A "CLOSE–UP" VIEW OF BREAKAGE AND SCATTERING KERNELS FOR THE DYNAMICS OF LIQUID DISPERSIONS: THEORY AND NUMERICAL SIMULATIONS

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In theoretical papers about the dynamics of liquid dispersions, the integral kernels appearing in the evolution equation for the droplet size distribution functions are usually given some generic properties, leaving their analytic structure unspecified. Moreover breakage is always described as a cumulative effect with no reference to the influence of the various breakage modes. Here we want to show how the effort of better understanding these integral kernels, guided by their physical meaning, helps significantly to identify a set of rather simple hypotheses guaranteeing the well–posedness of the problem. On the basis of the explicit structure of these kernels, we show examples of functions that fit perfectly the hypotheses of the existence–uniqueness theorem appearing in Refs. 1, 2 and present some numerical simulations.

1. Some forms of the evolution equation for the dynamics of liquid droplets

1.1. *Classical model*

Until a few years ago, the evolution equation for the dynamics of droplets in a liquid dispersion in the simplest case (i.e. homogeneous medium, no diffusion) used to be written as

$$\frac{\partial f}{\partial t} = L_c f + L_b f, \tag{1}$$

where $f(v,t)$ denotes the volume distribution function f, so that $f(v,t)\,\mathrm{d}v$ represents the number of droplets having volume in the interval $(v, v+\,\mathrm{d}v)$

*Work partially supported by the F.I.R.B. project of the italian M.I.U.R.

at time t, per unit volume of dispersion. The symbols L_c and L_b stay respectively for the *coalescence* and *breakage* operators and in the standard literature (see, e.g., Ref. 3) write as follows:

$$L_c f = \int_0^{v/2} \tau_c(w, v-w) f(w,t) f(v-w,t) \, \mathrm{d}w - f(v,t) \int_0^{+\infty} \tau_c(w,v) f(w,t) \, \mathrm{d}w,$$

$$L_b f = \int_v^{+\infty} \tau_b(s,v) f(s,t) \, \mathrm{d}s - f(v,t) \int_0^v \frac{s}{v} \tau_b(v,s) \, \mathrm{d}s.$$

Here τ_c is a symmetric non–negative function defined over $\mathbb{R}^+ \times \mathbb{R}^+$ with suitable properties. Similarly τ_b is a non–negative function defined over $\{(\xi, \eta) \mid 0 < \eta < \xi\}$ with suitable properties too (see, e.g., Refs. 3, 4, 5 for further details). The unbounded upper integration limit in both L_c and L_b was criticized in Ref. 5 on the basis of the experimental evidence and of the fact that posing a finite upper bound to the size of droplets is not only in agreement with the physics but may also imply a simpler mathematical treatment of the whole problem (see Ref. 6 for this point).

1.2. *A recent model which includes the "volume scattering effect"*

If, in agreement with the physics, we place a finite upper bound v_m to the droplet size, the model can be consistently modified only if we add at the r.h.s. of (1) the so–called *volume scattering effect* (see Refs. 5, 7) and rewrite (1) as follows

$$\frac{\partial f}{\partial t} = L_c f + L_b f + L_s f, \tag{2}$$

where

$$L_s f = \int_{v_m}^{v_m+v} \int_{s-v_m}^{s/2} \tau_s(s,v,w) f(w,t) f(s-w,t) \, \mathrm{d}w \, \mathrm{d}s - f(v,t) \int_{v_m}^{v_m+v} \tau_s(s,v,s-v) f(s-v,t) \, \mathrm{d}s.$$

The other two operators need to be modified accordingly. Indeed we write

$$L_c f = \int_0^{v/2} \tau_c(w, v-w) f(w,t) f(v-w,t) \, \mathrm{d}w - f(v,t) \int_0^{v_m - v} \tau_c(w,v) f(w,t) \, \mathrm{d}w,$$

$$L_b f = \int_v^{v_m} \tau_b(s,v) f(s,t) \, \mathrm{d}s - f(v,t) \int_0^v \frac{s}{v} \tau_b(v,s) \, \mathrm{d}s.$$

For simplicity we do not consider here further complications like the presence of an efficiency factor depending nonlocally on f (as it was done in

Refs. 5, 7) or like the so–called *collisional breakage* introduced in Refs. 8, 9. Instead we wish to focus on a more realistic structure of both the breakage and scattering kernel. Most Authors (of mathematical inspiration) look at $\tau_b(s,w)$ as the rate at which a particle of size s decays into a particle of size w regardless of how many particles contribute to the difference volume $s-w$. In other words the sintetic form of τ_b does not allow to distinguish among the various breakage modes *each having possibly its own rate and probability.* The same remark holds true for τ_s since the decay of droplets above the threshold limit v_m may occur with various rates and probability too. It is quite natural that this issue has been underestimated because of the objective difficulty of describing the single breakage modes. On the contrary, trying to clarify as much as we can the real structure of τ_b and τ_c will prove fruitful, when approaching the central problem of the well–posedness of the Cauchy problem, to select hypotheses which fit better the physics. In the next sections we shortly recall the results presented in Refs. 1, 2 and work out explicitly an example in which breakage modes up to the fourth order included are taken into account.

