is bounded. Notice also that $D_k(v,v)$ has zero measure and that $\beta_2(v,0) = 0$. Then, recalling all hypotheses from (H1) to (H5) and the estimate (3.15) we have that

$$\left|\frac{\mathrm{d}\mathfrak{S}_1^{(i,n)}}{\mathrm{d}v}\right|, \left|\frac{\mathrm{d}\mathfrak{S}_3^{(i,n)}}{\mathrm{d}v}\right| \leq C_1 \tag{3.26}$$

for a suitable positive constant C_1. Furthermore, being

$$\frac{\mathrm{d}\psi_{+,i}^{(n)}}{\mathrm{d}v} \leq \left|\frac{\mathrm{d}\psi_i^{(n)}}{\mathrm{d}v}\right|, \tag{3.27}$$

we also get

$$\left|\frac{\mathrm{d}\mathfrak{S}_2^{(i,n)}}{\mathrm{d}v}(v)\right| \leq C_2\left(1 + \left\|\frac{\mathrm{d}\psi_i^{(n)}}{\mathrm{d}v}\right\|\right) \tag{3.28}$$

for a suitable constant $C_2 > 0$.

Let us set

$$\frac{\partial\psi^{(n)}}{\partial v}(v,t) := U_n(v,t),$$

$$\Omega^{(i,n)}(v) := \mathfrak{S}_1^{(i,n)}(v) + \phi_i \sum_{k=2}^{N} \alpha_k(v),$$

$$\Lambda^{(i,n)}(v,t) := \frac{\mathrm{d}\mathfrak{S}_2^{(i,n)}(v)}{\mathrm{d}v} + \frac{\mathrm{d}\mathfrak{S}_3^{(i,n)}(v)}{\mathrm{d}v} - \frac{\mathrm{d}\mathfrak{S}_1^{(i,n)}(v)}{\mathrm{d}v}\psi^{(n)}(v,t) - \phi_i \sum_{k=2}^{N} \alpha_k'(v)\psi^{(n)}(v,t)$$

so that (3.22) rewrites as

$$\frac{\partial U_n(v,t)}{\partial t} + \Omega^{(i,n)}(v)U_n(v,t) = \Lambda^{(i,n)}(v,t), \qquad t \in (\vartheta_i, \vartheta_{i+1}), \quad i = 0,\ldots,n, \tag{3.29}$$

being $U_n(v,0) = f_o'(v)$. By integrating (3.29) with respect to time we obtain

$$U_n(v,t) = U_n(v,\vartheta_i)\exp\left[-\Omega^{(i,n)}(v)(t-\vartheta_i)\right] + \int_{\vartheta_i}^{t} \Lambda^{(i,n)}(v,\tau)\exp\left[-\Omega^{(i,n)}(v)(t-\tau)\right]\,\mathrm{d}\tau. \tag{3.30}$$

From estimates (3.26), (3.28), we obtain

$$|\Lambda^{(i,n)}(v,\tau)| \leq C_3 \left(1 + |U_n(v,\vartheta_i)| + |\psi(v,\tau)| + \sum_{k=2}^{N} \alpha'_k(v)|\psi(v,\tau)| \right), \tag{3.31}$$

for a suitable positive constant C_3.

We now prove that the sequence

$$D^{(k)}_{n,i} = \sup_{(0,v_m)} \left(\alpha'_k(v) \left| \psi^{(n)}_i(v) \right| \right) \tag{3.32}$$

is bounded. For $i = 0$ we have $D^{(k)}_{n,0} = \sup\limits_{(0,v_m)} (\alpha'_k(v) f_o(v))$ which is bounded by assumption. Multiply now (3.9) by $\alpha'_k(v)$ to get

$$\begin{aligned} \alpha'_k(v)\psi^{(n)}(v,t) \leq \alpha'_k(v)\psi^{(n)}_{+,i}(v) \exp\left[-\mathcal{A}_i(v)(t-\vartheta_i)\right] \\ + |\mathcal{B}_i(v)|\, \alpha'_k(v) \int_{\vartheta_i}^{t} \exp\left[-\mathcal{A}_i(v)(t-\tau)\right]\, \mathrm{d}\tau. \end{aligned} \tag{3.33}$$

Recalling (3.18), inequality (3.33) implies

$$\alpha'_k(v)\psi^{(n)}(v,t) \leq \alpha'_k(v) \left|\left| \psi^{(n)}_i \right|\right| (1 + C_4\varepsilon), \tag{3.34}$$

where, once more, we made use of (3.12), and $C_4 > 0$ is a suitable constant. From (3.34) we get

$$D^{(k)}_{n,i+1} \leq D^{(k)}_{n,0}(1 + C_4\varepsilon)^{i+1}, \qquad (k = 2, \ldots, N) \tag{3.35}$$

for all $i = 0, \ldots, n-1$. Thus, for $\varepsilon = 1/n$, it follows that, for each $k = 2, \ldots, N$, the sequence $\alpha'_k(v) \left|\psi^{(n)}(v,t)\right|$ with $i = 0, \ldots, n$ and $n \in \mathbb{N}$ is uniformly bounded.

As a consequence – and recalling once again the boundedness of $\psi(v,\tau)$ – we can rewrite (3.31) as follows

$$|\Lambda^{(i,n)}(v,\tau)| \leq C_5(1 + |U_n(v,\vartheta_i)|), \tag{3.36}$$

for a suitable constant $C_5 > 0$.

Notice now that, being $\Omega^{(i,n)} > 0$, (3.30) and (3.36) immediately imply

$$|U_n(v,t)| \leq |U_n(v,\vartheta_i)| + C_5 \int_{\vartheta_i}^{t} (1 + |U_n(v,\vartheta_i)|)\, \mathrm{d}\tau, \qquad \forall t \in [\vartheta_{i+1}, \vartheta_i]; \tag{3.37}$$

therefore

$$|U_n(v,\vartheta_{i+1})| \leq |U_n(v,\vartheta_i)| + C_5\left(1+|U_n(v,\vartheta_i)|\right)(\vartheta_{i+1}-\vartheta_i) \tag{3.38}$$

Recalling the meaning of U_n, estimate (3.38) leads to the following

$$\left|\frac{\partial\psi^{(n)}(v,t)}{\partial v}\right| \leq \left\|\frac{\mathrm{d}\psi_i^{(n)}(v)}{\mathrm{d}v}\right\| + C_5\left(1+\left\|\frac{\mathrm{d}\psi_i^{(n)}}{\mathrm{d}v}\right\|\right)(\vartheta_{i+1}-\vartheta_i), \qquad \forall t\in[\vartheta_{i+1},\vartheta_i]. \tag{3.39}$$

