Gravity-driven separation of oil-water dispersions

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Abstract. A model for the separation kinetics of a dispersion of two immiscible liquids under the action of gravity is presented. The scalar case (one family of equally sized drops), which is treated first, naturally suggests the guidelines for the vectorial case (n families of droplets of different sizes). The general model is governed by a non–symmetric system which is investigated for diluted dispersions and concentrated ones as well. In both cases and under very reasonable hypotheses, the system is proved to be strictly hyperbolic which guarantees local existence and uniqueness.

1 Introduction

A *dispersion* is a continuous medium formed by two immiscible liquid, one of which is fragmented in drops (variable in dimension and size) in the other. The drops are usually referred to as "dispersed phase", while the second liquid is called "continuous (or host) phase".

A typical case occurs in petroleum industry during processing operation on crude oil recovered in offshore well-bores ([8]); one deals there with oil in water (O/W) and water in oil (W/O) dispersions. Indeed, oil is usually recovered by using the original reservoir pressure: however, after some years of well-bore activity it is necessary to maintain pressure by injection of sea water, to assist in oil displacement. This leads to undesired O/W or W/O (depending on the hold-up) dispersions or emulsions, and makes it necessary to treat the product before shipping or pumping it to land, in order to separate phases and meet product specifications. In this connection it should be noticed that the separated sea water is

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generally re-dispersed in the surrounding environment and therefore, to minimize pollution, it should contain no more than a few p.p.m. (parts per million) of crude oil. Nevertheless, oil needs to contain as less sea water as possible, because of the corrosive action of the latter when oil is stocked in tanks or pumped through pipelines.

A rather standard procedure for treatment is to place the product in gravity-driven static separators (settlers), until phase separation is complete. Indeed, the separation of oil and water phases occurs spontaneously at rest because of the density difference; sometimes the process needs to be sped up by adding chemical additives, but we will not consider this complication here. Therefore the geometry for this problem is typically one-dimensional in the vertical direction. In this paper we present a model for the separation process which works well for both O/W and W/O dispersions, the only difference being whether denser water drops settle or lighter oil drops rise up. This indifference of the model to phase exchange may be useful for another problem which is also relevant to oil industry. Indeed, it is well known that pure crude oil is very difficult or even impossible to pump directly through a pipeline, because of its high viscosity. Thus it is generally necessary to produce an O/W dispersion in order to drastically reduce viscosity (even up to a factor $10^{-3} \div 10^{-4}$). In this case another scenario opens up to research, due to the instability features of this kind of dispersion. However we do not consider here any shearing in the plane orthogonal to the separation direction. This is a possible future development to be carried out.

The separation kinetics is examined in two subsequent steps; in the former we treat the scalar case (equally sized drops, i.e. mono-dispersed oil in water), while in the latter we deal with n (arbitrarily large integer) families of spherical droplets of different sizes; in both cases, the unknown functions will be the local oil concentrations, which in turn depend on time and space. In the mono-dispersed case we show exact concentration profiles (satisfying the model equations in the sense of distributions), corresponding to different initial distributions of oil. In the poly-dispersed case local existence and uniqueness of the solution are established. In the latter two different situations need to be considered; that of a diluted dispersion (where droplets are supposed to ascend according to a Stokes-like velocity) and that of a concentrated dispersion (where Stokes's law has to be modified to take into account the interactions between the droplets). The same existence and uniqueness results are achieved in both cases.

From the mathematical point of view the physical problem is expressed in terms of a hyperbolic equation (in the mono-dispersed case) or of a non-symmetric hyperbolic system (in the poly-dispersed case). The scalar case is rather elementary and can be approached by classical arguments ([1]). On the contrary the vectorial case (which models better the physical reality) presents some peculiar difficulties that make it nontrivial and original. Because of the structure of the droplets velocity, where the coefficients depend on the relative concentrations of oil droplets (which are the unknowns), the strict hyperbolicity of the system is far from being obvious. A complication also arises from the fact that the system might show parabolic features in some subregions of the domain where a solution is sought. Nevertheless, under very reasonable and not restrictive assumptions on the data this condition can be overcome, and existence and uniqueness are achieved in a suitable neighbourhood of the line carrying the data.

The mono-dispersed case and some exact solutions are presented in section 2, while section

3 is fully devoted to the poly-dispersed case.

2 Mono-dispersed case

Consider a sample of O/W dispersion at rest in a typical settling device, which can be simply modelled as a parallelepiped (see Fig. 1); we denote by $\underline{x}$ the position in the bulk with respect to some frame of reference, and by S_o and S_w the normalized local densities of oil and water in the dispersion, so that

(i) $S_o(\underline{x}, t)$ denotes the volume fraction of the unit cell around $\underline{x}$ occupied by oil at time t,

(ii) $S_w(\underline{x}, t)$ denotes the volume fraction of the unit cell around $\underline{x}$ occupied by water at time t.

S_o and S_w may depend a priori on the three spatial directions and time, but if gravity is the only macroscopic force driving the separation process, we may assume that variations of density occur only in the vertical direction y, and write

$$S_o = S_o(y, t), S_w = S_w(y, t).$$

We assume a frame of reference like in Fig.1, where H is the height of the dispersion. We have by definition

$$\begin{cases} 0 \leq S_o(y,t) \leq 1 \\ \\ 0 \leq S_w(y,t) \leq 1 \end{cases} \quad \forall (y,t) \in [0, H] \times [0, \infty) \tag{1}$$

and

$$S_o(y,t) + S_w(y,t) = 1, \qquad \forall (y,t) \in [0, H] \times [0, \infty). \tag{2}$$

We assume the volume of the container to be much larger than the mean drop diameter, in order to neglect wall effects.

After some time the situation in the bulk will change: the density difference between oil and water leads the oil droplets to flow upwards and form a compact water-free layer at the top of the bulk, while water tends to move towards the bottom and form a layer with no oil droplets; between these two single-phased regions we will find a layer of dispersion (Fig. 1).

2.1 Mathematical model

The governing equations are simply those expressing mass conservation for both phases

$$\frac{\partial (S_o)}{\partial t} + \frac{\partial (S_o V_o)}{\partial y} = 0, \tag{3}$$

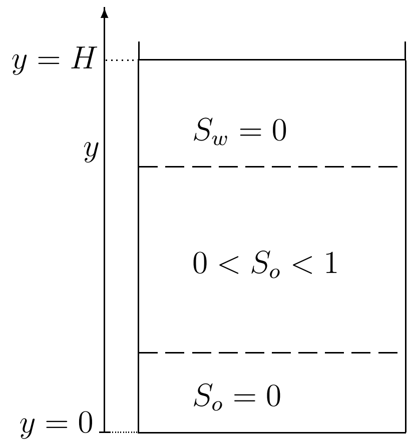

Figure 1: The separation process

$$\frac{\partial(S_w)}{\partial t} + \frac{\partial(S_w V_w)}{\partial y} = 0, \tag{4}$$

where

$$\begin{aligned} V_o &= \text{upward oil phase velocity,} \\ V_w &= \text{downward water phase velocity.} \end{aligned}$$

We assume V_o to depend only on local concentration through the following law (see [4])

$$V_o = V_o(S_o) = k(1 - S_o), \tag{5}$$

where k is a positive constant depending on the particular physical features of the dispersion.

The total flux rate must be equal to zero, that is

$$S_o V_o + S_w V_w = 0, \tag{6}$$

from which we get

$$V_w = -\frac{S_o}{S_w} V_o; \tag{7}$$

the latter, together with (2) and (5) yields

$$V_w = -k S_o = -k(1 - S_w).$$

Note that (3) and (4) are not independent; indeed, substitution of (6) and (2) into (3) yields exactly (4); we may therefore restrict our analysis to one only of the two continuity equations, e.g. the oil one; the behaviour of water phase, once the function $S_o(y, t)$ is known, will follow from relation (2). Substitution of (5) into (3) gives

$$\frac{\partial S_o}{\partial t} + k(1 - 2S_o)\frac{\partial S_o}{\partial y} = 0. \tag{8}$$

The initial condition for (8) is

$$S_o(y,0) = \overline{S_o}(y) \qquad \forall y: \qquad 0 < y < H, \tag{9}$$

where $\overline{S_o}(y)$ is some regular function positive in $(0, H)$.

