Breakage, Coalescence and Volume Scattering in Dispersed Phase Systems

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Abstract. In [**?**] a new model for the evolution of a system of droplets dispersed in an agitated liquid was presented, with the inclusion of the so–called *volume scattering effect* (a combination of coalescence and breakage). In that paper droplets breakage was considered to be binary, in order to simplify exposition. Here we remove that substantial limitation, considering multiple breakage and scattering.

1 Introduction

A system of two immiscible liquids agitated in a batch under the action of impellers gives rise to a set of droplets of one phase dispersed in the other phase. The resulting system is called a *dispersion* (finer dispersions are called *emulsions*) and its evolution is caused by the fact that droplets during their motion may break up in smaller droplets or they may coalesce (an essentially binary process), producing larger elements. Dispersions are commonly encountered in industrial products like food, cosmetics, pharmacology, photography and many others. This justifies the large amount of scientific papers devoted to this subject during the last century (see, for example, [**?**,**?**,**?**,**?**] for the main relevant literature). However, many basic questions are still pending so that research is still very active in this area.
Dealing with the specific case of the batch reactor, it is commonly assumed that spatial homogeneity is achieved, so that the droplet system is described by a volume distribution function $f(v,t)$, $f\,dv$ representing the number of droplets having volume in the interval $(v, v+dv)$ at time t, per unit volume of dispersion. When we come to the question of describing the evolution of f, there are essentially two kinds of difficulties somehow related to each other: the first is that the main processes influencing the evolution of f, namely *coalescence* and *breakage*, are indeed complex phenomena, not completely understood (particularly at high rotational speeds) and for which various descriptions have been proposed in the experimental literature. Experiments and observations are also very delicate so possible insights have to be taken very carefully. However – and this is mainly the second difficulty – the proposed mathematical models seem to require further refinements. Indeed a typical feature usually adopted in the construction of the mathematical model is that v is allowed to take any positive value: this is clearly meaningless from the physical point of view, but writing an evolution model is generally much simpler if v ranges over $(0,+\infty)$. On the other hand it is true that coalescence tends naturally to produce *large* droplets. However, it is not a controversial point that (for a given agitation speed of the mixture) the maximum observable size of droplets is *finite* (see *e.g.* [**?**]). In the mathematical literature this aspect of the problem is generally underestimated or at most by–passed in some artificial way like placing a cut–off in the coalescence kernel and allowing v to go to infinity. Recently we proposed a model (see [**?**,**?**]) for the dynamics of droplets including in a consistent way a constraint on the droplet size and pointing out that this requires the presence of a third physical mechanism in the evolution of the system, called *volume scattering*. This effect consists in an immediate decay by rupture of a droplet resulting from coalescence and exceeding the threshold value $v_{\max}$ so that all daughters remain in the allowed size range. Scattering is represented by a specific operator (with gain and loss terms) in the balance equation which adds to classical coalescence and breakage operators. The main advantage of this approach is that it is based on natural assumptions,

reflecting the real physics. Also the mathematics appears to be simpler since subtle questions regarding summability in unbounded domains are automatically eliminated. The resulting model consists in an initial value problem for a Boltzmann–like equation for the function $f(v,t)$.
Our first contribution dealt with the simpler case of *binary* events: this means that all ruptures of a parent droplet due to breakage or scattering produce precisely *two* daughters.
In [**?**,**?**] we proved that the problem is well–posed under rather general hypotheses and with a bounded fragmentation kernel. The extension to the case of an unbounded fragmentation kernel has been subsequently performed in [**?**].
Here we show how to remove the main limitation that we put in [**?**,**?**], namely the hypothesis of "binary" rupture. We notice that since the scattering operator involves a breakage event also this operator needs to be modified accordingly in order to allow the volume scattering with multiple exit. Of course, multiple breakage has been considered by many Authors (see *e.g.* [**?**,**?**,**?**]) but with the philosophy of capturing a global information about breakage, in view of the difficulty of analyzing the single modes. Here we emphasize the contribution of each breakage class to the rate of change of the distribution function, confining ourselves to a brief description of the model. A complete analysis of our equation will be published elsewhere.