Inequality (3.39) implies

$$1+\left\|\frac{\mathrm{d}\psi_{i+1}^{(n)}}{\mathrm{d}v}\right\| \leq \left(1+\left\|\frac{\mathrm{d}\psi_i^{(n)}}{\mathrm{d}v}\right\|\right) + C_5\left(1+\left\|\frac{\mathrm{d}\psi_i^{(n)}}{\mathrm{d}v}\right\|\right)(\vartheta_{i+1}-\vartheta_i). \tag{3.40}$$

which, recalling (3.12), in turn implies that

$$\left(1+\left\|\frac{\mathrm{d}\psi_{i+1}^{(n)}}{\mathrm{d}v}\right\|\right) \leq \left(1+\left\|\frac{\mathrm{d}\psi_i^{(n)}}{\mathrm{d}v}\right\|\right)(1+C_6\varepsilon) \leq \ldots \leq \left(1+\left\|\frac{\mathrm{d}f_o}{\mathrm{d}v}\right\|\right)(1+C_6\varepsilon)^{i+1}.$$

Since $\varepsilon = 1/n$, being also $i \leq n$, the boundedness of $\left\|\frac{\mathrm{d}\psi_{i+1}^{(n)}}{\mathrm{d}v}\right\|$ follows immediately. From (3.39) we conclude that all functions $\frac{\partial\psi^{(n)}(v,t)}{\partial v}$ are uniformly bounded over $[0,v_m]\times[0,T)$, which justifies *a posteriori* the formal procedure applied up to now. Ascoli–Arzelà's Theorem finally applies to guarantee that a converging subsequence can be extracted from $\psi^{(n)}(v,t)$ which is a solution of problem (2.6) in $[0,T)$ and the proof is complete.

□

Proposition 3.1. *Let f be a bounded solution to problem (2.5); then, under assumptions from* (H1) *to* (H5)*, all products*

$$\alpha_k(v)f(v,t), \qquad k=2,\ldots,N \tag{3.41}$$

are bounded for all $v\in[0,v_m]$.

Proof. Write problem (2.5) as follows

$$\frac{\partial f}{\partial t}(v,t) + \phi(t)\sum_{k=2}^{N}\alpha_k(v)f(v,t) = K(v,t) \tag{3.42}$$

where

$$K(v,t) = \phi(t) \left\{ \int_0^{v/2} \tau_c(w, v-w) f(w,t) f(v-w,t)\ \mathrm{d}w - f(v,t) \int_0^{v_m - v} \tau_c(w,v) f(w,t)\ \mathrm{d}w \right.$$

$$+ \int_v^{v_m} \alpha_2(s) \beta_2(s,v) f(s,t)\ \mathrm{d}s$$

$$+ \sum_{k=3}^{N} \int_v^{v_m} \alpha_k(s) f(s,t)\ \mathrm{d}s \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s - v - U_{k-2}\right)\ \mathrm{d}\sigma_{k-2}$$

$$+ \int_{v_m}^{v_m+v} \lambda_2(s) \beta_2(s, s-v)\ \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w, w) f(w,t) f(s-w,t)\ \mathrm{d}w$$

$$+ \sum_{k=3}^{N} \left[\int_{v_m}^{v_m+v} \lambda_k(s)\ \mathrm{d}s \int_{s-v_m}^{s/2} \tau_c(s-w,w) f(w,t) f(s-w,t)\ \mathrm{d}w \right.$$

$$\left. \left. \times \int_{D_k(s,v)} \beta_k\left(s, u_1, \ldots, u_{k-2}, s-v-U_{k-2}\right)\ \mathrm{d}\sigma_{k-2} \right] - f(v,t) \int_{v_m - v}^{v_m} \tau_c(v,w) f(w,t)\ \mathrm{d}s \right\}.$$

As in the proof of Lemma 6.2 in Part I, the hypotheses we made imply

$$|K(v,t)| \leq \overline{K} < +\infty$$

for a suitable positive constant $\overline{K}$.

Now integrate (3.42) and multiply by α_k to get

$$\begin{aligned} \alpha_k(v) f(v,t) = \alpha_k(v) f_o(v) \exp\left(-\sum_{h=2}^{N} \alpha_h(v) F(t) \right) \\ + \alpha_k(v) \int_0^t A(v,\tau) \exp\left[-\sum_{h=2}^{N} \alpha_h(v) (F(t) - F(\tau)) \right]\ \mathrm{d}\tau, \qquad v \in [0, v_m). \end{aligned} \tag{3.43}$$

Because of (3.4) we have

$$0 \leq \alpha_k(v) f_o(v) = \alpha_k(v) \left[f_o(v) - f_o(v_m) \right] \leq L \alpha_k(v) (v_m - v), \tag{3.44}$$

where $L > 0$ is a suitable constant. Thus, recalling (H3), we also have

$$0 \leq \alpha_k(v) f_o(v) \leq \widetilde{L} (v_m - v)^{1-\mu}, \qquad \mu \in (0,1). \tag{3.45}$$

Moreover, since $F(t) \geq \widehat{\Psi} t \geq 0$, we obtain

$$0 \leq \alpha_k(v) f_o(v) \exp\left(-\sum_{h=2}^{N} \alpha_h(v) F(t)\right) \leq \widetilde{L} v_m^{1-\mu}, \qquad \mu \in (0,1). \tag{3.46}$$

Therefore

$$\begin{aligned} \alpha_k(v) f(v,t) &\leq \widetilde{L} v_m^{1-\mu} + \overline{K} \alpha_k(v) \int_0^t \exp\left[-\sum_{h=2}^{N} \alpha_h(v) \left(F(t) - F(\tau)\right)\right] \mathrm{d}\tau \\ &\leq \widetilde{L} v_m^{1-\mu} + \overline{K} \alpha_k(v) \int_0^t \exp\left[-\sum_{h=2}^{N} \alpha_h(v) \widehat{\Psi}\,(t-\tau)\right] \mathrm{d}\tau, \qquad \mu \in (0,1). \end{aligned} \tag{3.47}$$

If we put $y = \sum_{k=2}^{N} \alpha_k(v) \widehat{\Psi}(t-\tau)$ we obtain

$$\begin{aligned} \alpha_k(v) f(v,t) &\leq \sum_{k=2}^{N} \alpha_k(v) f(v,t) \\ &\leq (N-1)\widetilde{L} v_m^{1-\mu} + \frac{\overline{K}}{\widehat{\Psi}} \int_0^{+\infty} \exp(-y)\, \mathrm{d}y \\ &= (N-1)\widetilde{L} v_m^{1-\mu} + \frac{\overline{K}}{\widehat{\Psi}}, \qquad \mu \in (0,1), \end{aligned} \tag{3.48}$$

which completes the proof.