The boundary conditions for (8) are obtained observing the following: from the beginning of the separation process there will be no oil droplets close to the bottom of the bulk, since those that might be there at $t = 0$ will immediately migrate upwards, due to their nonzero ascending velocity (of course we do not consider the trivial cases $S_o(y,0) = 1 \;\; \forall y \in [0, H]$ or $S_o(y,0) = 0 \;\; \forall y \in [0, H]$). Similarly, the top of the bulk is free from water for any $t > 0$: therefore we write

$$S_o(0,t) = 0, \qquad \forall t: \qquad 0 < t < \infty, \tag{10}$$

$$S_w(H,t) = 0, \qquad \forall t: \qquad 0 < t < \infty. \tag{11}$$

The latter means (remember (2)) that

$$S_o(H,t) = 1 \qquad \forall t: \qquad 0 < t < \infty. \tag{12}$$

Of course more general choices of the initial data are possible (for example $\overline{S_o}(y)$ vanishing over a sub–interval of $[0, H]$) but, in that case, the boundary data $S_o(H, t)$ cannot be assigned any longer in a completely independent manner.

Multiply (8) by $-2k$ and use the linear transformation

$$\Sigma(y,t) = k[1 - 2S_o(y,t)] \tag{13}$$

to get

$$\frac{\partial \Sigma}{\partial t} + \Sigma \frac{\partial \Sigma}{\partial y} = 0. \tag{14}$$

The transformed initial and boundary conditions for (14) turn out to be

$$\Sigma(y,0) = \overline{\Sigma}(y) = k[1 - 2\overline{S_o}(y)], \qquad \forall y: \qquad 0 \le y \le H, \tag{15}$$

$$\Sigma(0,t) = k, \qquad \forall t: \qquad 0 < t < \infty, \tag{16}$$

$$\Sigma(H,t) = -k, \qquad \forall t: \qquad 0 < t < \infty, \tag{17}$$

and the bounds on S_o take for Σ the form

$$|\Sigma(y,t)| \le k, \qquad \forall \; (y,t) \;\in\; [0, H] \times [0, \infty). \tag{18}$$

Equation (14) is the well-known *nonlinear wave equation*. Analytical solutions to problem (14)...(18) are easily obtained, provided a physically meaningful choice of the function (15) is done (see for example [1]); the solution $\Sigma(y,t)$ is often referred to as a *traveling wave*.

The regularity of the solution of the mixed (initial *and* boundary) problem (14) ... (17) depends on the choice of the initial function (15); in particular, discontinuities may propagate from the points $(y, t) = (0, 0)$ and $(y, t) = (H, 0)$ if $\Sigma(y, 0)$ does not join smoothly the boundary conditions there (see Fig. 2). In this case (and also if $\overline{\Sigma}(y)$ fails to be continuous in some subset of $(0, H)$) only generalized solutions can exist. Moreover, the nonlinearity of (14) can cause the breakdown of the solution even if the data are smooth; it is well-known that this happens when different characteristic curves (which are straight lines in our case) intersect after some finite time (breaking time).

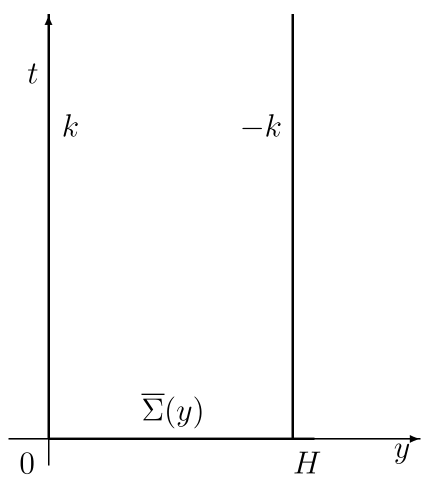

Figure 2: Initial and boundary data

For what concerns the regularity of the data in dependence of $\overline{\Sigma}(y)$, observe that they will be discontinuous if

$$\lim_{y \to 0^+} \overline{\Sigma}(y) \neq k \tag{19}$$

and/or

$$\lim_{y \to H^-} \overline{\Sigma}(y) \neq -k, \tag{20}$$

while they will be at least C^0 if

$$\begin{cases} \lim_{y \to 0^+} \overline{\Sigma}(y) = k, \\ \lim_{y \to H^-} \overline{\Sigma}(y) = -k, \end{cases} \tag{21}$$

and at least C^n if $\overline{\Sigma}(y)$ fulfills (21) *and*

$$\begin{cases} \lim_{y \to 0^+} \overline{\Sigma}^{(h)}(y) = 0, \\ \lim_{y \to H^-} \overline{\Sigma}^{(h)}(y) = 0, \end{cases} \tag{22}$$

for $h = 1, 2, \dots, n$; of course in the latter we assume that $\overline{\Sigma}(y)$ is smooth inside $(0, H)$.

2.2 Exact solutions for particular initial data

<u>*Linear data*</u>

Continuous linear initial data have necessarily the form

$$\overline{\Sigma}(y) = k\big(1 - \frac{2y}{H}\big), \qquad y \in (0, H), \tag{23}$$

which corresponds to an initial oil-phase density

$$\overline{S}_0(y) = \frac{y}{H}, \qquad y \in (0, H). \tag{24}$$

A solution in the classical sense does not exist, due to the non-differentiability of the data at the points $(0, 0)$ and $(H, 0)$, but we can obtain the following weak solution;

$$\begin{cases} \Sigma(y,t) = k, & 0 \le y \le kt, \quad 0 \le t < \dfrac{H}{2k}, \\ \Sigma(y,t) = k\dfrac{H-2y}{H-2kt}, & kt \le y \le H - kt, \quad 0 \le t < \dfrac{H}{2k}, \\ \Sigma(y,t) = -k, & H - kt \le y \le H, \quad 0 \le t < \dfrac{H}{2k} \end{cases} \tag{25}$$

for $t < \dfrac{H}{2k}$ and

$$\begin{cases} \Sigma(y,t) = k, & 0 \le y < \dfrac{H}{2}, \quad \dfrac{H}{2k} \le t, \\ \Sigma(y,t) = -k, & \dfrac{H}{2} < y \le H \quad \dfrac{H}{2k} \le t, \end{cases} \tag{26}$$

for $t \ge \dfrac{H}{2k}$. It can be shown that other analitycal solutions to problem (14)...(18), (23) exist, but (25)-(26) is the only one with physical meaning since it satisfies the so-called *entropy condition* (see [5], [6]).

Note that (25) is not differentiable along the straight lines $y = H - kt$ and $y = kt$, while (26) is discontinuous and shows that complete phase separation occurs for $t \ge \dfrac{H}{2k}$ (see Fig. 3)

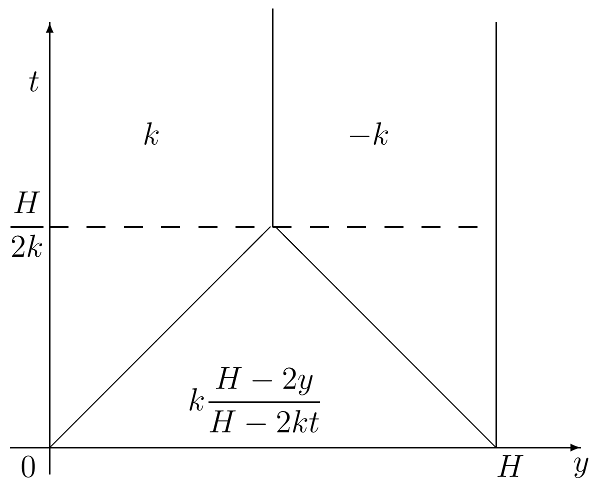

Figure 3: Weak solution for linear initial data

Constant data with a small perturbation

It is certainly more interesting and realistic to work out the case of slightly perturbed constant data, which amounts to consider a nearly homogeneous dispersion for $t = 0$. In this case $\overline{\Sigma}(y)$ has the form

$$\Sigma(y,0) = \overline{\Sigma}(y) = \sigma_0 + \delta F(y), \qquad 0 \le y \le H, \tag{27}$$

where $-k < \sigma_0 < k$, $F(y)$ is a periodic function such that $F(0) = F(H) = 0$, e.g.

$$F(y) = \sin \frac{n\pi}{H} y, \tag{28}$$

n is an arbitrary positive integer and δ is the small (positive) perturbation parameter.