2 Mathematical Model

We assume droplets to be uniformly distributed in the reactor so that f does not depend on spatial coordinates. We also assume that the system is isolated, so that there is no heat or mass exchange. Thus it is possible to formulate the following evolution equation:

$$\frac{\partial f}{\partial t} = \phi(t)\left(L_{\mathrm{c}} f + L_{\mathrm{b}} f + L_{\mathrm{s}} f\right) \ . \tag{1}$$

Terms appearing at the r.h.s of equation (**??**) have been clarified in various papers (see [**?**,**?**] for example): essentially $\phi(t) = \Phi\left[\mathcal{N}(t), \mathcal{S}(t)\right]$ with

$$\mathcal{N}(t) = \int_0^{v_{\max}} f(v,t)\, dv \ , \qquad \mathcal{S}(t) = \int_0^{v_{\max}} v^{(2/3)} f(v,t)\, dv \ ,$$

represents what we called an *efficiency factor*, $\mathcal{N}$ and $\mathcal{S}$ having, respectively, the meaning of the instantaneous total *number of droplet* and *inter–facial area* per unit volume of dispersion. The role of ϕ has been described in the quoted papers.
The operators at the r.h.s. of (**??**) have a rather complex structure: L_{c} is the *coalescence* operator and depends on a coalescence kernel τ_c which is a known function of the sizes of the two colliding droplets; L_{b} is the *breakage* operator summing up the contributions of the rupture various modes (binary, ternary, etc.), having defined for each breakage mode its *frequency* α_i and the *probability density* β_i of its outcome. Finally L_{s} is the *scattering* operator and the kernel of the i-th mode is just the product of β_i and τ_c. Natural size limitations among droplets impose particular care when the integration domains of the various terms on the r.h.s. of (**??**) are specified. Here, in the next section, we work out with the necessary details the ternary mode of rupture. Higher orders becomes formally more complicated but the procedure we present is general and applies in all cases.

3 Breakage and scattering: the ternary mode made explicit

Ternary breakage of a droplet of volume w produces three droplets with preservation of volume. We select the volumes u_1, u_2 as *ordered* independent variables ($u_1 \le u_2$) and we distinguish three cases depending on the size of the third droplet $\widetilde{u} = w - u_1 - u_2$ compared to u_1, u_2. In the following the

subscript "3" refers to the specific mode we are considering. We first discuss the *breakage operator* L_b so that w does not exceed $v_{\max}$.

(b_1) Suppose that $u_1 \le u_2 \le \widetilde{u} \le v_{\max}$; these relations imply $u_1 \in (0, w/3)$ and $u_2 \in (u_1, (w-u_1)/2)$, defining the region

$$T_{3,1}(w) = \left\{ (u_1, u_2) \mid 0 < u_1 \le \frac{w}{3}, \quad u_1 \le u_2 \le \widetilde{u} \le v_{\max} \right\}.$$

(b_2) Suppose that $u_1 \le \widetilde{u} \le u_2 \le v_{\max}$; in this case, from these inequalities we still get $u_1 \in (0, w/3)$ but now $u_2 \in ((w-u_1)/2, w-2u_1)$. Accordingly

$$T_{3,2}(w) = \left\{ (u_1, u_2) \mid 0 < u_1 \le \frac{w}{3}, \quad \frac{w-u_1}{2} \le u_2 \le w - 2u_1 \right\}.$$

(b_3) Let us finally assume that $0 < \widetilde{u} \le u_1 \le u_2 \le v_{\max}$; in this case u_1 is allowed to range between 0 and $w/2$ and these inequalities imply that for $u_1 \in (0, w/3)$ it must be $u_2 \in (w-2u_1, w-u_1)$ while, if $u_1 \in (w/3, w/2)$, the correct bounds are $u_2 \in (u_1, w-u_1)$. Then let $T_{3,3}(w)$ be the region

$$\left\{ (u_1, u_2) \mid 0 < u_1 \le \frac{w}{3},\ w - 2u_1 \le u_2 \le w - u_1 \right\} \bigcup \left\{ (u_1, u_2) \mid \frac{w}{3} < u_1 \le \frac{w}{2},\ u_1 \le u_2 \le w - u_1 \right\}.$$