□

Remark 1. We notice explicitly that the upper bound on $\alpha_k(v) f(v,t)$ depends on the upper bound on $f(v,t)$.

4. Global existence in time

In [3] we proved that the local solution f to (2.5) can be extended over an arbitrary large time interval remaining bounded. To this aim we made use of a global time estimate of the solution $\mathcal{N}$ of (the binary version of) equation (4.28) in Part I Indeed, for $k = 2$, that equation implies

$$\dot{\mathcal{N}}(t) \leq \phi(t) \int_0^{v_m} \alpha_2(s) f(s,t)\, \mathrm{d}s, \tag{4.1}$$

which, for α_2 bounded, gives $\mathcal{N}(t) \leq \mathcal{N}(0) \exp\left[\left(||\alpha_2|| \sup_{t\in[0,+\infty]} \phi\right) T\right]$. Being this estimate independent of the bound for f in $[0,T]$, directly from (2.4) one obtains

$$\{f\}_t \leq ||f_o|| + C(T) \int_0^t \{f\}_\vartheta \, \mathrm{d}\vartheta, \tag{4.2}$$

where $\{\bullet\}_t \equiv \sup_{(0,v_m)\times(0,t)} |\bullet|$ and $C(T)$ depends, besides on T, *only* on the bounds for $\alpha_2, \beta_2, \tau_c$ and ϕ. From (4.2) we immediately obtain that

$$|f(v,t)| \leq C^*(T), \tag{4.3}$$

and from this, with standard arguments, the possibility to make use of $f(v,T)$ as a new initial data for (2.4).

This approach however does not work in the present case mainly for two reasons:

(a) the scattering term gives a *non–negative* (quadratic in f) contribution to $\dot{\mathcal{N}}$ (see equation (4.28) in Part I). Therefore that equation appears unable to provide an estimate of $\mathcal{N}$ *independent* of the bound for f furnished in the proof of Theorem 3.2.
(b) Being α_k unbounded in $(0, v_m)$, also the non–negative linear term in (4.28) of Part I cannot be estimated through $\mathcal{N}$.

To overcome these difficulties and achieve a global existence result we need to follow a different approach. Not surprisingly, the right idea is directly suggested by the very physics of the problem.

A large number of droplets of (possibly) arbitrarily small size appearing because of breakage from larger droplets are generally observed only in exceptional situations (like the instantaneous break–up of a rather long filament of fluid, see for example Fig. 8 in [6]). If we disregard this and similar cases, it is reasonable to assume, in addition to all previous hypotheses made for α_k, also the following

$$\alpha_k(v) \equiv 0, \qquad \forall v \in [0, v^*_{\text{crit}}], \quad \forall k \geq 2 \tag{4.4}$$

where $0 < v^*_{\text{crit}} \ll v_m$ is a (small) threshold value (see Remark 3 in Section 2 of Part I). The same request is needed for τ_c: indeed two very small droplets are very unlikely to coalesce (see figures 4.1 and 4.2 taken from [5]), because of the large energy needed to drain and break the interposed separating film[1].

Therefore we also assume

[1] As we already mentioned in Remark 3 in Section 2 of Part I, one should eventually distinguish among various lower threshold values related to the different existing mechanisms that control evolution. This further complication has not been considered here although the mathematics involved does not change very much.

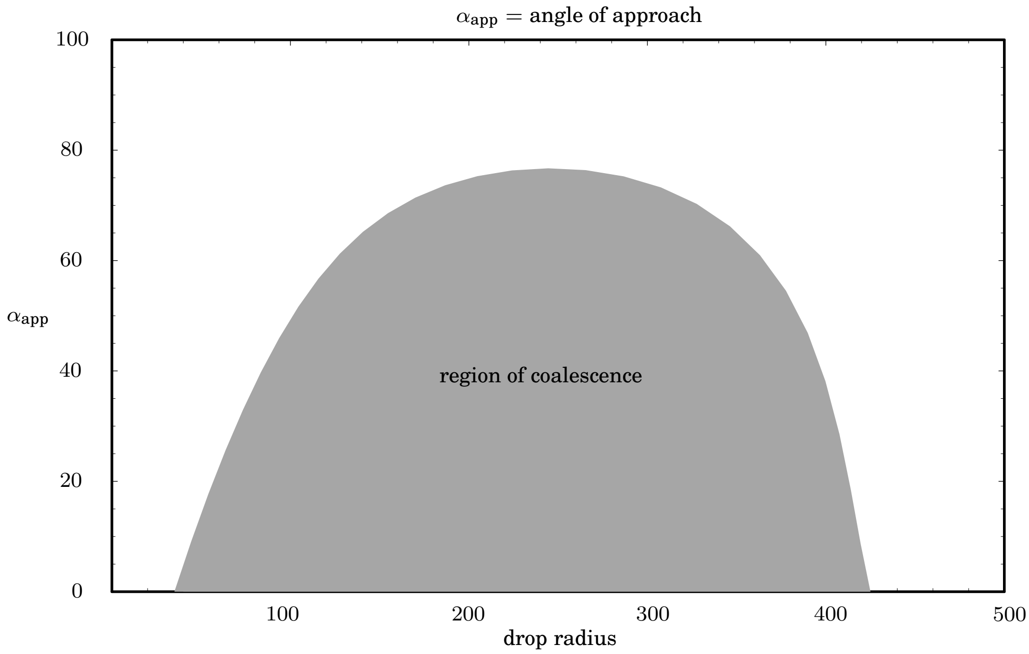

Fig. 4.1. Coalescence region for colliding drops of *equal size* (see [5]): very large and very small droplets do not coalesce regardless of the mutual angle of approach ($\alpha_{app} = 0°$ means "head-on collision", $\alpha_{app} = 90°$ means "grazing droplets")

$$\tau_c \equiv 0, \qquad \text{in} \quad [0, v^*_{\rm crit}] \times [0, v^*_{\rm crit}]. \tag{4.5}$$

Similarly, considering there is no chance to get a droplet of subcritical size as the final product of either a breakage or a scattering event, we need to impose

$$\beta_k \equiv 0, \qquad \text{if} \quad v \in [0, v^*_{\rm crit}]. \tag{4.6}$$

As a consequence the only physical mechanism remaining active for $v \in [0, v^*_{\rm crit}]$ is the loss of small droplets due to coalescence with ones of ordinary size, *i.e.* above threshold (see figures 4.1 and 4.2 again).