1.3. *Expanding the integral kernels*

The key point of our approach is that *each mode of drops rupture has its own frequency and probability.* With this in mind, we have recently proposed (see Refs. 1, 2) the following forms for L_b and L_c:

$$
\begin{aligned}
L_b f &= \int_v^{v_m} \mathfrak{O}_{\mathrm{b}}^{+}(s,v) f(s,t)\,\mathrm{d}s - \mathfrak{O}_{\mathrm{b}}^{-}(v) f(v,t),\\
L_s f &= \int_{v_m}^{v_m+v} \int_{s-v_m}^{s/2} \mathfrak{O}_{\mathrm{s}}^{+}(s,v,w) f(w,t) f(s-w,t)\,\mathrm{d}w\,\mathrm{d}s\\
&\quad - f(v,t) \int_{v_m}^{v_m+v} \tau_c(v,s-v) f(s-v,t)\,\mathrm{d}s,
\end{aligned}
$$

where $\mathfrak{O}_{\mathrm{b}}^{-}(v) = \sum\limits_{k=2}^{N} \alpha_k(v)$ and

$$
\begin{aligned}
\mathfrak{O}_{\mathrm{b}}^{+}(s,v) &= \alpha_2(s)\beta_2(s,v) + \sum_{k=3}^{N} \alpha_k(s) \int_{D_k(s,v)} \beta_k\left(s,\mathbf{u}_{k-2}, s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2},\\
\mathfrak{O}_{\mathrm{s}}^{+}(s,v,w) &= \tau_c(s-w,w)\lambda_2(s)\beta_2(s,s-v)\\
&\quad + \tau_c(s-w,w) \sum_{k=3}^{N} \lambda_k(s) \int_{D_k(s,v)} \beta_k\left(s,\mathbf{u}_{k-2}, s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}
\end{aligned}
$$

The functions appearing in $\mathfrak{O}^{\pm}_{\mathrm{b}}, \mathfrak{O}^{\pm}_{\mathrm{s}}$ have the following meaning:

- $\alpha_k(s)$ is the *breakage rate* of droplets with volume $s \in (v^{(1)}_{\mathrm{crit}}, v_m)$ into k droplets, $v^{(1)}_{\mathrm{crit}}$ being a lower non–negative threshold.
- $\lambda_k(s)$ is a suitable *weight* (to be chosen conveniently) measuring the probability of the parent droplet $s \in (v_m, 2v_m]$ to break exactly in k pieces within the scattering process; of course $\sum_{k=2}^{N} \lambda_k(s) = 1$.
- $U_n = \sum_{h=1}^{n} u_h, \;\; \mathbf{u}_n = (u_1, \ldots, u_n)$
- $\beta_k\left(s, \mathbf{u}_{k-1}\right)$ is the probability density of drops with volume $s \in (0, 2v_m)$ to generate by breakage k fragments with prescribed volumes u_j $(j = 1, \ldots, k-1)$ in increasing order, $v^{(2)}_{\mathrm{crit}} \le u_1 \le u_2 \le \ldots \le u_{k-1}$ (the volume of the remaining drop is the complement to s and may occupy any position in the size order), $v^{(2)}_{\mathrm{crit}}$ being a lower non–negative threshold. The inequality $v^{(2)}_{\mathrm{crit}} \le u_1$ is replaced with $v^{(2)}_{\mathrm{crit}} < u_1$ in the case $v^{(2)}_{\mathrm{crit}}$ vanishes.
- $\tau_c(v, w)$ is the *coalescence kernel*, that is proportional to the probability that two colliding droplets of respective volumes v and w coalesce to form a single droplet of volume $v + w$.