Regions $T_{3,i}$ are shown in figure **??**. Now we introduce the function $\beta_3(w, u_1, u_2)$ such that $\beta_3(w, u_1, u_2)\ du_1\ du_2$ is the probability that two among the daughters have volumes in the intervals $(u_1, u_1 + du_1)$, $(u_2, u_2 + du_2)$. We impose the natural normalization

$$\int_{T_{3,1}(w)} \beta_3(w, u_1, u_2)\ du_1 du_2 = 1.$$

The probability density β_3 is then extended over the other two regions. Indeed the probability must remain the same if $\widetilde{u}$ becomes the middle droplet or the smallest one. Consider then the plane coordinate transformation

$$\mathcal{C} : (u_1, u_2) \mapsto (\xi, \eta)$$

given by $\xi = w - u_1 - u_2$, $\eta = u_1$ and which is directly suggested by the the definition of $\widetilde{u}$. Transformation $\mathcal{C}$ is area–preserving and such that $\mathcal{C}(T_{3,1}) = T_{3,2}$ and $\mathcal{C}(T_{3,2}) = T_{3,3}$. Consequently

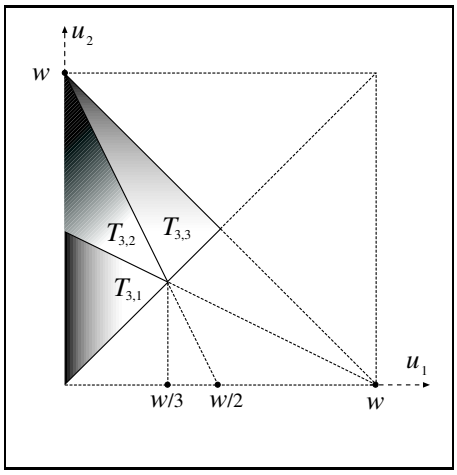

Fig. 1. Triangular regions $T_{3,i}$ where β is defined. The three triangles have the same area

$$\begin{aligned}
&\int_{T_{3,1}(w)} \beta_3(w, u_1, u_2)\ du_1 du_2 = \qquad 1 \\
&= \int_{\mathcal{C}(T_{3,1}(w))} \beta_3 \circ \mathcal{C}^{-1}(w, \eta, \xi)\ d\eta d\xi = \int_{T_{3,2}(w)} \beta_3(w, u_1, w - u_1 - u_2)\ du_1 du_2 \qquad (1) \\
&= \int_{\mathcal{C}(T_{3,2}(w))} \beta_3 \circ \mathcal{C}^{-1}(w, \eta, \xi)\ d\eta d\xi = \int_{T_{3,3}(w)} \beta_3(w, w - u_1 - u_2, u_2)\ du_1 du_2.
\end{aligned}$$

The *gain part* of the breakage operator now writes as follows ($\widetilde{u}$ playing the role of v)

$$L^{(3)}_{\mathrm{b,gain}} f(v,t) = \int_v^{v_{\max}} dw \int_0^{(w-v)/2} \alpha_3(w)\beta_3(w,u_1,w-v-u_1) f(w,t)\, du_1. \tag{2}$$

Let us set $T_3 = \bigcup_{i=1}^{3} T_{3,i}$; due to the normalization property (**??**) it is easy to check that

$$\begin{aligned}
\int_0^{v_{\max}} L^{(3)}_{\mathrm{b,gain}} f(v,t)\, dv &= \int_0^{v_{\max}} dw \int_0^{w} dv \int_0^{(w-v)/2} \alpha_3(w)\beta_3(w,u_1,w-v-u_1) f(w,t)\, du_1 \\
&= \int_0^{v_{\max}} \left[\int_{T_3(w)} \beta_3(w,u_1,u_2)\, du_1 du_2 \right] \alpha_3(w) f(w,t) dw \qquad (3) \\
&= 3 \int_0^{v_{\max}} \alpha_3(w) f(w,t) dw,
\end{aligned}$$

which can be interpreted as the overall production rate due to ternary breakage; of course, since the *loss part* of the breakage operator writes $L^{(3)}_{\mathrm{b,loss}} f(v,t) = \alpha_3(v) f(v,t)$, the net production amounts to $2\int_0^{v_{\max}} \alpha_3(w) f(w,t) dw$. It can also be checked that within the above scheme the total volume is preserved. Indeed we have

$$\int_0^{v_{\max}} v\, dv \int_v^{v_{\max}} \alpha_3(w) f(w,t)\, dw \int_0^{(w-v)/2} \beta_3(w,u_1,w-v-u_1)\, du_1$$

$$= \int_0^{v_{\max}} \alpha_3(w) f(w,t)\, dw \left[\int_{T_3(w)} (w-u_1-u_2)\beta_3(w,u_1,u_2)\, du_1\, du_2 \right]$$

and

$$\int_{T_3(w)} w\beta_3(w,u_1,u_2)\, du_1\, du_2 = 3w.$$

Moreover, by means of transformation $\mathcal{T}$,

$$\int_{T_{3,2}(w)} (u_1+u_2)\beta_3(w,u_1,u_2)\, du_1\, du_2 = w - \int_{T_{3,1}(w)} \xi\beta_3(w,\eta,\xi)\, d\eta\, d\xi$$

and

$$\int_{T_{3,3}(w)} (u_1+u_2)\beta_3(w,u_1,u_2)\, du_1\, du_2 = w - \int_{T_{3,1}(w)} \xi\beta_3(w,\eta,\xi)\, d\eta\, d\xi.$$