The additional assumptions (4.4), (4.5) and (4.6) have an immediate consequence on the behaviour of f in a right neighbourhood of $v = 0$. Indeed from (2.1), (2.2), (2.3), we get that

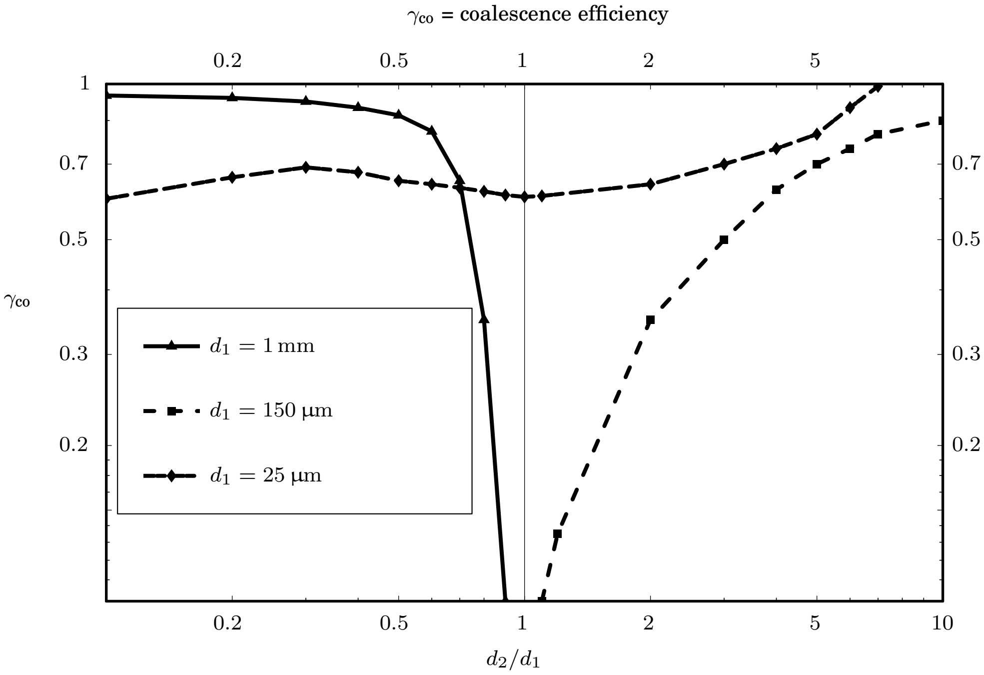

Fig. 4.2. Coalescence efficiency *vs.* droplets ratio (see [5]). This graph is in strict agreement that shown in Figure 4. Indeed the coalescence efficiency reduces to zero as the droplet ratio goes to one only for relatively "small" and "large" droplets

$$v \in [0, v^*_{\text{crit}}] \Rightarrow \begin{cases} L_c f(v,t) = -f(v,t) \displaystyle\int_{v^*_{\text{crit}}}^{v_m - v} \tau_c(w,v) f(w,t)\, \mathrm{d}w, \\ L_b f(v,t) = 0, \\ L_s f(v,t) = -f(v,t) \displaystyle\int_{v_m - v}^{v_m} \tau_c(w,v) f(w,t)\, \mathrm{d}w, \end{cases} \tag{4.7}$$

so that

$$\frac{\partial f}{\partial t} = -\phi(t) f(v,t) \int_{v^*_{\text{crit}}}^{v_m} \tau_c(w,v) f(w,t)\, \mathrm{d}w < 0, \qquad \forall v \in [0, v^*_{\text{crit}}]. \tag{4.8}$$

Consequently

$$f(v,t) \leq f_o(v), \qquad (v,t) \in [0, v^*_{\text{crit}}] \times [0,T]. \tag{4.9}$$

Relation (4.9) implies that, because of the conservation of volume (Theorem 5.1 in Part I), also the number of droplets cannot go to infinity because of a possible non–integrable singularity of f near $v = 0$. Indeed from

$$\int_0^{v^*_{\rm crit}} vf(v,t)\,\mathrm{d}v + \int_{v^*_{\rm crit}}^{v_m} vf(v,t)\,\mathrm{d}v \equiv \mathcal{V}(t) = \mathcal{V}(0), \tag{4.10}$$

we obtain

$$v^*_{\rm crit} \int_{v^*_{\rm crit}}^{v_m} f(v,t)\,\mathrm{d}v \le \mathcal{V}(0), \tag{4.11}$$

and also

$$\int_0^{v^*_{\rm crit}} vf(v,t)\,\mathrm{d}v \le \int_0^{v^*_{\rm crit}} vf_o(v)\,\mathrm{d}v \le v^*_{\rm crit} \int_0^{v^*_{\rm crit}} f_o(v)\,\mathrm{d}v. \tag{4.12}$$

Therefore

$$\begin{aligned} \mathcal{N}(t) &= \int_0^{v_m} f(v,t)\,\mathrm{d}v = \int_0^{v^*_{\rm crit}} f(v,t)\,\mathrm{d}v + \int_{v^*_{\rm crit}}^{v_m} f(v,t)\,\mathrm{d}v \\ &\le \int_0^{v^*_{\rm crit}} f_o(v)\,\mathrm{d}v + \frac{\mathcal{V}(0)}{v^*_{\rm crit}} \le \mathcal{N}(0) + \frac{\mathcal{V}(0)}{v^*_{\rm crit}}. \end{aligned} \tag{4.13}$$

Estimate (4.13) for $\mathcal{N}$ is *a priori*, global and independent of any bound for f in the local time of existence. Now we can go back to (2.4): from (4.13), hypotheses (H1) to (H5) and Lemma 3.1 we easily get that

$$\left(\frac{\partial f}{\partial t}\right)_+ \le C\left(1 + \{f\}_t\right), \tag{4.14}$$

where C does not depend on f. We can thus proceed as in [3] to prove that (4.14) implies the global existence of f. In conclusion we have

Theorem 4.1. (GLOBAL EXISTENCE) *If the hypotheses of Theorem 3.2 are supplemented with (4.4), (4.5) and (4.6), then the solution to problem (2.5) given by Theorem 3.2 can be extended beyond $t = T$ over any finite time interval with the same regularity properties.*