The problem (14)...(18), (27) can only have weak solutions , as it is evident from the discontinuities in the data at $(0,0)$ and $(H,0)$ (see [2]); the determination of the discontinuity curves is not as straightforward as in the previous case, and we must proceed with an approximate method (see [9]). We look for a solution of type

$$\Sigma(y,t) = \sigma_0 + \delta\sigma_1(y,t) + \delta^2\sigma_2(y,t) + \cdots = \sum_{i=0}^{\infty} \delta^i \sigma_i(y,t); \tag{29}$$

by formal substitution of this series into

$$\partial_t \Sigma + \Sigma \partial_y \Sigma = 0, \tag{30}$$

on collecting equal powers of δ and equating their coefficients to zero, we get a recursive family of partial differential equations in the unknowns $\sigma_i(y,t),\ i=1,2,\dots$ and we obtain the corresponding initial conditions by imposing

$$\lim_{t\to 0^+} \sigma_0+\delta\sigma_1(y,t)+\delta^2\sigma_2(y,t)+\cdots=\overline{\Sigma}(y)=\sigma_0+\delta F(y); \tag{31}$$

these problems are easily solved for $i=1,2$ and yield

$$\begin{cases} \sigma_1(y,t)=F(y-\sigma_0 t), \\ \\ \sigma_2(y,t)=-tF(y-\sigma_0 t)F'(y-\sigma_0 t); \end{cases} \tag{32}$$

therefore the second order approximated solution in the influence domain of the initial data is

$$\Sigma(y,t)=\sigma_0+\delta F(y-\sigma_0 t)-\delta^2 tF(y-\sigma_0 t)F'(y-\sigma_0 t)+o(\delta^2). \tag{33}$$

If we choose $F(y)$ as in (28), we obtain

$$\begin{aligned} \Sigma(y,t) &= \sigma_0+\delta\sin\left[\frac{n\pi}{H}(y-\sigma_0 t)\right] \\ &- \frac{n\pi}{H}\delta^2 t\sin\left[\frac{n\pi}{H}(y-\sigma_0 t)\right]\cos\left[\frac{n\pi}{H}(y-\sigma_0 t)\right]+o(\delta^2). \end{aligned} \tag{34}$$

The discontinuity curves $g(t)$, $h(t)$ are defined by the differential equations

$$\begin{cases} g'(t)=\dfrac{1}{2}\left[k+\lim\limits_{y\to g(t)^+}\Sigma(y,t)\right], \\ \\ g(0)=0, \end{cases} \tag{35}$$

and

$$\begin{cases} h'(t)=\dfrac{1}{2}\left[-k+\lim\limits_{y\to h(t)^-}\Sigma(y,t)\right], \\ \\ h(0)=H, \end{cases} \tag{36}$$

which are solved using (34) and give the approximate solutions

$$g(t)\simeq\frac{kH}{n\pi\delta}\left[e^{\frac{n\pi\delta}{2H}t}-1\right]+\frac{\sigma_0}{2}t, \tag{37}$$

$$h(t)\simeq He^{\frac{n\pi\delta}{2H}t}+(\sigma_0+k)\frac{H}{n\pi\delta}\left[1-e^{\frac{n\pi\delta}{2H}t}\right]+\sigma_0 t. \tag{38}$$

The domain of existence of the approximate solution (34) is

$$0\le t<H/2k,\quad g(t)\le y\le h(t),$$

provided δ is small enough. In the subdomains

$$0 \leq t < H/2k, \quad 0 \leq y \leq g(t)$$

and

$$0 \leq t < H/2k, \quad h(t) \leq y \leq H,$$

we have respectively $\Sigma(y,t) = k$ and $\Sigma(y,t) = -k$. The solution may clearly be extended beyond the breaking time $H/2k$, in an analogous manner to (26), to describe complete phase separation.
Note that when $\delta \to 0$ the approximate expressions for $\Sigma(y,t)$, $g(t)$ and $h(t)$ become the exact ones corresponding to unperturbed constant initial data.

3 Poly-dispersed case

In the previous model we implicitly assumed the droplets to be equally sized, i.e. spheres with the same radius. Formula (5) for the ascending velocity only expresses a macroscopic feature of the process, and does not involve any geometrical or rheological parameter of the O/W dispersion. In the practical case, the droplets may have a high *poly-dispersion degree* and their radii may vary a lot; this would affect the way the single droplet moves upwards.

To improve the accuracy of the model, we assume therefore that the droplets volumes range in the discrete set $\{v_1, v_2, ...v_n\}$, n being a positive arbitrary integer. Then we define $f_i(y,t)$ to be the number of droplets with volume v_i in the cell of unit volume centered in y at time t. Let v_0 be the volume of the unit cell, and $S_i(y,t) = \dfrac{v_i f_i(y,t)}{v_0}$ be the volume fraction (of the unit cell at (y,t)) occupied by droplets with volume v_i. If S_o denotes the local volume fraction of oil as before, we have

$$\sum_{i=1}^{n} S_i(y,t) = S_o(y,t), \tag{39}$$

with

$$S_o(y,t) + S_w(y,t) = 1 \tag{40}$$

and

$$\begin{cases} 0 < S_i(y,t) < 1, \\ 0 < S_w(y,t) < 1. \end{cases} \tag{41}$$

Note that the inequalities in (41) differ from the corresponding limitations (1) of the scalar case, since in (41) we assume strict inequality signs. This is just a formal restriction, which leaves the physical situation unchanged but simplifies a bit the proof of the existence and uniqueness results of this section (in particular, it prevents the problem to show parabolic features, as it will be clear later). The unknown functions S_i and S_w can be allowed to take the limiting values in (41), but this requires a modification of the existence and uniqueness proof techniques presented below: for the sake of brevity we do not show this case here.

The above relations are valid in the domain

$$0 \le y \le H, \quad 0 \le t \le T,$$

$(0 < T < \infty)$. Let finally $\eta_d,\ \eta_w,\ \eta_o$ (respectively $\rho_d,\ \rho_w,\ \rho_o$) be the viscosities (respectively the densities) of dispersion, water and oil; as already pointed out, $\rho_o < \rho_w$ is the driving force of the separation process.
The following relation between densities holds

$$\rho_d = S_o\rho_o + S_w\rho_w = S_o\rho_o + (1 - S_o)\rho_w, \qquad \rho_o < \rho_d < \rho_w; \tag{42}$$

we assume η_d to be strictly positive and to depend smoothly on the local oil concentration, and to be influenced in a different way from droplets of different volumes; in particular we assume η_d to be a strictly increasing function of S_i, $i = 1, \ldots, n$. Further, when oil is nearly absent, η_d must be equal to the water viscosity; summarizing we set

$$\eta_d = \eta_d(S_1, S_2, \cdots, S_n) > 0, \quad \frac{\partial \eta_d}{\partial S_i} > 0, \quad \eta_d\big|_{S_o << 1} \simeq \eta_w. \tag{43}$$

We remind that in practice η_o is 2-3 order of magnitude larger than η_w, while differences in density between oil and water are much smaller.