Therefore

$$\int_{T_3(w)} (w-u_1-u_2)\beta_3(w,u_1,u_2)\, du_1\, du_2$$

$$= 3w - 2w - \int_{T_{3,1}(w)} (\xi+\eta)\beta_3(w,\eta,\xi)\, d\eta\, d\xi$$

$$+ \int_{T_{3,1}(w)} \xi\beta_3(w,\eta,\xi)\, d\eta\, d\xi + \int_{T_{3,1}(w)} \eta\beta_3(w,\eta,\xi)\, d\eta\, d\xi = w.$$

Let us now consider the *scattering operator* L_s; in this case the parent droplet s is greater that $v_{\max}$ but less than $2v_{\max}$. As before we think of s and v as fixed and distinguish three cases according to the size of $\widetilde{u}$.

(s_1) Suppose that $u_1 \le u_2 \le \widetilde{u} \le v_{\max}$; these relations imply that $u_1 \in (0, s/3)$, $u_2 \in (u_1, s-u_1-u_2)$ and $s - u_1 - u_2 < v_{\max}$. Let us define $T^*_{3,1}$ to be the region of $\mathbb{R}^2$

$$T^*_{3,1}(s) = \{(u_1,u_2) \mid 0 < u_1 \le u_2, \quad u_1 + 2u_2 \le s, \quad u_1 + u_2 \ge s - v_{\max}\}.$$

We notice that $T^*_{3,1} \neq \varnothing$ since for $s \in (v_{\max}, 2v_{\max})$ the straight line $u_1 + u_2 = s - v_{\max}$ remains always below the line $u_1 + u_2 = 2s/3$.

(s_2) suppose that $u_1 \le \widetilde{u} \le u_2 \le v_{\max}$; in this case, from these inequalities we get $u_1 \in (0, s/3)$ and $u_2 \in ((s-u_1)/2, s-2u_1)$. Let us define $T^*_{3,2}$ to be the region of $\mathbb{R}^2$

$$T^*_{3,2}(s) = \left\{(u_1,u_2) \mid 0 < u_1 \le \frac{s}{3}, \quad \frac{s-u_1}{2} \le u_2 \le s - 2u_1, \quad u_2 \le v_{\max}\right\}.$$

We notice that $T^*_{3,1} \neq \varnothing$ since for $s \in (v_{\max}, 2v_{\max})$ the horizontal line $u_2 = v_{\max}$ is always above the line $u_2 = s/3$.

(s_3) Let us finally assume that $0 < \widetilde{u} \le u_1 \le u_2 \le v_{\max}$; in this case u_1 is allowed to range between 0 and $s/2$ and these inequalities imply that for $u_1 \in (0, s/3)$ it must be $u_2 \in (s-2u_1, s-u_1)$ while, if $u_1 \in (s/3, s/2)$, the correct bounds are $u_2 \in (u_1, s-u_1)$. Then let $T^*_{3,3}(s)$ be the region

$$\left\{(u_1,u_2) \mid \frac{s-v_{\max}}{2} < u_1 \le \frac{s}{3},\ s-2u_1 \le u_2 \le v_{\max}\right\} \bigcup \left\{(u_1,u_2) \mid \frac{s}{3} < u_1 \le \frac{s}{2},\ u_1 \le u_2 \le s-u_1\right\},$$

Regions $T^*_{3,i}$ are shown in figure **??**. It is not difficult to see that regions $T^*_{3,i}$ still have the same area. The same argument developed for the breakage case still applies: β_3 is extended for $s \in (v_{\max}, 2v_{\max})$ in such a way that

$$\int_{T^*_{3,1}(s)} \beta_3(s,u_1,u_2)\, du_1 du_2 = 1.$$

Transformation $\mathcal{C}$ still shows that

$$\begin{aligned} &\int_{T^*_{3,1}(s)} \beta_3(s,u_1,u_2)\, du_1 du_2 = 1 \\ &= \int_{T^*_{3,2}(s)} \beta_3(s,u_1,u_2)\, du_1 du_2 \qquad (4) \\ &= \int_{T^*_{3,3}(s)} \beta_3(s,u_1,u_2)\, du_1 du_2. \end{aligned}$$