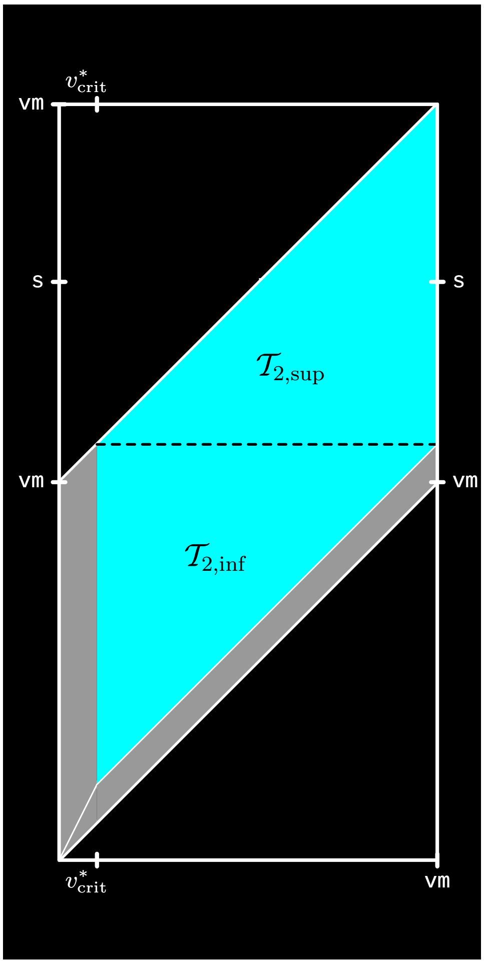

Fig. 5.1. The domain $\mathcal{T}_2$ when $v^*_{\rm crit} > 0$. The probability density β_2 is identically zero not only in the black region but also in the gray one (the size of $v^*_{\rm crit}$ is exaggerated for visualization purposes)

5. The case $N = 3$: an example of functions β_2 and β_3

If $v^*_{\rm crit}$ is strictly positive, region $\mathcal{T}_2$ modifies as shown in figure 5.1. Recall that for $s \in (0, v_m]$, β_2 is normalized as follows

$$\int_0^{s/2} \beta_2(s, v)\, {\rm d}v = 1, \tag{5.1}$$

while, for $s \in (v_m, 2v_m]$

$$\int_{s-v_m}^{s/2} \beta_2(s,u)\,\mathrm{d}u = 1. \tag{5.2}$$

Also recall that, for $k = 3, \ldots, N$, we have set

$$\int_{T_{k,h}(s)} \beta_k\left(s, u_1, \ldots, u_{k-1}\right)\,\mathrm{d}\sigma_{k-1} = 1, \qquad \forall h = 1, \ldots, k. \tag{5.3}$$

Moreover

$$\int_{\max\{0,s-v_m\}}^{s/2} \beta_2(s,u)\,\mathrm{d}u = \int_{s/2}^{\min\{s,v_m\}} \beta_2(s,u)\,\mathrm{d}u = 1, \qquad \forall s \in (0, 2v_m).$$

Thus, if $s \to 2v_m$, both intervals $(\max\{0, s-v_m\}, s/2)$ and $(s/2, \min\{s, v_m\})$ degenerate to a single point and β_2 behaves like a Dirac's δ–function (see Figures 5.2 and 5.3).

We show an example of such a function which meets the whole set of hypotheses we made so far. Although we could work out most calculations with generic parameters v^*_{crit} and v_m some of them (and even the representation of the involved functions) turn out to be very heavy. Thus we decided to leave v^*_{crit} and v_m unspecified as far as exposition takes advantage from this choice and to turn to simple numerical values (like $v^*_{\text{crit}} = 0$ and $v_m = 1$) when this is no longer true.

Let us choose β_2 as follows: first notice that $\mathcal{T}_2$ (with $v^*_{\text{crit}} > 0$, see Figure 5.1) can be represented as $\mathcal{T}_{2,\text{inf}} \cup \mathcal{T}_{2,\text{sup}}$ where

$$\mathcal{T}_{2,\text{inf}} = \{(s,u) \mid 2v^*_{\text{crit}} \le s \le v_m + v^*_{\text{crit}} \wedge v^*_{\text{crit}} \le u \le v_m\},$$

$$\mathcal{T}_{2,\text{sup}} = \{(s,u) \mid v_m + v^*_{\text{crit}} \le s \le 2v_m \wedge s - v_m \le u \le v_m\},$$

Then consider the characteristic functions $\chi_{\mathcal{T}_{2,\text{inf}}}, \chi_{\mathcal{T}_{2,\text{sup}}}$ and put

$$\beta_{2,a}(s,u) = A_0(s)(u - v^*_{\text{crit}})[(s-u) - v^*_{\text{crit}}], \tag{5.4}$$

$$\beta_{2,b}(s,u) = -\frac{A_1(s)}{2}\left(u - \frac{s}{2}\right)^2 + A_2(s), \tag{5.5}$$

$$\beta_{2,c}(s,u) = A_2(s). \tag{5.6}$$

Finally put

$$\beta_2 = \begin{cases} \beta_{2,a}\, \chi_{\mathcal{T}_{2,\inf}}, \text{ if } s \in (2v^*_{\text{crit}}, v_m + v^*_{\text{crit}}), \\ \beta_{2,b}\, \chi_{\mathcal{T}_{2,\sup}}, \text{ if } s \in (v_m + v^*_{\text{crit}}, \bar{s}), \\ \beta_{2,c}\, \chi_{\mathcal{T}_{2,\sup}}, \text{ if } s \in (\bar{s}, 2v_m), \end{cases} \tag{5.7}$$

Functions $A_i(s)$ are chosen in such a way to satisfy all the normalization and regularity hypotheses we made so far. Indeed we found that for $\bar{s} = 3v_m/2$, $v^*_{\text{crit}} = 0$, $v_m = 1$

$$A_0(s) = \frac{12}{s^3}, \qquad A_1(s) = 72 - 48s$$

$$A_2(s) = \frac{-\left(624 - 1248\, s + 1008\, s^2 - 360\, s^3 + 48\, s^4\right)}{24\, (-2 + s)}$$

(5.1) and (5.2) are both satisfied. Let us now check hypotheses (H5) concerning β_2: request (i) of (H5) is obvious from the very definition of β_2. Laborious but straightforward calculations then show that β_2 turns out to be of class C^2 in $\mathring{\mathcal{T}}_{2,\inf} \cup \mathring{\mathcal{T}}_{2,\sup}$ and in particular across the line $s = v_m + v^*_{\text{crit}}$. Thus (ii) of (H5) is also satisfied. As far the list (ii-a) to (ii-e) of (H5) is concerned, we now choose

$$\lambda_2(s) = m_2 \left(\frac{2v_m - s}{2v_m}\right)^{\varepsilon_2}$$

where the coefficients $m_2, \varepsilon_2 > 0$ are to be chosen conveniently. If $N = 3$, necessarily $\lambda_3(s) = 1 - \lambda_2(s)$ (remind that $\sum_{k=2}^{N} \lambda_k(s) = 1$). By noticing that A_2 behaves like $(2v_m - s)^{-1}$ as $s \to 2$, hypothesis (ii–a) is manifestly satisfied if $\varepsilon_2 \geq 1$. Concerning (ii-b), being λ_2 a bounded function, it sufficies to examine the integral

$$\int_{v_m}^{v_m+v} \frac{\partial \beta_2}{\partial v}\, \mathrm{d}s;$$

being β_2 independent of v in $(\bar{s}, 2v_m)$ we have

$$\left| \int_{v_m}^{v_m+v} \frac{\partial \beta_2}{\partial v}\, \mathrm{d}s = \right| \leq \int_{v_m}^{\bar{s}} \left| \frac{\partial \beta_2}{\partial v} \right|\, \mathrm{d}s.$$