We will distinguish the case of a diluted dispersion from that of a concentrated one; in the former, in fact, a Stokes-like model for the ascending droplet velocity is realistic, while in the latter we need to modify our model to take into account the possible interactions between droplets.

3.1 A model for diluted dispersions

For a single oil droplet rising up in a dispersion of viscosity η_d we propose the following generalization of the classical Stokes's law for the ascending velocity:

$$V_{ai}(y,t) = \gamma \frac{v_i^{2/3}}{\eta_d(S_1(y,t), \ldots, S_n(y,t))}. \tag{44}$$

Here g is gravity and

$$\gamma = \frac{2g}{9\zeta}\left(\rho_d - \rho_o\right), \ \zeta = \left(\frac{4\pi}{3}\right)^{2/3}, \ v_i^{2/3} = \zeta r_i^2,$$

r_i being radius of the (spherical) droplet with volume v_i.

A considerable simplification is achieved if the term $\rho_d - \rho_o$ in γ is replaced by $\rho_w - \rho_o$; this is not a strong assumption, since the difference $(\rho_w - \rho_d)$ is small compared to the other terms. In what follows, we omit the subscript in the viscosity of the dispersion. Then (44) takes the form

$$V_{ai}(y,t) = c\frac{v_i^{2/3}}{\eta(S_1(y,t),\cdots,S_n(y,t))}, \tag{45}$$

with

$$c = \frac{2g}{9\zeta}(\rho_w - \rho_o). \tag{46}$$

The equations governing the poly-dispersed case are analogous to those of the mono-dispersed one;

- **CONTINUITY**

$$\frac{\partial S_i}{\partial t} + \frac{\partial (S_i V_{ai})}{\partial y} = 0, \qquad i = 1,\dots,n, \tag{47}$$

$$\frac{\partial S_w}{\partial t} + \frac{\partial (S_w V_w)}{\partial y} = 0. \tag{48}$$

- **FLUX CONTINUITY**

$$S_w V_w + \sum_{i=1}^{n} S_i V_{ai} = 0. \tag{49}$$

Note that (39), (40), (47), (48) and (49) are not independent; for example, (48) may be obtained by summing (47) over i,

$$\frac{\partial\left(\sum_{i=1}^{n} S_i\right)}{\partial t} + \frac{\partial\left(\sum_{i=1}^{n} S_i V_{ai}\right)}{\partial y} = 0, \tag{50}$$

and using (39), (40) and (49); indeed we get

$$\frac{\partial S_o}{\partial t} + \frac{\partial(-S_w V_w)}{\partial y} = 0 \Rightarrow -\frac{\partial S_w}{\partial t} - \frac{\partial (S_w V_w)}{\partial y} = 0, \tag{51}$$

which is (48). We may therefore confine ourselves to consider only (39), (40), (47) and (49); once $S_1, S_2, \dots, S_n$ are known, the function $S_w(y,t)$ can be obtained directly from (40). To this aim we substitute (45) into (47) to get

$$\frac{\partial S_i}{\partial t} + V_{ai}\frac{\partial S_i}{\partial y} - \frac{c v_i^{2/3}}{\eta^2}\left(\frac{\partial \eta}{\partial S_1}\frac{\partial S_1}{\partial y} + \cdots + \frac{\partial \eta}{\partial S_n}\frac{\partial S_n}{\partial y}\right) S_i = 0. \tag{52}$$

The n equations (52) can be written in the compact form

$$S_t + A(S)S_y = 0, \tag{53}$$

where

$$[A(S)]_{ij} = \begin{cases} \dfrac{1}{\eta^2}\left[-a_i S_i \dfrac{\partial \eta}{\partial S_j}\right] & \text{for } i \neq j \\ \\ \dfrac{1}{\eta^2}\left[a_i(\eta - S_i \dfrac{\partial \eta}{\partial S_i})\right] & \text{for } i = j. \end{cases} \tag{54}$$

$i, j = 1, \dots, n$, where $a_i = c{v_i}^{2/3}$ and $S = (S_1, \dots, S_n)^t$.

3.1.1 System (53) is strictly hyperbolic

We now prove the main result of this paper, that is, the first order quasi-linear system (53) is strictly hyperbolic.

Theorem 1. *The matrix $A(S)$ in (53) has n real and distinct eigenvalues among which at most one may not be positive; further, if the function $\eta(S_1, \dots, S_n)$ satisfies the inequality*

$$\eta - S_1 \frac{\partial \eta}{\partial S_1} - \dots - S_n \frac{\partial \eta}{\partial S_n} > 0, \tag{55}$$

then all the eigenvalues are positive.

Proof. Without loss of generality, we may assume $v_1 > v_2 > \dots > v_n (> 0$, obviously), and therefore $a_1 > a_2 > \dots > a_n$. We consider the matrix $\tilde{A} = \eta^2 A$, whose eigenvalues are real and positive if and only if A's eigenvalues are real and positive. Let us put, for simplicity, $\dfrac{\partial \eta}{\partial S_i} = \eta_i$, the characteristic polynomial $P_n(\lambda)$ of $\tilde{A}$ is given by

$$\begin{vmatrix} (a_1\eta - \lambda - a_1 S_1 \eta_1) & -a_1 S_1 \eta_2 & \dots & -a_1 S_1 \eta_{n-1} & -a_1 S_1 \eta_n \\ -a_2 S_2 \eta_1 & (a_2 \eta - \lambda - a_2 S_2 \eta_2) & \dots & -a_2 S_2 \eta_{n-1} & -a_2 S_2 \eta_n \\ \vdots & & & & \\ -a_{n-1} S_{n-1} \eta_1 & -a_{n-1} S_{n-1} \eta_2 & \dots & (a_{n-1}\eta - \lambda - a_{n-1} S_{n-1} \eta_{n-1}) & -a_{n-1} S_{n-1} \eta_n \\ -a_n S_n \eta_1 & -a_n S_n \eta_2 & \dots & -a_n S_n \eta_{n-1} & (a_n \eta - \lambda - a_n S_n \eta_n) \end{vmatrix} \tag{56}$$

Multiplying the first column of $(\tilde{A} - \lambda I)$ by $(-\eta_n/\eta_1)$ and adding the result to the n-th column, we obtain the following expression for $P_n(\lambda)$:

$$\begin{vmatrix} (a_1\eta - \lambda \\ -a_1S_1\eta_1\ ) & -a_1S_1\eta_2 & \dots & -a_1S_1\eta_{n-1} & (\lambda - a_1\eta)(\eta_n/\eta_1) \\ -a_2S_2\eta_1 & (a_2\eta - \lambda \\ & -a_2S_2\eta_2\ ) & \dots & -a_2S_2\eta_{n-1} & 0 \\ & \dots & & & \\ \vdots & & & & \\ -a_{n-1}S_{n-1}\eta_1 & -a_{n-1}S_{n-1}\eta_2 & \dots & (a_{n-1}\eta - \lambda - \\ & & & a_{n-1}S_{n-1}\eta_{n-1}\ ) & 0 \\ -a_nS_n\eta_1 & -a_nS_n\eta_2 & \dots & -a_nS_n\eta_{n-1} & a_n\eta - \lambda \end{vmatrix} \tag{57}$$

Therefore

$$P_n(\lambda) = (a_n\eta - \lambda)P_{n-1}(\lambda) + (-1)^{n+1}(-\eta_n/\eta_1)(a_1\eta - \lambda)\det\mathfrak{A}, \tag{58}$$

where $P_{n-1}(\lambda)$ is the characteristic polynomial of $\tilde{A}_{(n-1)\times(n-1)}$ and $\mathfrak{A}$ is the $(n-1)\times(n-1)$ matrix given by

$$[\mathfrak{A}]_{ij} = \begin{cases} -a_{i+1}S_{i+1}\eta_j & \text{for } i+1 \neq j \\ & i,j = 1,\dots,n-1 \\ a_j\eta - \lambda - a_jS_j\eta_j & \text{for } i+1 = j. \end{cases} \tag{59}$$