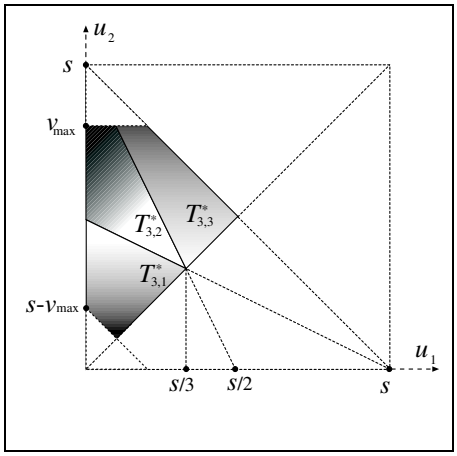

Fig. 2. Triangular regions $T^*_{3,i}$. The three triangles have the same area

We are now ready to make the scattering operator explicit in the ternary case: if we set $L_s = L_{s,gain} - L_{s,loss}$ then

$$L_{s,gain} f(v,t) = \int_{\Omega(v)} S_g(s,u,u_1,u_2 \mid v = s-u_1-u_2)\, d\boldsymbol{\omega}\,, \qquad L_{s,loss} = \int_{\Lambda(v)} S_\ell(s,v,u_1,u_2)\, d\boldsymbol{\lambda}$$

where

$$S_g = \tau_c(u, s-u)\beta_3(s, u_1, u_2)f(u,t)f(s-u,t), \qquad S_\ell = f(v,t)f(s-v,t)\tau_c(v, s-v)\beta_3(s, u_1, u_2)$$

and

$$\int_{\Omega(v)} d\boldsymbol{\omega} = \int_{v_{\max}}^{v_{\max}+v} ds \int_{s-v_{\max}}^{s/2} du \int_0^{(s-v)/2} du_1 + \int_{v_{\max}+v}^{2v_{\max}} ds \int_{s-v_{\max}}^{s/2} du \int_{s-v-v_{\max}}^{(s-v)/2} du_1,$$

$$\int_{\Lambda(v)} d\boldsymbol{\lambda} = \int_{v_{\max}}^{v_{\max}+v} ds \iint_{T_3^*(s)} du_1 du_2.$$

In particular, because of (**??**),

$$\int_{\Lambda(v)} S_\ell(s, v, u_1, u_2)\, d\boldsymbol{\lambda} = 3f(v,t) \int_{v_{\max}}^{v_{\max}+v} f(s-v,t)\tau_c(v, s-v)\, ds.$$

The same argument we used for the breakage operator shows that also the scattering operator is volume preserving.

4 Choice of the kernels

The model presented applies to any kind of fluid-fluid dispersion (with no chemical reactions) since it is based on the *mechanics* of breakage and coalescence. However the selection of the parameters of the model (the *coalescence kernel* τ_c, the *breakage-frequency* α and the probability distribution β) able to fit the effective behavior of a real dispersion is really a hard problem.
In our approach, we used *the simplest* possible equations for the kernels based just on geometrical and mechanical considerations. Moreover, for simplicity, only the binary mode of rupture is taken into account since even the addition of the sole ternary mode increases considerably the computational time.
We used two very simple expressions of the initial distribution $f_o(v)$:

- a Gaussian distribution function centered at 0.4

$$f_o(v) = k \exp(-((v-0.40)(v-0.40)) * 128)/0.1567$$

- a piecewise constant function

$$f_o(v) = \begin{cases} k & \text{if} \quad v \in [0.2, 0.6], \\ 0 & \text{otherwise,} \end{cases}$$

where k is determined by assigning the volume fraction $\lambda \in (0,1]$ of dispersed phase: indeed

$$\lambda = \int_0^1 v f_o(v)\, dv.$$

The function β is just a probability distribution function. A simple form satisfying the requirements (ii) and (iii) is

$$d(s) = \left(\frac{v_{\max}}{2} - v_b\right)_+ = \left(\frac{1}{2} - v_b\right)_+ = \text{const.},$$