The precise definition of the functions $\beta_k\left(s, \mathbf{u}_{k-1}\right)$ and of the domains $D_k(s, v)$ is more complex and needs several preliminaries: we devote the following Section just to this topic. Before doing this we prefer to specify some properties of the functions appearing in the kernels of L_c, L_b, L_s:

(H1) τ_c is non–negative, symmetric and continuously differentiable in $[v^{(3)}_{\mathrm{crit}}, v_m] \times [v^{(3)}_{\mathrm{crit}}, v_m]$, $v^{(3)}_{\mathrm{crit}}$ being a lower non–negative threshold.

(H2) For $k = 2, \ldots, N$, α_k is non–negative, continuously differentiable and non–decreasing in $[v^{(1)}_{\mathrm{crit}}, v_m)$, unbounded as v tends to v_m. We also assume $\sum_{k=2}^{N} \alpha_k > 0$ for all $v \in (v^{(1)}_{\mathrm{crit}}, v_m)$ and

   a) $\alpha_k \simeq (v_m - v)_+^{-\mu_k}$ with $\mu_k \in (0, 1)$ in a left neighbourhood of $v = v_m$,

   b) $\alpha_k \simeq (v - v^{(1)}_{\mathrm{crit}})_+^{\delta_k}$ with $\delta_k > 0$ in a right neighbourhood of $v = v^{(1)}_{\mathrm{crit}}$, being $(\bullet)_+ := \max\{\bullet, 0\}$.

(H3) For $k = 2, \ldots, N$, functions λ_k are continuous in $[v_m, 2v_m]$.

The thresholds $v_{\rm crit}^{(1)}, v_{\rm crit}^{(2)}, v_{\rm crit}^{(3)}$ have an important physical meaning and play also a role in the question of global existence in time (see Refs. 1, 2 for all relevant details). For the sake of simplicity we assume, unless explicitly stated, all these lower thresholds equal to zero. However it may be interesting to notice that, if $v_{\rm crit}^* = \min\{v_{\rm crit}^{(1)}, v_{\rm crit}^{(2)}, v_{\rm crit}^{(3)}\} > 0$, the highest number of allowable rupture modes N (that in our model is a free parameter) can be roughly estimated from above through the ratio $2v_m/v_{\rm crit}^*$. It is also worth noticing that in the exceptional case of binary ruptures only (that is $\alpha_k = \lambda_k = 0$ for all $k \geq 3$) the model we propose coincides with the one presented in Refs. 5, 7.

2. Probability functions and their domains

The function β_2 is such that $\beta_2(s,u) = \beta_2(s, s-u)$, and $\beta_2(s,u) = 0$, if $s \leq u$. In other words, for each $s \in (0, v_m]$, we only need to define β_2 in $[0, s/2]$. If $s \in (v_m, 2v_m]$ being $u = s - (s-u) > s - v_m$, the function is defined in $(s - v_m, s/2)$. We set

$$T_{2,1}(s) = \{u_1 \mid 0 < u_1 \leq s - u_1 \leq v_m\} = \left(\max\{0, s - v_m\}, \frac{s}{2}\right),$$

$$T_{2,2}(s) = \{u_1 \mid 0 < s - u_1 < u_1 \leq v_m\} = \left(\frac{s}{2}, \min\{s, v_m\}\right),$$

Notice that the map $C_0 : s - u \mapsto u$, transforms $T_{2,2}(s)$ one–to–one onto $T_{2,1}(s)$. Therefore, for any value of $s \in (0, 2v_m)$, we assign β_2 on $T_{2,1}(s)$ in such a way that $\int_{T_{2,1}(s)} \beta_2(s,u)\, {\rm d}u = 1$, and think of $\beta_2 \circ C_0$ as its extension on $T_{2,2}(s)$. Also notice that $C_0 = C_0^{-1}$; moreover C_0 is measure–preserving, so that

$$\int_{T_{2,2}(s)} \beta_2(s, s-u)\, {\rm d}u = \int_{T_{2,1}(s)} \beta_2(s,u)\, {\rm d}u = 1, \tag{3}$$

and we first consider the case $s \in (0, v_m]$. Now, for a given $k \geq 3$, let us define the set of $\mathbb{R}^{k-1}$

$$T_{k,1}(s) = \{\mathbf{u}_{k-1} \mid 0 < u_1 \leq \ldots \leq u_{k-1} \leq s - U_{k-1} \leq v_m\}. \tag{4}$$

Clearly $\widetilde{u} = s - U_{k-1}$ identifies one of the k daughters and $T_{k,1}$ is characterized by the circumstance of $\widetilde{u}$ being the volume of the *largest* daughter(s). Function β_k is assigned on $T_{k,1}$ in such a way that

$$\int_{T_{k,1}(s)} \beta_k\left(s, \mathbf{u}_{k-1}\right)\, {\rm d}\sigma_{k-1} = 1. \tag{5}$$