We now multiply the i-th column of $\mathfrak{A}$ by $(-\eta_{i-1}/\eta_i)$ and add the result to the $(i-1)$-th column, $i = 2,\dots,n-1$. This yields an upper triangular matrix C, where the first $n-2$ entries on the diagonal are

$$(\lambda - a_{i+1}\eta)\frac{\eta_i}{\eta_{i+1}}, \quad i = 1,\dots,n-2$$

and the last diagonal term is $-a_nS_n\eta_{n-1}$; evidently det $C =$ det $\mathfrak{A}$, and so

$$\begin{aligned} \det\ \mathfrak{A} &= -a_nS_n\eta_{n-1}\prod_{i=1}^{n-2}\frac{\eta_i}{\eta_{i+1}}(\lambda - a_{i+1}\eta) \\ &= (-1)^{n-1}\eta_1 a_n S_n\prod_{i=2}^{n-1}(a_i\eta - \lambda) \end{aligned} \tag{60}$$

Substituting the latter in $P_n(\lambda)$ we get

$$\begin{aligned} P_n(\lambda) &= (a_n\eta - \lambda)P_{n-1}(\lambda) - a_nS_n\eta_n\prod_{i=1}^{n-1}(a_i\eta - \lambda) \\ &= (a_n\eta - \lambda)P_{n-1}(\lambda) - a_nS_n\eta_nQ_{n-1}(\lambda), \end{aligned} \tag{61}$$

where we have set

$$Q_{n-1}(\lambda) = (a_1\eta - \lambda)(a_2\eta - \lambda)\cdots(a_{n-1}\eta - \lambda). \tag{62}$$

We now assume that

$$sgn\ P_{n-1}(a_j\eta) = (-1)^{n+j}, \quad j = 1, 2, \ldots, n-1, \tag{63}$$

and apply the induction principle. It is easily checked that (63) is true for $P_2(a_j\eta)$; we now verify that the same equality holds for n, i.e. that

$$sgn\ P_n(a_j\eta) = (-1)^{n+j+1}, \quad j = 1, 2, \ldots, n. \tag{64}$$

Thanks to the previously found expression (58) it is quite straightforward to get

$$\begin{cases} P_n(a_n\eta) = \overbrace{-a_nS_n\eta_n}^{<0} \cdot \overbrace{(a_1 - a_n)\eta}^{>0} \cdot \overbrace{(a_2 - a_n)\eta}^{>0} \cdots \overbrace{(a_{n-1} - a_n)\eta}^{>0} < 0, \\ P_n(a_{n-1}\eta) = \overbrace{(a_n - a_{n-1})\eta}^{<0} \cdot \overbrace{P_{n-1}(a_{n-1}\eta)}^{<0 \text{ (ind. hp.)}} - a_nS_n\eta_n \overbrace{Q_{n-1}(a_{n-1}\eta)}^{=0} > 0, \\ \vdots \\ P_n(a_j\eta) = \underbrace{\overbrace{(a_n - a_j)\eta}^{<0} \cdot \overbrace{P_{n-1}(a_j\eta)}^{\text{has sign } (-1)^{n+j} \text{ (ind.hp.)}}}_{\text{has sign } (-1)^{n+j+1}} - a_nS_n\eta_n \overbrace{Q_{n-1}(a_j\eta)}^{=0}, \end{cases} \tag{65}$$

where we have used the induction hypothesis (ind. hp.) on P_{n-1}; we end up with

$$sgn\ P_n(a_j\eta) = (-1)^{n+j+1}\ \forall\ j = 1, 2, \ldots, n,$$

i.e.(64) turns out to be true.

We have now important indications about the values of $P_n(\lambda)$ when

$$\lambda = a_n\eta,\ a_{n-1}\eta, \ldots, a_1\eta.$$

These indications are shown qualitatively in Fig. 4 (n even) and 5 (n odd).

Due to our assumptions on η, $P_n(\lambda)$ has continuous coefficients. Therefore we may say that for any n, $P_n(\lambda)$ has a real zero in each interval $(a_{i+1}\eta, a_i\eta)$. Moreover, since

$$P_n(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}(tr\tilde{A})\lambda^{n-1} + \ldots - \ldots + \det\tilde{A}, \tag{66}$$

we get

$$\lim_{\lambda\to\pm\infty} P_n(\lambda) = +\infty, \qquad \text{if } n \text{ is even},$$

$$\lim_{\lambda\to\pm\infty} P_n(\lambda) = \mp\infty \qquad \text{if } n \text{ is odd}.$$

Therefore $P_n(\lambda)$ must intersect exactly n times the λ-axis, and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\tilde{A}$ are all real; furthermore, if we rearrange them in such a way that $\lambda_n < \lambda_{n-1} < \ldots < \lambda_2 < \lambda_1$, we find that for all n

$$\begin{cases} \lambda_i \in (a_{i+1}\eta, a_i\eta) \quad i = 1, \ldots, n-1, \\ \lambda_n \in (-\infty, a_n\eta). \end{cases} \tag{67}$$

This implies that:

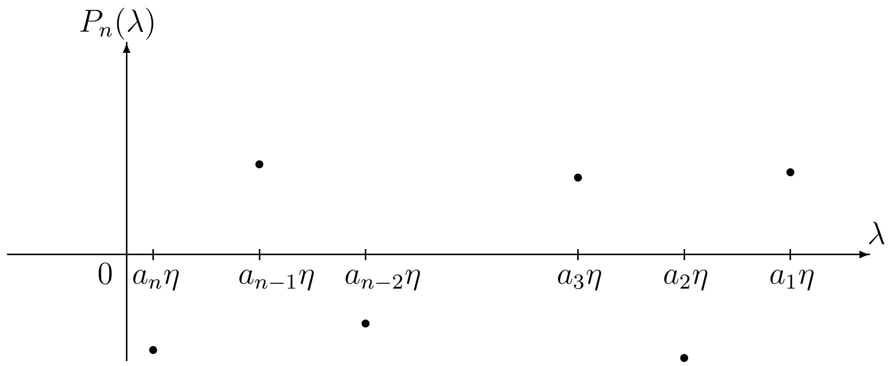

Figure 4: $P_n(a_j\eta)$, n even

i) $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct because they belong to disjoint intervals:

ii) $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ are strictly positive (since a_i and η are positive quantities).

Recalling that $\tilde{A} = \eta^2 A$, it follows that the eigenvalues of $A(S)$, say $\mu_1, \mu_2, \dots, \mu_n$ ($\mu_i = \dfrac{\lambda_i}{\eta^2}$), satisfy

$$\begin{cases} \mu_i \in \left(\dfrac{a_{i+1}}{\eta}, \dfrac{a_i}{\eta}\right), \quad i = 1, \dots, n-1, \\ \mu_n \in \left(-\infty, \dfrac{a_n}{\eta}\right). \end{cases} \tag{68}$$

Up to a scaling factor $(1/\eta^2)$, the same properties (i) and (ii) hold for the eigenvalues μ_i, which are therefore real and distinct.