More specifically

$$\int_{v_m}^{\bar{s}} \frac{\partial \beta_2}{\partial v}\, \mathrm{d}s = \int_1^{3/2} A_0(s) \left| \frac{s}{2} - v \right|\, \mathrm{d}s$$

which is clearly finite.

Consider now the integral

$$\int_{v}^{v_m} \alpha_2(s)\beta_2(s,v)\ \mathrm{d}s;$$

recalling hypothesis (H3) in Part I,we assume

$$\alpha_2(s) = \bar{\alpha}_2(s - v^*_{\mathrm{crit}})_+^{\delta_2}(v_m - s)_+^{-\mu_2}, \tag{5.8}$$

where $\bar{\alpha}_2$ is a positive constant and $\delta_2 > 0$ and $\mu_2 \in (0,1)$ have to be conveniently chosen.
Thus we have

$$\left| \int_{v}^{v_m} \alpha_2(s)\beta_2(s,v)\ \mathrm{d}s \right| \leq C \int_{v}^{1} \left(v s^{\delta_2-2} + v^2 s^{\delta_2-3}\right)(1-s)^{-\mu_2}\ \mathrm{d}s$$

which remains finite for all $v \in [0,1]$, provided that $\delta_2 > 0$ and $\mu_2 \in (0,1)$.
Consider finally the integral

$$\int_{v}^{v_m} \alpha_2(s)\frac{\partial \beta_2}{\partial v}(s,v)\ \mathrm{d}s;$$

specifically we have

$$\left| \int_{v}^{v_m} \alpha_2(s)\frac{\partial \beta_2}{\partial v}(s,v)\ \mathrm{d}s \right| \leq C \int_{v}^{1} \left(s^{\delta_2-2} + v s^{\delta_2-3}\right)(1-s)^{-\mu_2}\ \mathrm{d}s;$$

which remains finite for all $v \in [0,1]$, provided that $\delta_2 > 1$ and $\mu_2 \in (0,1)$.

Figures 5.2 and 5.3 give, respectively, a two-dimensional and a three-dimensional view of β_2.

We now construct an example of function β_3 assuming definitely, for simplicity, $v^*_{\mathrm{crit}} = 0$ and $v_m = 1$. First of all let us put $f(s,u_1,u_2) = 720 s^{-5} u_1 u_2 (s - u_1 - u_2)$ and consider the characteristic functions $\chi_{T_{3,h}(s)}$ of the sets $T_{3,h}(s)$ ($h = 1,2,3$). Then define

$$\begin{cases} \beta_{3,1}(s,u_1,u_2) = f(s,u_1,u_2) \circ \chi_{T_{3,1}(s)}(u_1,u_2), & (u_1,u_2) \in T_{3,1}(s), \\ \beta_{3,2}(s,u_1,u_2) = f(s,u_1,s-u_1-u_2) \circ \chi_{T_{3,2}(s)}(u_1,u_2), & (u_1,u_2) \in T_{3,2}(s), \\ \beta_{3,3}(s,u_1,u_2) = f(s,s-u_1-u_2,u_1) \circ \chi_{T_{3,1}(s)}(u_1,u_2), & (u_1,u_2) \in T_{3,3}(s), \end{cases} \tag{5.9}$$

and

$$\beta_3 = \beta_{3,1} + \beta_{3,2} + \beta_{3,3}. \tag{5.10}$$

Notice that β_3 is constructed exactly as described in Sect. 3 of Part I since, for $k = 3$, the maps C_j reduces to the following

$$C_1(\xi_1,\xi_2) = \begin{cases} u_1 = \xi_1, \\ u_2 = s - \xi_1 - \xi_2, \end{cases} \qquad C_2(\xi_1,\xi_2) = \begin{cases} u_1 = s - \xi_1 - \xi_2, \\ u_2 = \xi_2, \end{cases}$$

Figure 5.4 shows the contour plot of β_3 while Figure 5.5 shows a three–dimensional view of the same function from two different perspectives.

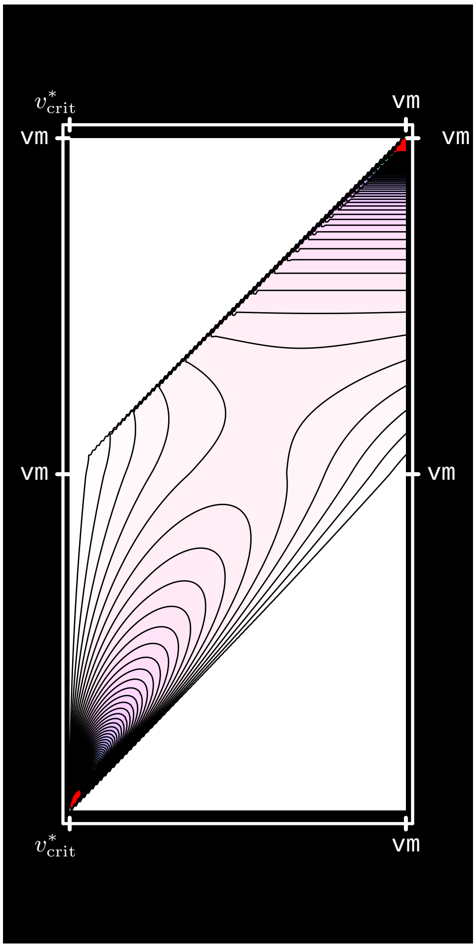

Fig. 5.2. The contour plot of the function $\beta_2(s, u)$

It is not difficult to see that $\int_{T_{3,1}} \beta_{3,1}(s, u_1, u_2)\ \mathrm{d}u_1\ \mathrm{d}u_2 = 1$. It is also obvious that (H5)–(i) is satisfied: indeed over $D_3(s, v)$ the function f writes $720s^{-5}u_1u_2v$. As (u_1, u_2) moves in $T_3(s)$ the droplet of size v just change meaning: in $T_{3,1}(s)$ is the largest among the three, while in $T_{3,3}(s)$ is the smallest. For $v \to 0$ the line $s - v = u_1 + u_2$ identifies with a portion of the boundary of $T_{3,3}(s)$ where β_3 vanishes.