We then define the following subsets in $\mathbb{R}^{k-1}$

$$T_{k,j}(s) = \{\mathbf{u}_{k-1} \mid 0 < u_1 \leq \ldots \leq u_{k-j} \leq s - U_{k-1} \leq u_{k-j+1} \leq \ldots \leq u_{k-1} \leq v_m\}, \quad j = 2, \ldots, k, \tag{6}$$

where, by definition, $u_0 = 0$ (*i. e.* in $T_{k,k}(s)$, $\widetilde{u}$ is the volume of the smallest drop(s)). If $s \leq v_m$, as we suppose for the moment, the last inequality in (4) and (6) is obviously redundant. Then we consider, again for a fixed s, the maps

$$C_j : (\xi_1, \ldots, \xi_{k-1}) \mapsto (u_1, \ldots, u_{k-1}), \ \ j = 1, \ldots, k-1,$$

defined by

$$\begin{cases} u_1 = \xi_1, \ldots, u_{k-j-1} = \xi_{k-j-1}, \qquad u_{k-j} = s - \sum_{i=1}^{k-1} \xi_i, \\ u_{k-j+1} = \xi_{k-j+1}, \ldots, u_{k-1} = \xi_{k-1}. \end{cases} \tag{7}$$

The purpose of maps (7) is to "re–locate" the *residual drop* $\widetilde{u}$ with respect to the ordered set of the other daughters. Indeed $U_{k-1} = s - \xi_{k-j}$ with $\widetilde{u}$ taking the place of ξ_{k-j}. It is easy to see that the Jacobian of each map C_j is equal to one and that

$$\left\{ C_j \left(T_{k,j}(s)\right) = T_{k,j+1}(s), \ \ C_j \left(T_{k,j+1}(s)\right) = T_{k,j}(s) \right.$$

so that $C_j = C_j^{-1}$. The main reason for introducing the maps C_j is to extend the probability density over all domains $T_{k,j}$. The procedure is the following. Indeed it can be proved (see Ref. 1) that, for all $k \geq 3$ and $i \neq j$, $\overset{\circ}{T}_{k,j} \bigcap \overset{\circ}{T}_{k,i} = \varnothing$ and that $\bigcap_{j=1}^{k} T_{k,j}$ reduces to a single point which can be identified with the event $u_1 = u_2 = \ldots = u_{k-1} = \frac{s}{k}$, that is "all droplets have the same volume". Now, by means of the maps C_j we extend β_k from $T_{k,1}$ to $T_{k,2}$, from $T_{k,2}$ to $T_{k,3}$ and so on, up to $T_{k,k}$. In other words we put

$$\widetilde{\beta}_k\left(s, \mathbf{u}_{k-1}\right) = \begin{cases} \beta_k\left(s, \mathbf{u}_{k-1}\right), & \text{if } \mathbf{u}_{k-1} \in T_{k,1}(s), \\ \beta_k \circ C_1\left(\mathbf{u}_{k-1}\right), & \text{if } \mathbf{u}_{k-1} \in T_{k,2}(s), \\ \vdots & \vdots \\ \beta_k \circ C_1 \circ C_2 \circ \ldots \circ C_{k-1}\left(\mathbf{u}_{k-1}\right), & \text{if } \mathbf{u}_{k-1} \in T_{k,k}(s). \end{cases} \tag{8}$$

Because of the properties of the maps C_j, we have

$$\int_{T_{k,1}(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = \int_{T_{k,2}(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = \ldots = \int_{T_{k,k}(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = 1; \tag{9}$$

if we define $T_k = \bigcup_{j=1}^{k} T_{k,j}$ and recall that $\overset{\circ}{T}_{k,j} \cap \overset{\circ}{T}_{k,i} = \varnothing$ for $i \neq j$, we also have

$$\int_{T_k(s)} \widetilde{\beta}_k \, \mathrm{d}\sigma_{k-1} = k. \tag{10}$$

We now put

$$D_k(s,v) = T_k(s) \cap \{U_{k-1} = s - v\}\,. \tag{11}$$

Thus in all the $T_{k,j}$ contributing to $D_k(s,v)$, the volume v is just that of the "residual drop". Notice that $D_k(s,v)$ is the intersection of the $(k-1)-$dimensional convex polytope $T_k(s)$ with a hyperplane in $\mathbb{R}^{k-2}$, so that $\partial D_k(s,v)$ is an orientable hypersurface in $\mathbb{R}^{k-3}$. From now on we drop the "tilde" above β_k in (8), i.e. we identify β_k with its extension over $T_k(s)$. Since we allow s in the interval $(0, 2v_m]$, function β_k is defined in the $k-$dimensional polytope

$$\mathcal{T}_k = \Big\{(s, \mathbf{u}_{k-1}) \in \mathbb{R}^k \mid s \in (0, 2v_m], 0 < u_1 \leq u_2 \leq \ldots \leq u_{k-1} \leq u_k, U_k = s\Big\}.$$