All eigenvalues, except at most the smallest μ_n, are positive, and it is readily seen that if det $\tilde{A} > 0$ then also λ_n, and therefore μ_n, is positive. Now, we have

$$\begin{cases} tr\tilde{A} = \displaystyle\sum_{i=1}^{n} a_i(\eta - S_i\eta_i), \\ \det \tilde{A} = a_1 a_2 \dots a_n \eta^{n-1} \left[\eta - \displaystyle\sum_{i=1}^{n} S_i\eta_i\right] \end{cases} \tag{69}$$

(the second equation is found through an induction argument); this shows that if the function $\eta(S_1, \dots, S_n)$ satisfies the inequality

$$\eta > \sum_{i=1}^{n} S_i\eta_i \tag{70}$$

then also the smallest eigenvalue μ_n is positive; the proof of the theorem is thus complete. □

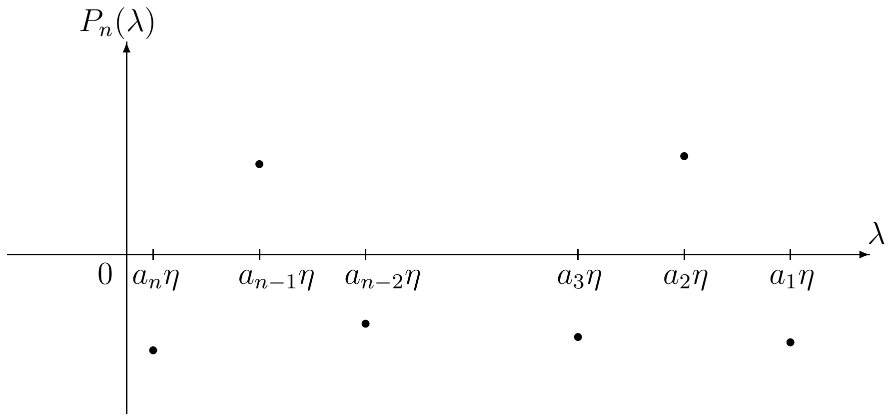

Figure 5: $P_n(a_j\eta)$, n odd

Remark 1. *A reasonable choice for* $\eta(S_1,\dots,S_n)$ *is*

$$\eta(S_1,S_2,\dots,S_n)=\xi_1S_1+\xi_2S_2+\dots+\xi_nS_n+\eta_wS_w, \tag{71}$$

where $\xi_i>0$, $i=1,\dots,n$, *and* $\xi_i\neq\xi_j$ *for* $i\neq j$, *(each* $S_i(y,t)$ *may influence the viscosity of the dispersion in a different way); we can easily check that (71) satisfies condition (70) for* $S_w>0$.

Remark 2. *Note that (71) can be taken as the linear expansion of* $\eta(S)=\eta(S_1,\dots,S_n)$ *around* $S_1=\ldots=S_n=0$. *Then we can infer from (69) and (70) that* $\mu_n=\mu_n(S_1,\dots,S_n)$ *is positive at points of the yt-plane where the oil density functions take arbitrarily small values.*

3.1.2 Existence and uniqueness

Since $A=A(S)$ n has real and distinct eigenvalues $\mu_1,\mu_2,\dots,\mu_n$, the corresponding right eigenvectors $r_1,r_2,\dots,r_n$ span $\mathbb{R}^n$. Recalling [3], C.2, p.64, we set

$$W=L^{-1}S, \tag{72}$$

where L is the (non singular) $n\times n$ matrix whose i-th column is r_i, and we can re-write system (53) in the following form:

$$W_t+DW_y+G=0, \tag{73}$$

or equivalently

$$\frac{\partial W_i}{\partial t}+\mu_i(W)\frac{\partial W_i}{\partial y}+(G)_i=0, \tag{74}$$

where D is the diagonal $n \times n$ matrix such that $D_{ii} = \mu_i$ and

$$G = (L^{-1}L_t + DL^{-1}L_y)W.$$

If $A(S)$ has Lipschitz continuous coefficients w.r.t. S and if $S \in C^1$, then (53) has n distinct characteristic directions. The initial conditions are a straightforward generalization of (9);

$$S_i(y,0) = S_{0i}(y) \qquad i = 1, \dots, n, \tag{75}$$

i.e. $S(y,0) = S_0(y)$: boundary conditions require some more care and we will formulate them later. The initial data for the system (73) follow from $W = L^{-1}S$;

$$W_i(y,0) = W_{0i}(y) = [L^{-1}(y,0)S(y,0)]_i, \quad i = 1, \dots, n, \tag{76}$$

that is, $W(y,0) = L^{-1}(y,0)S(y,0) = W_0(y)$.

We recall now that if $(\phi(y,t),1)$ is the direction of a curve Γ of the yt-plane and P is a point belonging to Γ, Γ is said to be spacelike in P with respect to system (53) if

$$\mu_n(P) \leq \phi(P) \leq \mu_1(P),$$

and timelike in P otherwise. Evidently the y-axis is always timelike and therefore never a characteristic direction. We now set

$$\Gamma_1 \ : \ \begin{cases} y = 0, \\ 0 \leq t \leq \infty, \end{cases} \tag{77}$$

$$\Gamma_2 \ : \ \begin{cases} y = H, \\ 0 \leq t \leq \infty, \end{cases} \tag{78}$$

$$M = \{0 \leq y \leq H, \quad 0 \leq t < +\infty\}, \tag{79}$$

and apply existence and uniqueness theorems shown in [3], C.2, by distinguishing two cases.

(1) $\mu_n|_{\Gamma_2} \leq 0$: in this case Γ_2 is a spacelike curve. Since $\mu_1, \dots, \mu_{n-1}$ are positive (theorem 1) and only one eigenvalue (μ_n) is non-positive along Γ_2, for any $P' \in \Gamma_2$ and close to the initial line there are exactly $n-1$ characteristic curves starting from the initial line that intersect Γ_2 in P', that means that we have to specify one extra condition on Γ_2; this will be obtained as a straightforward extension of the scalar condition (12), namely we put

$$\sum_{i=1}^{n} S_i(H,t) = 1 - \epsilon, \tag{80}$$

i.e. almost no water at the top of the bulk (ϵ arbitrarily small positive parameter); the corresponding condition for W's components is found through $W = L^{-1}S$ to be

$$\sum_{i=1}^{n} z_i W_i(H,t) = 1 - \epsilon \qquad (z_i = \sum_{j=1}^{n} L_{ji}). \tag{81}$$

We can easily see that Γ_1 is a timelike curve: indeed, along Γ_1 a condition analogous to (10) has to hold, i.e. $S_i(0,t)$ nearly zero, $i = 1, \dots, n$. By virtue of Remark 2 we can infer that the smallest eigenvalue of $A(S)$ is positive along Γ_1, and the latter is therefore timelike.

If the coefficients of $A(S)$ have Lipschitz continuous second derivatives, then the W_{0i} (i.e. the components of the initial vector $W_0(y)$) have Lipschitz continuous derivatives; in this case theorem 2.2 p.74 in [3] applies and we can state that, in a neighbourhood of the initial line $0 \le y \le H,\ t = 0$, *the system (*73*) with initial data (*76*) and boundary data (*81*) has a unique solution*: due to the one–to–one mapping (72), also system (53) with conditions (75), (80) has a unique solution.

(2) $\mu_n|_{\Gamma_2} > 0$: in this case neither Γ_1 nor Γ_2 are spacelike curves. The problem is then a purely initial value one, and (provided the coefficient of the matrix $A(S)$ are smooth functions) existence and uniqueness for this case are guaranteed by theorem 2.1, p.71 in [3].

3.2 A model for concentrated dispersion

The previous model is valid as long as dispersions with a low fraction (generally less than 10%) of dispersed phase are considered, since in the opposite case the mutual interactions of the droplets during their motion cannot be neglected. When the fraction of dispersed phase increases above these values the expression (45) for the ascending velocity has to be modified; this is achieved by introducing a crowding factor, like in the scalar case. So we replace (45) by

$$V_{ai}(y,t) = c\frac{v_i^{2/3}}{\eta(S_1,\dots,S_n)}(1 - S_o), \tag{82}$$

i.e., reminding (39),

$$V_{ai}(y,t) = c\frac{v_i^{2/3}}{\eta(S_1,\dots,S_n)}(1 - \sum_{i=1}^{n} S_i), \tag{83}$$

where c and v_i have the same meanings described in section 3 and in subsection 3.1. All the other equations found for the dilute dispersion still hold; again we set $\dfrac{\partial \eta}{\partial S_i} = \eta_i$, substitute (83) in each of the n equations

$$\frac{\partial S_i}{\partial t} + \frac{\partial (S_i V_{ai})}{\partial y} = 0, \qquad i = 1, \dots, n, \tag{84}$$

and get the system

$$S_t + B(S)S_y = 0, \tag{85}$$

where $S = (S_1, \dots, S_n)$ as before and $B(S)$ is given by

$$[B(S)]_{ij} = \begin{cases} -\dfrac{1}{\eta^2}[a_i S_i(\eta_j S_w + \eta)] & \text{for } i \neq j, \\ & i,j = 1, \dots, n, \\ -\dfrac{1}{\eta^2}[a_i(\eta_i S_w S_i + \eta S_i - \eta S_w)] & \text{for } i = j. \end{cases} \tag{86}$$

The system (85) is again a first order quasi-linear one.