$$\begin{cases} \beta(w,v) = 12\dfrac{(v-v_{\rm b})_+\,(w-v-v_{\rm b})_+}{(w-2v_{\rm b})_+^3} & \text{if} \quad w \in (2v_{\rm b}, v_{\max}+v_{\rm b})\,, \\ \beta(s,v) = -\dfrac{c(s)}{2}\left(v-\dfrac{s}{2}\right)^2 + d(s) & \text{if} \quad s \in (v_{\max}+v_{\rm b}, 2v_{\max})\,. \end{cases}$$

and

$$c(s) = \begin{cases} 0 & \text{if} \quad s \in (v_{\max}, v_{\max}+v_{\rm b}) = (1, 1+v_{\rm b}), \\ (s-v_{\max}-v_{\rm b}) = (s-1-v_{\rm b}) & \text{if} \quad s \in \left(v_{\max}+v_{\rm b}, \dfrac{3v_{\max}}{2}\right) = \left(1+v_{\rm b}, \dfrac{3}{2}\right), \\ g(s) & \text{if} \quad s \in \left(\dfrac{3v_{\max}}{2}, 2v_{\rm o}\right) = \left(\dfrac{3}{2}, 2\right). \end{cases}$$

In order to work out some significant numerical simulations we need to make some remarks about the integration domain. From the mathematical point of view $v_{\max}$ is a *fixed* upper bound for the volume size of droplets independent of time.

However the characteristic length scales for $f_{\rm o}$ and $f(v,\infty)$ may differ, in some cases, even by two or three orders of magnitude. To see one of these cases let us define

$$v^\star(t) = \sup \operatorname{supp} f(v,t);$$

clearly $v^\star$ depends, besides t, on the rotational speed Θ, the geometry of the container and impeller, the hold-up λ, the temperature and many other rheological parameters (see [**?**] for example): for a water in (light crude) oil dispersion, temperature $\simeq 60\ ^\circ C$, high rotational speeds (say 4000-6000 r.p.m.), a percentage of dispersed phase water about 60%, and an agitation time of about 15 minutes, the top size diameter is about $60 \div 70\ \mu$m. This can be identified with the asymptotic value since there is practically no change for $t > 15$ minutes. This means, for this case, that we can set $v^\star(\infty) \simeq 10^{-7}\,\text{cm}^3$; since typical values of the maximum diameter at the very early stages of agitation is about ten times larger than the initial ones, we have

$$\frac{v^\star(\infty)}{v^\star(0)} \simeq 10^{-3}.$$

Thus, in cases like the above, it is quite difficult to show graphically the evolution of f from $f_{\rm o}$ using the same length scale $v_{\max}$. In these cases we did not use the effective initial data but rather an intermediate configuration with a characteristic length much closer to that of the expected asymptotic distribution. Physically this corresponds to a *pre–mixing* period before examining the evolution.

Concerning a lower bound for the volume of breakable drops, we decided to set $v_{\rm b} = 0$ in all simulations. Indeed, if we refer to the expression found in literature for $v_{\rm b}$ (called *Weber relation*, see [**?**]), we get that the stable drop size is given by

$$v_{\rm b} = 10^{-4}\pi D^3 We^{-1.8}, \qquad \left(We := \frac{\Theta D^3 \rho_{\rm c}}{\sigma}\right)$$

where σ, $\rho_{\rm c}$ and D are the surface tension, the density of the dispersed phase and the impeller diameter respectively. The following table shows the critical lower droplet volume $v_{\rm b}$ for a real dispersion (for $\sigma = 29$ dyne/cm, $\rho \simeq 1$ gr/cm^3 and $D = 15$ cm) vs. the angular velocity Θ of the impeller.

Θ (in r.p.m.)	v_b (in cm^3)
1000	$8.094\,10^{-9}$
2000	$6.675\,10^{-10}$
3000	$1.550\,10^{-10}$
4000	$5.505\,10^{-11}$
5000	$2.465\,10^{-11}$
6000	$1.278\,10^{-11}$
7000	$7.342\,10^{-12}$
8000	$4.540\,10^{-12}$

Therefore $v^\star(\infty)/v_\mathrm{b} \simeq 10^4$ for $\Theta \simeq 4000 \div 6000$ r.p.m. (which is a rather standard rotational speed in industrial applications). This means that only with a very large sampling in the v axis (something like 10^5 or more nodes in the v direction) the numerical code is able to appreciate the effect of $v_\mathrm{b} \neq 0$. Being all the simulations done with nodes on the v axis spaced not less than .01 units, we consistently set $v_\mathrm{b} = 0$.