The domain $T_k(s)$ is the intersection of $\mathcal{T}_k$ with the plane $s =$constant. We now pass to the case $s \in (v_m, 2v_m]$, in which the last inequality appearing in the definitions (4) and (6) plays an effective role. We also extend the assumption (5): we put $\int_{T_{k,1}(s)} \beta_k\,(s, \mathbf{u}_{k-1}) \; \mathrm{d}\sigma_{k-1} = 1$, regardless of the size of s in $(0, 2v_m]$. The maps C_j then allow to extend β_k over the whole set $T_k(s)$ also for $s \in (v_m, 2v_m)$. Of course also (9) and (10) extend to this case. We notice explicitly that (9) is the natural extension to the case $k \geq 3$ of (3) and that, being $T_{2,1}(s) \cap T_{2,2}(s) = \{s/2\}$, $\overset{\circ}{T}_{2,1} \cap \overset{\circ}{T}_{2,2} = \varnothing$ and $\operatorname{meas} T_{2,1}(s) = \operatorname{meas} T_{2,2}(s)$, relation (10) also holds true for $k = 2$. The same conclusion concerns (11): for $k = 2$ this set reduces the single point of abscissa $s - v$ in the interval $(\max\{0, s - v_m\}, \min\{s, v_m\})$.

2.1. *Well-posedness*

In Refs. 1, 2 we proved the physical consistency of the model *a priori* (like the positivity of the solution and the conservation of volume) and also that if α_k and β_k obey some summability hypotheses the Cauchy problem for Eq. (2) is well posed for a suitable class of initial data. The precise statements of these hypotheses are as follows.

(H4) (*Regularity*): for all $k \geq 2$, β_k vanishes if the size of the smallest daughter goes to zero; $\beta_2(s,v)$ is piecewise continuously differentiable in $\overset{\circ}{\mathcal{T}_2} = \bigcup_{s\in(0,2v_m)} \{s\}\times \overset{\circ}{T_2}(s)$ and, in addition, there exists a suitable positive constant C such that, for all $v \in (0, v_m)$,

$$\lambda_2(v_m+v)\beta_2(v_m+v,v) \leq C, \quad \left|\int_{v_m}^{v_m+v} \lambda_2(s)\frac{\partial\beta_2(s,v)}{\partial v}\,\mathrm{d}s\right| \leq C,$$
$$\int_v^{v_m} \alpha_2(s)\beta_2(s,v)\,\mathrm{d}s \leq C, \quad \left|\int_v^{v_m} \alpha_2(s)\frac{\partial\beta_2(s,v)}{\partial v}\,\mathrm{d}s\right| \leq C.$$

Furthermore, for $k = 3,\ldots,N$, β_k is piecewise continuously differentiable in $\overset{\circ}{\mathcal{T}_k} = \bigcup_{s\in(0,2v_m)} \{s\}\times \overset{\circ}{T_k}(s)$ and, in addition, there exists a suitable positive constant C such that, for all $v \in (0, v_m)$,

(a) $\left(\int_{D_k(s,v)} \beta_k\left(s, \mathbf{u}_{k-2}, s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}\right)\Big|_{s=v_m+v} \leq C,$

(b) $\int_v^{v_m} \alpha_k(s)\int_{D_k(s,v)} \beta_k\left(s, \mathbf{u}_{k-2}, s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-2}\,\mathrm{d}s \leq C,$

(c) $\int_v^{v_m} \alpha_k(s)\int_{\partial D_k(s,v)} \beta_k\left(s, \mathbf{u}_{k-2}, s-v-U_{k-2}\right)\,\mathrm{d}\sigma_{k-3}\,\mathrm{d}s \leq C,$

(d) $\left|\int_v^{v_m} \alpha_k(s)\int_{D_k(s,v)} \left[\frac{\partial\beta_k}{\partial u_{k-1}}\right]_{u_{k-1}=s-v-U_{k-2}}\,\mathrm{d}\sigma_{k-2}\,\mathrm{d}s\right| \leq C.$