Consider now the integral (hypothesis (iii–a) of (H5))

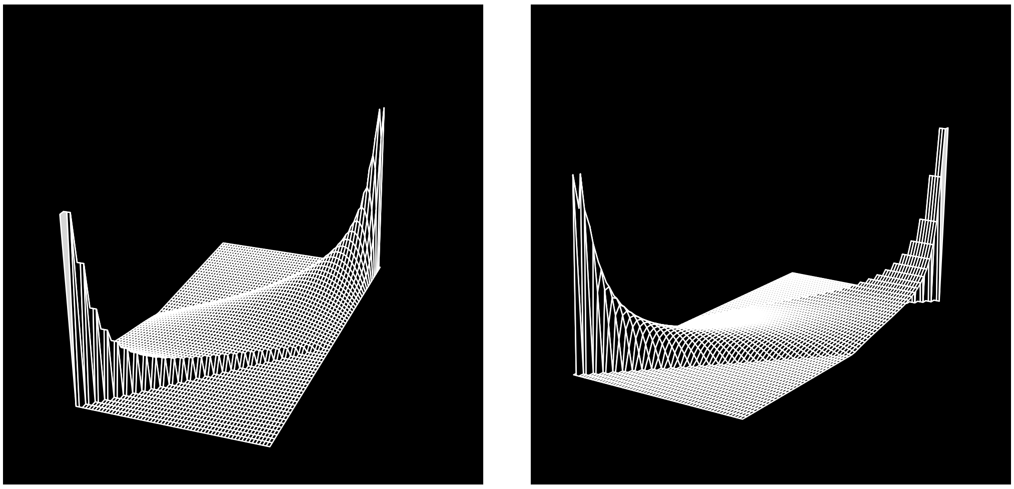

Fig. 5.3. The function $\beta_2(s,u)$ from two different viewpoints

$$\int_{D_3(v_m+v,v)} \beta_3(v_m+v,u_1,v_m-u_1)\,\mathrm{d}\sigma_1;$$

notice that, up to the sign, we have

$$\int_{D_3(s,v)} \beta_3(s,u_1,s-v-u_1)\,\mathrm{d}\sigma_1 = \sqrt{2}\int_{\max\{0,s-v_m\}}^{s/2} \beta_3(s,u_1,s-v-u_1)\,\mathrm{d}u_1.$$

Thus

$$\left|\int_{D_3(v_m+v,v)} \beta_3(v_m+v,u_1,v_m-u_1)\,\mathrm{d}\sigma_1\right| \le \sqrt{2}\int_v^{(v_m+v)/2} \beta_3(v_m+v,u_1,v_m-u_1)\,\mathrm{d}u_1$$

$$\le \sqrt{2}\int_v^{(v_m+v)/2} f(v_m+v,u_1,v_m-u_1) = \sqrt{2}\int_v^{(v_m+v)/2} \frac{720 v u_1(v_m-u_1)}{(v_m+v)^5}\,\mathrm{d}u_1$$

$$= 30\sqrt{2}\frac{v(v-v_m)^2(7v+2v_m)}{(v+v_m)^5}$$

The right hand side is uniformly bounded over $(0,v_m)$ and, for $v_m=1$, takes a maximum at $v\approx 0.214$ with value ≈ 7.44.

Consider then (iii–b) of (H5): up to a multiplying constant, we have

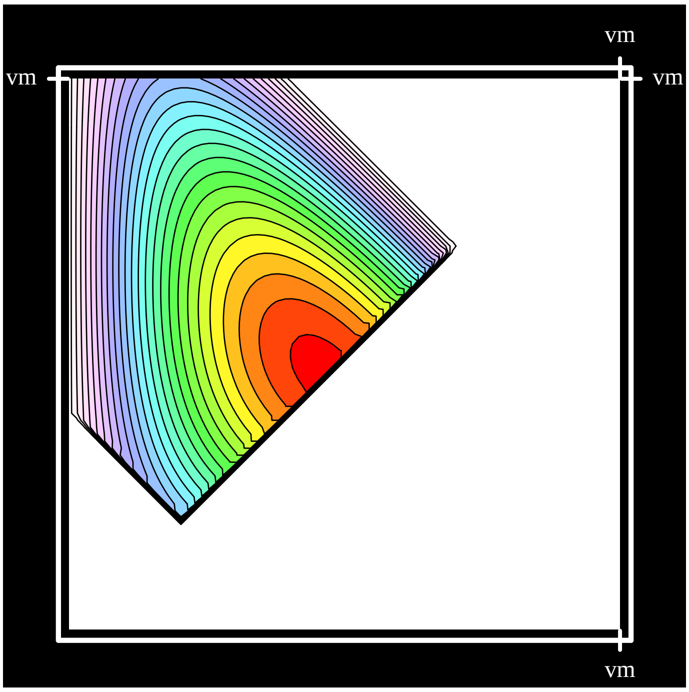

Fig. 5.4. Contour plot of the function $\beta_3(s,u_1,u_2)$ over $T_3(s)$ for a given value of s (greater than v_m in this case)

$$\int_v^{v_m} \alpha_3(s) \int_{D_3(s,v)} \beta_3(s,u_1,s-v-u_1)\, \mathrm{d}\sigma_1\, \mathrm{d}s$$

$$= \int_v^{v_m} \frac{s^{\delta_3}}{(v_m-s)^{\mu_3}} \int_{\max\{0,s-v_m\}}^{s/2} \beta_3(s,u_1,s-v-u_1)\, \mathrm{d}u_1\, \mathrm{d}s$$

$$= v \int_v^{v_m} \frac{s^{\delta_3-5}}{(v_m-s)^{\mu_3}} \left(\frac{-s^3}{24} + \frac{s^2\ (s-v)}{8} - \frac{(s-v)\ \max\{0,s-v_m\}^2}{2} + \frac{\max\{0,s-v_m\}^3}{3} \right) \mathrm{d}s$$

Also for $v_m = 1$, $\delta_3 = 2$ and $\mu_3 = 1/2$ the right hand side of the above relation is a rather complicated function of v. However it is not difficult to check that, for a suitable constant $C > 0$

$$v \int_v^1 \frac{s^{-3}}{\sqrt{1-s}} \left(\frac{-s^3}{24} + \frac{s^2\ (s-v)}{8} - \frac{(s-v)\ \max\{0,s-1\}^2}{2} + \frac{\max\{0,s-1\}^3}{3} \right) \mathrm{d}s \leq Cv\sqrt{1-v}$$

and thus it is uniformly bounded over $(0,v_m) = (0,1)$.