3.2.1 System (85) is strictly hyperbolic

We will go on proving that the eigenvalues of $B(S)$ are real and distinct by a similar argument as in the case of diluted dispersions. This is done in the following

Theorem 2. *The matrix $B(S)$ in (85) has n real and distinct eigenvalues. Moreover, if the inequality*

$$S_w \sum_{i=1}^{n} \eta_i S_i + \eta\Big(\sum_{i=1}^{n} S_i - S_w\Big) < 0$$

holds, then all the eigenvalues are positive.

Proof. The eigenvalues $\lambda_1, \dots, \lambda_n$ of $\tilde{B} = \eta^2 B$ differ from those of B, say $\mu_1, \dots, \mu_n$, only by a real and positive factor η^2. The characteristic polynomial $P_n(\lambda)$ of $\tilde{B}$ is

$$\begin{vmatrix} -a_1(\eta_1 S_w S_1 + \eta S_1 - \eta S_w) - \lambda & -a_1 S_1(\eta_2 S_w + \eta) & -a_1 S_1(\eta_3 S_w + \eta) & \dots & -a_1 S_1(\eta_n S_w + \eta) \\ -a_2 S_2(\eta_1 S_w + \eta) & -a_2(\eta_2 S_w S_2 + \eta S_2 - \eta S_w) - \lambda & -a_2 S_2(\eta_3 S_w + \eta) & \dots & -a_2 S_2(\eta_n S_w + \eta) \\ -a_3 S_3(\eta_1 S_w + \eta) & -a_3 S_3(\eta_2 S_w + \eta) & -a_3(\eta_3 S_w S_3 + \eta S_3 - \eta S_w) - \lambda & \dots & -a_3 S_3(\eta_n S_w + \eta) \\ \vdots & & & & \\ -a_n S_n(\eta_1 S_w + \eta) & -a_n S_n(\eta_2 S_w + \eta) & -a_n S_n(\eta_3 S_w + \eta) & \dots & -a_n(\eta_n S_w S_n + \eta S_n - \eta S_w) - \lambda \end{vmatrix} \tag{87}$$

Multiply the first column of $(\tilde{B} - \lambda I)$ by $-\dfrac{\eta_n S_w + \eta}{\eta_1 S_w + \eta}$ and add the result to the last column; we get

$$\begin{pmatrix} -a_1(\eta_1 S_w S_1 + & \dots & -a_1 S_1(\eta_{n-1} S_w + \eta) & (\lambda - a_1 S_w \eta)\dfrac{\eta_n S_w + \eta}{\eta_1 S_w + \eta} \\ \eta S_1 - \eta S_w) - \lambda & & & \\ -a_2 S_2(\eta_1 S_w + \eta) & \dots & -a_2 S_2(\eta_{n-1} S_w + \eta) & 0 \\ \vdots & & & \\ -a_{n-1} S_{n-1} & \dots & -a_{n-1}(\eta_{n-1} S_w S_{n-1} + & 0 \\ (\eta_1 S_w + \eta) & & \eta S_{n-1} - \eta S_w) - \lambda & \\ -a_n S_n(\eta_1 S_w + \eta) & \dots & -a_n S_n(\eta_{n-1} S_w + \eta) & a_n S_w \eta - \lambda \end{pmatrix} \tag{88}$$

We then evaluate the determinant along the last column to get

$$\begin{aligned} \det(\tilde{B} - \lambda I) &= (a_n S_w \eta - \lambda) P_{n-1}(\lambda) \\ &+ (-1)^{n+1}(\lambda - a_1 S_w \eta)\frac{\eta_n S_w + \eta}{\eta_1 S_w + \eta} \det \mathfrak{B}, \end{aligned} \tag{89}$$

where $\mathfrak{B}$ is the $(n-1) \times (n-1)$ matrix given by

$$[\mathfrak{B}]_{ij} = \begin{cases} -a_{i+1} S_{i+1}(\eta_j S_w + \eta) & i+1 \neq j \\ -a_j(\eta_j S_w S_j + \eta S_j - \eta S_w) - \lambda & i+1 = j, \end{cases} \tag{90}$$

$i, j = 1, \dots, n-1$. Multiply the $(i+1)$-th column of $\mathfrak{B}$ by $-\dfrac{\eta_i S_w + \eta}{\eta_{i+1} S_w + \eta}$ and add the result to the i-th column, $i = 1, \dots, n-2$: we end up with the upper triangular matrix K where the first $n-2$ entries on the diagonal are

$$(\lambda - a_{i+1}\eta S_w)\frac{\eta_i S_w + \eta}{\eta_{i+1} S_w + \eta}, \quad i = 1, \dots, n-2$$

and the last diagonal term is $-a_n S_n(\eta_{n-1} S_w + \eta)$. We obviously have $\det K = \det \mathfrak{B}$ and so

$$\det \mathfrak{B} = (-1)^{n-1} a_n S_n(\eta_1 S_w + \eta) \prod_{i=1}^{n-2}(a_{i+1}\eta S_w - \lambda). \tag{91}$$

By substitution of the latter in (89) we find

$$\begin{aligned} P_n(\lambda) &= \det(\tilde{B} - \lambda I) \\ &= (a_n S_w \eta - \lambda) P_{n-1}(\lambda) - a_n S_n(\eta_n S_w + \eta) \underbrace{\prod_{i=1}^{n-1}(a_i \eta S_w - \lambda)}_{=:\, Q_{n-1}(\lambda)} \\ &= (a_n S_w \eta - \lambda) P_{n-1}(\lambda) - a_n S_n(\eta_n S_w + \eta) Q_{n-1}(\lambda). \end{aligned} \tag{92}$$

Now we assume as an induction argument that

$$sgn\ P_{n-1}(a_j S_w \eta) = (-1)^{n+j}, \quad j = 1, \dots, n-1 \tag{93}$$

(which is easily checked for $n = 2$), and show that it holds for n, i.e. that

$$sgn\ P_n(a_j S_w \eta) = (-1)^{n+j+1}, \quad j = 1, \dots, n. \tag{94}$$

Let us evaluate $P_n(\lambda)$ in the points $\lambda = a_j S_w \eta, \quad j = 1, \dots, n$: we find

$$\begin{cases} P_n(a_n S_w \eta) = \overbrace{-a_n S_n}^{<0} \cdot \overbrace{(\eta_n S_w + \eta)}^{>0} \overbrace{\prod_{i=1}^{n-1} (a_i - a_n)\eta S_w}^{>0} < 0, \\ P_n(a_{n-1} S_w \eta) = \overbrace{(a_n - a_{n-1})}^{<0} \cdot \overbrace{S_w \eta}^{>0} \cdot \overbrace{P_{n-1}(a_{n-1} S_w \eta)}^{<0 \text{ (ind.hp.)}} > 0, \\ \vdots \\ P_n(a_j S_w \eta) = \underbrace{\overbrace{(a_n - a_j)\eta}^{<0} \overbrace{S_w \eta}^{>0} \cdot \overbrace{P_{n-1}(a_j S_w \eta)}^{\text{has sign } (-1)^{n+j}}}_{\text{has sign } (-1)^{n+j+1}} - a_n S_n \eta_n \overbrace{Q_{n-1}(a_j \eta)}^{=0}, \end{cases} \tag{95}$$