We recall that $\partial D_k(s,v)$ is an orientable hypersurface in $\mathbb{R}^{k-3}$. We mean that all integrals of type (iii-c) are positive. Concerning the initial data we assume that $f_o(0) = f_o(v_m) = 0$, that $f_o(v)$ is non–negative and piecewise $C^1[0, v_m]$, and finally that

$$|\alpha_k'(v)f_o(v)| < +\infty, \qquad \forall k = 2,\ldots,N.$$

The unique solution of the Cauchy problem turns out to be at least Lipschitz continuous in $[0, v_m]\times[0,T)$ for a suitable $T > 0$. To achieve global existence we need, in addition, that α_k, β_k and τ_c vanish identically in a right neighbourhood of the origin: this forbids the blow-up of the number of droplets of arbitrary small size and is perfectly justifiable on the basis of experimental observations.

3. Numerical simulations

Guided by the detailed structure of the breakage and scattering kernels it is not difficult to give examples of functions β_k which fit all the assumptions we made. In this section we present possible forms of $\beta_k, \alpha_k, \lambda_k$ for $k = 2, 3, 4$ which meet all the hypotheses stated in the previous sections. All simulations are then carried out taking into account effects up to the fourth mode included. Computing solutions including higher modes presents no other difficulty but longer computational time. Let us define $\beta_{2,a}(s,u) = A_0(s)u(s-u)$, and $\beta_{2,b}(s,u) = -\frac{A_1(s)}{2}\left(u-\frac{s}{2}\right)^2 + A_2(s)$, $\beta_{2,c}(s,u) = A_2(s)$. Then set

$$\beta_2 = \begin{cases} \beta_{2,a}\,\chi_{\mathcal{T}_{2,\mathrm{inf}}}, & \text{if } s \in (0,1), \\ \beta_{2,b}\,\chi_{\mathcal{T}_{2,\mathrm{sup}}}, & \text{if } s \in (1,\bar{s}), \\ \beta_{2,c}\,\chi_{\mathcal{T}_{2,\mathrm{sup}}}, & \text{if } s \in (\bar{s},2), \end{cases} \tag{12}$$

being $\chi_{\mathcal{A}}$ the characteristic function of the set $\mathcal{A}$. For simplicity, we have set (in (12) and in the sequel) $v^* = 0$ and $v_m = 1$. If $\bar{s} = 3/2$, the hypotheses made in the previous section are all satisfied, provided that $A_0(s) = \frac{12}{s^3}$, $A_1(s) = 72 - 48s$ and that $A_2(s) = -\left(26 - 52\,s + 42\,s^2 - 15\,s^3 + 2\,s^4\right)(-2+s)^{-1}$. All hypotheses (H4) concerning β_2 can be checked by taking, e.g., $\alpha_2(s) = s^{\delta_2}(1-s)_+^{-\mu_2}$ with $\mu_2 \in (0,1)$, $\delta_2 > 0$, and $\lambda_2(s) = \frac{2-s}{2}$. Since we consider breakage events up to the fourth order mode, we define $\lambda_4(s) = \lambda_3(s) = (1-\lambda_2(s))/2$. Recall that $C_1 : (\xi_1,\xi_2) \mapsto (\xi_1, s-\xi_1-\xi_2)$, $C_2 : (\xi_1,\xi_2) \mapsto (s-\xi_1-\xi_2, \xi_2)$ and, being $f(s,\mathbf{u}_2) = 720 s^{-5} u_1 u_2 (s - U_2)$, write

$$\begin{cases} \beta_{3,1}(s,\mathbf{u}_2) = f(s,\mathbf{u}_2) \circ \chi_{T_{3,1}(s)}(\mathbf{u}_2), & \mathbf{u}_2 \in T_{3,1}(s), \\ \beta_{3,2}(s,\mathbf{u}_2) = f(s,u_1,s-U_2) \circ \chi_{T_{3,2}(s)}(\mathbf{u}_2), & \mathbf{u}_2 \in T_{3,2}(s), \\ \beta_{3,3}(s,\mathbf{u}_2) = f(s,s-U_2,u_1) \circ \chi_{T_{3,1}(s)}(\mathbf{u}_2), & \mathbf{u}_2 \in T_{3,3}(s), \end{cases}$$