Let us notice now that

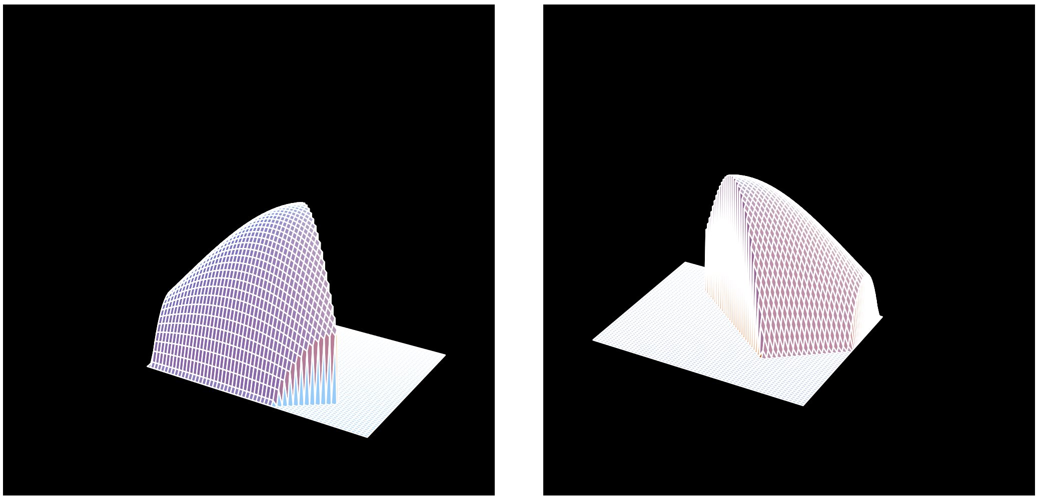

Fig. 5.5. Three-dimensional view of the function $\beta_3(s, u_1, u_2)$ a over $T_3(s)$ for a given value of s (greater than v_m in this case) from two opposite perspectives

$$\int_v^{v_m} \alpha_3(s) \left| \int_{\partial D_3(s,v)} \beta_3(s, u_1, s - v - u_1)\, \mathrm{d}\sigma_1 \right| \mathrm{d}s$$

$$= \int_v^{v_m} \alpha_3(s) \left| \beta_3(s, \max\{0, s - v_m\}, s - v - \max\{0, s - v_m\}) - \beta_3(s, s/2, s/2 - v) \right| \mathrm{d}s$$

$$= 180 \int_v^{v_m} \left| \frac{(2v - s)\, v}{s^2 \sqrt{v_m - s}} \right| \mathrm{d}s \leq 180\, v \int_v^{v_m} \frac{\max\{v, v_m - 2v\}}{s^2 \sqrt{v_m - s}}\, \mathrm{d}s$$

$$= 180 \left(v^2 \int_v^{\max\{v_m/3, v\}} \frac{\mathrm{d}s}{s^2 \sqrt{v_m - s}} + v\,(v_m - 2v) \int_{\max\{v_m/3, v\}}^{v_m} \frac{\mathrm{d}s}{s^2 \sqrt{v_m - s}} \right)$$

$$\leq 180 \left(\int_v^{\max\{v_m/3, v\}} \frac{\mathrm{d}s}{\sqrt{v_m - s}} + \frac{v\,(v_m - 2v)}{\max^2\{v_m/3, v\}} \int_{\max\{v_m/3, v\}}^{v_m} \frac{\mathrm{d}s}{\sqrt{v_m - s}} \right)$$

$$\leq 360 \left(\sqrt{v_m - v} + \frac{v\,(v_m - 2v)}{\max^2\{v_m/3, v\}} \sqrt{v_m - \max\{v, v_m/3\}} \right)$$

It can be easily checked that the right hand side of the above inequality is uniformly bounded in $(0, v_m)$. This proves that also hypothesis (iii–c) of the list (H5) is satisfied.

We finally come to the integral

$$\int_{v}^{v_m} \alpha_3(s) \int_{D_3(s,v)} \left[\frac{\partial \beta_3(s, u_1, \xi)}{\partial \xi}\right]_{\xi = s - v - u_1} \mathrm{d}\sigma_1 \; \mathrm{d}s;$$

up to a multiplying constant, this turns out to be equal to

$$\int_{v}^{v_m} \frac{s^2}{\sqrt{v_m - s}} \int_{\max\{0, s - v_m\}}^{s/2} \left(\frac{u_1 v - u_1(s - v - u_1)}{s^5}\right) \mathrm{d}u_1 \; \mathrm{d}s.$$

This integral can be calculated exactly and, for $v_m = 1$, turns out to be equal to

$$\frac{-\sqrt{1-v} + 3\,v \; \mathrm{arctanh}(\sqrt{1-v})}{6};$$

this function is uniformly bounded over $(0, 1)$. Thus also hypothesis (iii-d) of (H5) is satisfied.

References

1. I. Borsi. Dynamics of liquid–liquid dispersions with unbounded fragmentation kernel. *Adv. Math. Sci. Appl.*, 11(2):571–591, 2001.
2. A. Fasano and F. Rosso. Dynamics of droplets in an agitated dispersion with multiple breakage. Part I: formulation of the model and physical consistency. *Math. Meth. Appl. Sc.*
3. A. Fasano and F. Rosso. Analysis of the dynamics of liquid–liquid dispersions. In L. Arkeryd, J. Bergh, P. Brenner, and R. Pettersson, editors, *Progress in Industrial Mathematics at ECMI 98*, pages 214–221, Stuttgart Leipzig, 1999. Teubner.
4. A. Fasano and F. Rosso. A new model for the dynamics of dispersions in a batch reactor. In H. J. Bungartz, R. H. W. Hoppe, and Ch. Zenger, editors, *Proceedings of the Symposium Organized by the Sonderforschungsbereich 438 on the Occasion of Karl-Heinz Hoffman's 60th Birthday*, Lectures in Applied Mathematics, pages 123–141, Berlin, 2000. Springer Verlag.
5. S. Kumar, R. Kumar, and K. S. Gandhi. A new model for coalescence efficiency of drops in stirred dispersions. *Chemical Engineering Science*, 48(11):2025–2038, 1993.
6. H. A. Stone. Dynamics od drop deformation and breakup in viscous fluids. *Ann. Rev. Fluid Mech.*, 26:65–102, 1994.