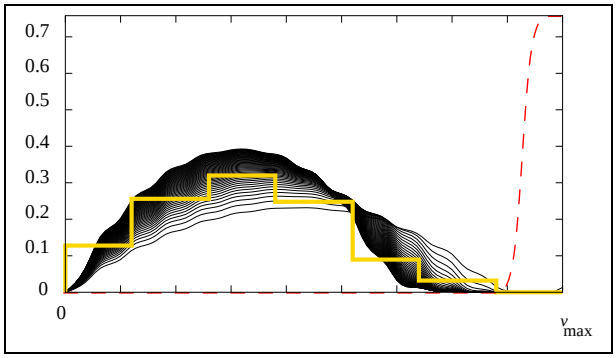

Fig. 3. Evolution from a stepwise initial data (dashed) towards a final distribution (solid lines). The asymptotic shape fits one of the experimental curves (histogram) shown in [**?**] and obtained with a high-speed rotational impeller (ultra-Turrax at 8000 r.p.m.); v_max is about 70 μm

As far as the breakage frequency is concerned we follow [**?**] and [**?**]: thus, for $v_\mathrm{b} = 0$, we write

$$\alpha(w) = A_\mathrm{b} w, \tag{1}$$

and

$$\tau_\mathrm{c}(v,w) = A_\mathrm{c} \exp\left[-\left(\frac{v^{1/3}+w^{1/3}}{v^{1/3}w^{1/3}}\right)^4\right] \times \left(v^{\frac{1}{3}} + w^{\frac{1}{3}}\right)^2, \tag{2}$$

although other laws are commonly encountered in the literature (for example $\alpha(w) = A_\mathrm{b} w^q$ with a positive $q \neq 1$).
The proportionality factors depend on the rotational speed: we set $A_\mathrm{b}(\Theta) = \overline{A}_\mathrm{b}\Theta$, $A_\mathrm{c}(\Theta) = \overline{A}_\mathrm{c}(\overline{\Theta}-\Theta)\Theta$, where $\overline{A}_\mathrm{b}$, $\overline{A}_\mathrm{c}$, $\overline{\Theta}$ are constants which typically depend on the rheology, geometry and the hold-up λ. The expression for A_c is suggested by the fact that in most cases (see [**?**]) the coalescence efficiency increases with Θ up to a maximum and then reduces drastically for high rotational speeds.

5 Numerical simulations

In our simulations we allowed Θ to range from few hundreds to some thousands although, for brevity, only few cases are reported here. We made only one comparison with a real experiment (see figure **??**) to match some constants and scale measures. Indeed our aim here is mainly to show the qualitative behavior of solutions and the distinctive features of the model.

(i) Figure **??** shows a comparison with some available experimental data. Data are taken from [**?**]. The agreement is rather satisfactory.
(ii) Figure **??** and shows the relevance of scattering. Indeed they show the contribution (gain and loss) related to breakage, coalescence and scattering respectively at a mean range rotational speed. The scattering effect plays a significant role being of the same order of magnitude of the other two terms. Data for these figures are all the same as those for figure **??** but Θ that is now 4000 r.p.m.

(iii) Figure **??** shows that the limit configuration depends only on the hold-up while it is totally independent of the specific initial configuration. In our simulations we used the same amount of dispersed phase and the same scale parameters, but we changed the shape of the initial configuration. The final configuration turned out to be always the same.

6 Additional remarks

Further numerical simulations (that we skip for shortness) also confirmed some expected properties of the solution.

(iv) Possible initial discontinuities are rapidly smoothed out by the dynamics and most of the changes in the distribution occurs in the first time steps. Indeed we can prove that the jump discontinuities decrease exponentially.

(v) When coalescence is the dominating effect, volume scattering becomes of great relevance. This is justifies why although the support of the initial distribution grows with time, the asymptotic distribution vanishes at $v_{\max}$.