Then assume $\beta_3 = \beta_{3,1} + \beta_{3,2} + \beta_{3,3}$. For $v \in (0,1)$ fixed, the domain $D_3(s,v)$ is defined as $D_3(s,v) = T_3(s) \cap \{U_2 = s - v\}$. It is easy to check that $\int_{T_{3,1}} \beta_{3,1}(s,\mathbf{u}_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 = 1$ and that, over $D_3(s,v)$, the function f writes $720 s^{-5} u_1 u_2 v$. Concerning hypothesis (a) of the set (H4) notice that, up to the sign, $\int_{D_3(s,v)} \beta_3(s,u_1,s-v-u_1)\,\mathrm{d}\sigma_1 = \sqrt{2}\int_{\max\{0,s-1\}}^{s/2} \beta_3(s,u_1,s-v-u_1)\,\mathrm{d}u_1$. Thus

$$\left|\int_{D_3(1+v,v)} \beta_3(1+v,u_1,1-u_1)\,\mathrm{d}\sigma_1\right| \le \sqrt{2}\int_v^{(1+v)/2} \beta_3(1+v,u_1,1-u_1)\,\mathrm{d}u_1$$

$$\le \sqrt{2}\int_v^{(1+v)/2} f(1+v,u_1,1-u_1) = \sqrt{2}\int_v^{(1+v)/2} \frac{720 v u_1 (1-u_1)}{(1+v)^5}\,\mathrm{d}u_1$$

$$= 30\sqrt{2}\,\frac{v(v-1)^2(7v+2)}{(v+1)^5}$$

which is uniformly bounded in $(0,1)$. Similarly for hypothesis (b) we need to consider the integral $\int_v^1 \alpha_3(s) \int_{D_3(s,v)} \beta_3(s,u_1,s-v-u_1)\ \mathrm{d}\sigma_1\ \mathrm{d}s$ which rewrites $\int_v^1 s^{\delta_3}(1-s)^{-\mu_3} \int_0^{s/2} \beta_3(s,u_1,s-v-u_1)\ \mathrm{d}u_1\ \mathrm{d}s$. For $\delta_3 = 2$ and $\mu_3 = 1/2$ it turns out that this integral is bounded by $Cv\sqrt{1-v}$ for a suitable positive constant C and thus is uniformly bounded over $(0,1)$. Similarly, concerning hypothesis (c), the integral $\int_v^1 \alpha_3(s) \int_{\partial D_3(s,v)} \beta_3(s,u_1,s-v-u_1)\ \mathrm{d}\sigma_1\ \mathrm{d}s$ reduces, up to a multiplying positive constant, to $\int_v^1 v\,|2v-s|s^{-2}(1-s)^{-1/2}\ \mathrm{d}s$. It can be easily checked that this integral is bounded by the function $2\left(\sqrt{1-v} + \frac{v(1-2v)\sqrt{1-\max\{1/3,v\}}}{\max^2\{1/3,v\}}\right)$, which is in turn uniformly bounded over $(0,1)$. Concerning hypothesis (d) we have that the integral

$$\begin{aligned}
&\int_v^1 \alpha_3(s) \int_{D_3(s,v)} \left[\frac{\partial \beta_3(s,u_1,\xi)}{\partial \xi}\right]_{\xi = s-v-u_1} \mathrm{d}\sigma_1\ \mathrm{d}s \\
&= M \int_v^1 s^{-3}(1-s)^{-1/2} \int_0^{s/2} u_1 v - u_1(s-v-u_1)\ \mathrm{d}u_1\ \mathrm{d}s.
\end{aligned}$$

being M a suitable positive constant. This integral can be calculated exactly and, up to a multiplying constant, turns out to be equal to

$$-\sqrt{1-v} + 3\,v\ \mathrm{arctanh}(\sqrt{1-v});$$

this function is uniformly bounded over $(0,1)$.

If $k=4$ the procedure is the same: the function β_4 is first defined over $T_{4,1}(s)$ and then extended by means of the maps C_j. We first define the function $g(s,\mathbf{u}_3) = 120960 s^{-7} u_1 u_2 u_3 (s - U_3)$ and

$$\begin{aligned}
C_1 &: (\xi_1,\xi_2,\xi_3) \mapsto (\xi_1,\xi_2,s-\xi_1-\xi_2-\xi_3), \\
C_2 &: (\xi_1,\xi_2,\xi_3) \mapsto (\xi_1,s-\xi_1-\xi_2-\xi_3,\xi_3), \\
C_3 &: (\xi_1,\xi_2,\xi_3) \mapsto (s-\xi_1-\xi_2-\xi_3,\xi_2,\xi_3).
\end{aligned}$$

Thus

$$\begin{cases}
\beta_{4,1}(s,\mathbf{u}_3) = g(s,\mathbf{u}_3) \circ \chi_{T_{4,1}(s)}(\mathbf{u}_3), & \text{if } \mathbf{u}_3 \in T_{4,1}(s), \\
\beta_{4,2}(s,\mathbf{u}_3) = g(s,u_1,u_2,s-U_3) \circ \chi_{T_{4,2}(s)}(\mathbf{u}_3), & \text{if } \mathbf{u}_3 \in T_{4,2}(s), \\
\beta_{4,3}(s,\mathbf{u}_3) = g(s,u_1,s-U_3,u_2) \circ \chi_{T_{4,3}(s)}(\mathbf{u}_3), & \text{if } \mathbf{u}_3 \in T_{4,3}(s), \\
\beta_{4,4}(s,\mathbf{u}_3) = g(s,s-U_3,u_1,u_2) \circ \chi_{T_{4,4}(s)}(\mathbf{u}_3), & \text{if } \mathbf{u}_3 \in T_{4,4}(s)
\end{cases}$$

and $\beta_4 = \beta_{4,1} + \beta_{4,2} + \beta_{4,3} + \beta_{4,4}$. For $v \in (0,1)$ fixed, the domain $D_4(s,v)$ is defined as $D_4(s,v) = T_4(s) \cap \{U_3 = s-v\}$. It can be easily checked that $\int_{T_{4,1}(s)} \beta_{4,1}(s,\mathbf{u}_3)\ \mathrm{d}u_1\ \mathrm{d}u_2\ \mathrm{d}u_3 = 1$. As before hypotheses (H4) can be verified via straightforward calculations (too long anyway to be proposed here).

The following Figs. show the effects of the various terms in a cumulative way. The last Fig. shows the expected independence of the asymptotic configuration from the initial data, provided that the initial volume of the dispersed phase remains the same. Indeed in Ref. 10 it was recently proved that the stationary version of Eq. (2) possesses a non–trivial solution provided that the kernels (in usual "closed" form) satisfy some suitable growth conditions.

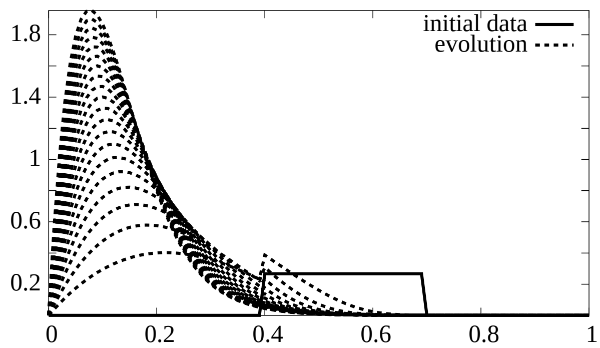

Figure 1. Evolution of $f(v,t)$ from a given initial datum: in this case only binary events are considered.

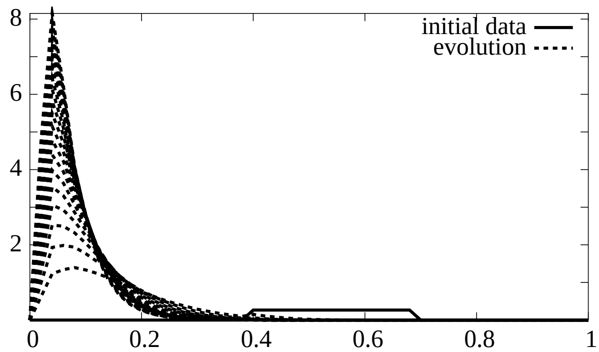

Figure 2. Evolution of $f(v,t)$ from a given initial datum: events up to the fourth order included are considered. The difference with respect to Fig. 1 should be noted.

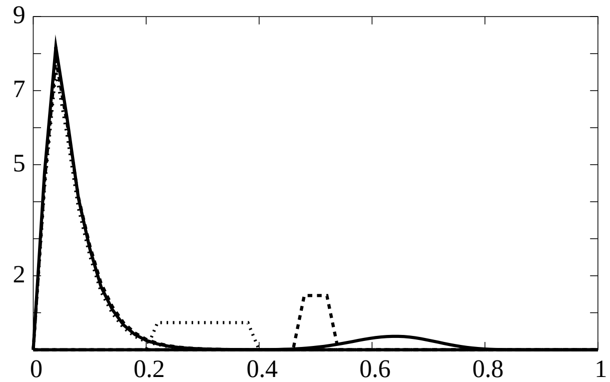

Figure 3. Evolution of $f(v,t)$ from three different initial data with the same total mass: the asymptotic limits coincide.

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