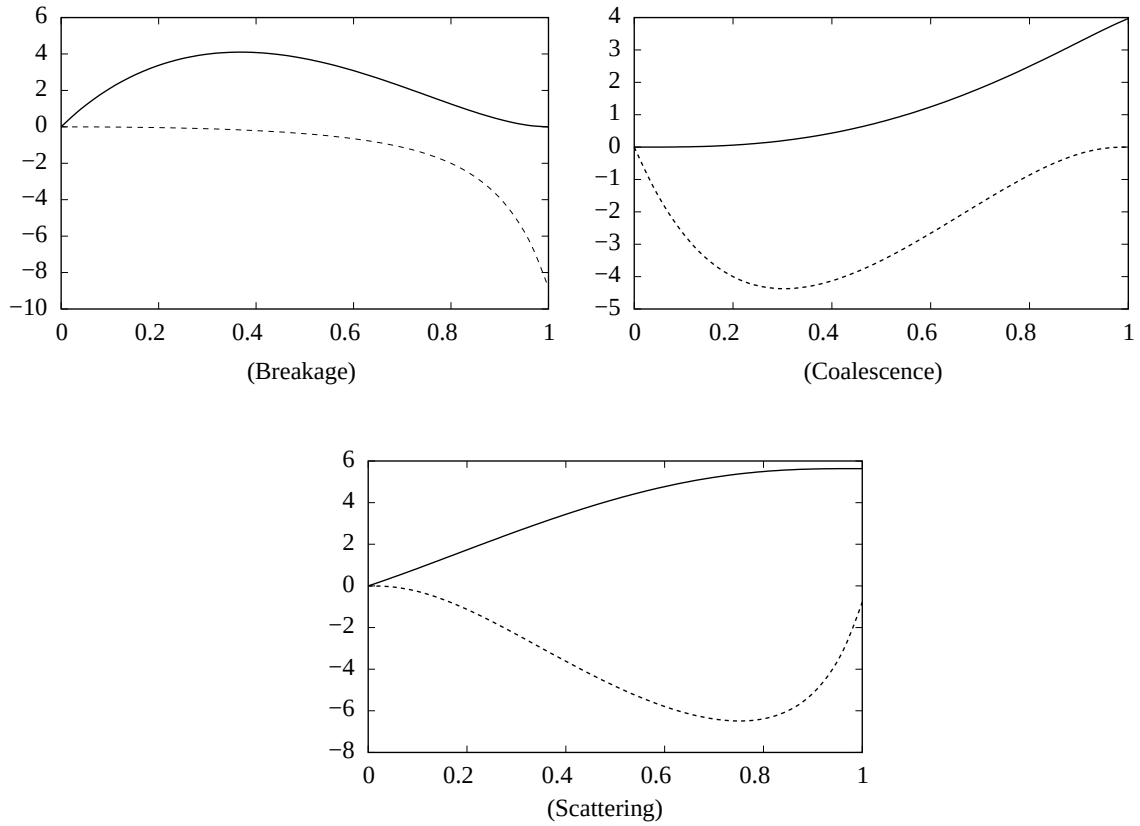

Fig. 4. Mutual importance at equilibrium (large t) of breakage, coalescence and scattering terms at low rotational speed; the dashed line represents the loss term, the solid one the gain term

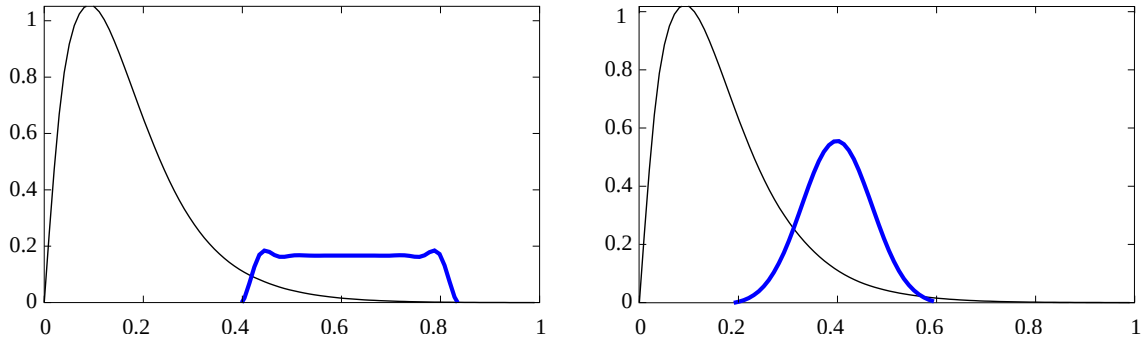

Fig. 5. Invariance of the asymptotic configuration with respect to $f_o(v)$: in the two cases considered the volume of dispersed phase (*hold–up)*) is the same but the shape of the initial distribution is totally different

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