where as in the diluted case we have used the induction argument (ind. hp.) for the sign of P_{n-1}: it then follows that $sgn\ P_n(a_j S_w \eta) = (-1)^{n+j+1}\ \forall\ j = 1, 2, \dots, n$, and (94) is proved. By observing that $P_n(a_j S_w \eta)$ has the opposite sign of $P_n(a_{j+1} S_w \eta), \quad j = 1, \dots, n-1$, and using the relation

$$P_n(\lambda) = (-1)^n \lambda^n + (-1)^{n-1}(tr\tilde{B})\lambda^{n-1} + \dots - \dots + \det \tilde{B} \tag{96}$$

to determine $\lim_{\lambda \to \pm\infty} P_n(\lambda)$, we can see that $P_n(\lambda)$ intersects the λ-axis exactly n times, namely in the intervals $(a_{j+1} S_w \eta, a_j S_w \eta), \quad j = 1, \dots, n-1$, and $(-\infty, a_n S_w \eta)$; $P_n(\lambda)$ has n *real and distinct* roots, and system (85) is therefore *strictly hyperbolic*. The relation $\mu_i = \dfrac{\lambda_i}{\eta^2}$ between the eigenvalues of $\tilde{B}$ and those of B shows immediately that the ranges of the eigenvalues μ_i are (assume $\mu_n < \mu_{n-1} < \dots < \mu_2 < \mu_1$)

$$\begin{cases} \mu_i \in \left(\dfrac{a_{i+1} S_w}{\eta}, \dfrac{a_i S_w}{\eta} \right), \qquad i = 1, 2, \dots, n-1, \\ \mu_n \in \left(-\infty, \dfrac{a_n S_w}{\eta} \right). \end{cases} \tag{97}$$

All the eigenvalues, except at most the smallest μ_n, are positive; as in the diluted case, we can see that the signs of λ_n and μ_n coincide with the sign of det $\tilde{B}$. By letting $\lambda = 0$ in (92) we get

$$\det\ \tilde{B} = a_n S_w \eta P_{n-1}(0) - a_n S_n (\eta_n S_w + \eta)(\eta S_w)^{n-1} a_1 \cdots a_{n-1}, \tag{98}$$

which suggests the following expression for det $\tilde{B}$:

$$\det\ \tilde{B} = -a_1 a_2 \cdots a_n (\eta S_w)^{n-1} \left[S_w \sum_{i=1}^{n} \eta_i S_i + \eta \sum_{i=1}^{n} S_i - S_w \eta \right] \tag{99}$$

Relation (99) can be proved to hold true by induction, and the sign of μ_n is then the opposite of the quantity in square brackets in (99). This concludes the proof of the theorem. □

3.2.2 Existence and uniqueness

As in the dilute case, the correct formulation of existence and uniqueness results depends on the sign of the quantity in square brackets in (99) along the boundary. We use the same notation as in 3.1.2 for the boundary lines, and therefore denote by Γ_1 and Γ_2 respectively the curves $\{y = 0, \quad t \geq 0\}$ and $\{y = H, \quad t \geq 0\}$.

It is readily seen by an argument similar to that in Remark 2 that Γ_1 is always timelike, while again Γ_2 turns out to be spacelike if μ_n is non–positive (i.e. if the quantity in square brackets in (99) is non–negative) and timelike otherwise.

In the first case data are specified partially on a timelike curve (Γ_1 and the initial line $t = 0, \quad 0 \leq y \leq H$) and partially on a spacelike curve (Γ_2), and the boundary conditions on Γ_2 are assigned exactly like in the diluted case. The system (85) represents then a *mixed* problem, for which existence and uniqueness of the solution are ensured by theorem 2.2 p.74, in [3].

When also the smallest eigenvalue μ_n is positive, none of the boundary curves is spacelike, so that the problem is a pure initial value one; existence and uniqueness of the solution rely both on theorem 2.1, p.71 in [3].

Remark 3. *Comparison between (97) and (68) shows that the eigenvalues for dilute dispersions are larger than those for concentrate dispersions; this is simply a consequence of the fact that in dilute dispersions the droplets' ascending velocities are larger because of reduced interaction among them.*

Remark 4. *The mono-dispersed model may be re-obtained from the poly-dispersed one (concentrated case) as follows. Let us first define (see [7])*

$$r_m^2 = \frac{\sum_{i=1}^{n} S_i r_i^2}{\sum_{i=1}^{n} S_i} = \text{ quadratic mean radius,} \tag{100}$$

and remind that, for the i-th family of droplets, equation (47) holds. By summing over i and using $\sum_{i=1}^{n} S_i = S_o$ *we also get*

$$\frac{\partial S_o}{\partial t} + \frac{\partial (\sum_{i=1}^{n} S_i V_{ai})}{\partial y} = 0. \tag{101}$$

If we define

$$\overline{V_o} := \frac{\sum_{i=1}^{n} V_{ai} S_i}{\sum_{i=1}^{n} S_i} =: \text{ ascending mean velocity,} \tag{102}$$

(101) may be written as

$$\frac{\partial S_o}{\partial t} + \frac{\partial(\overline{V_o} S_o)}{\partial y} = 0. \tag{103}$$

The ascending mean velocity $\overline{V_o}$ has the same meaning of the scalar ascending velocity

$$V_o(S_o) = k(1 - S_o)$$

in (5), where the constant k introduced there had not been characterized any further. However, from (82) we have

$$\begin{aligned} V_{ai}(y,t) &= \frac{2g}{9\zeta}(\rho_w - \rho_o)\frac{v_i^{2/3}}{\eta(S_1, \dots, S_n)} S_w \\ &= \frac{2g}{9}(\rho_w - \rho_o)\frac{r_i^2}{\eta} S_w, \end{aligned} \tag{104}$$

and so we get

$$\begin{aligned} \overline{V_o} &= \frac{\sum_{i=1}^{n} V_{ai} S_i}{\sum_{i=1}^{n} S_i} = \frac{2g}{9\eta}(\rho_w - \rho_o)\frac{\sum_{i=1}^{n} S_i r_i^2}{\sum_{i=1}^{n} S_i} S_w \\ &= \frac{2g}{9\eta}(\rho_w - \rho_o) r_m^2 (1 - S_o) = V_o, \end{aligned} \tag{105}$$

provided that k has been chosen to be

$$k := \frac{2g}{9\eta}(\rho_w - \rho_o) r_m^2. \tag{106}$$

Note that the viscosity in (106) is constant. The equal size of all the droplets in the scalar case is given by the mean radius defined in (100), while the way the constant k is related to the physical properties of the dispersion is shown by (106).

References

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Interscience, New York, 1953.

[2] A. Jeffrey, *The Evolution of Discontinuities of Homogeneous Nonlinear Hyperbolic Equations Smooth Initial Data*, J. Math. and Mech. **17** (1967), 331–352.

[3] ———, *Quasilinear Hyperbolic Systems and Waves*, Pitman, Boston, 1976.

[4] G. J. Kynch, *A Theory of Sedimentation*, Trans. Faraday Soc. **48** (1952), 166–176.

[5] P. D. Lax, *Nonlinear Hyperbolic Equations*, Comm. Pure and Applied Mathematics **6** (1953), 231–258.

[6] ______, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM Regional Conference Series in Applied Mathematics, no. 11, SIAM, Philadelphia, 1973.

[7] C. Orr., *Emulsion Droplet Size Data*, Encyclopedia of Emulsion Technology (P. Becher, ed.), vol. I, Basic Theory, Marcel Dekker Inc., 1979, pp. 369–404.

[8] K. Panoussopoulos, *Separation of Crude Oil-Water Emulsions: Experimental Techniques and Models*, Ph.D. thesis, Swiss Federal Institute of Technology, 1998.

[9] E. Zauderer, *Partial Differential Equations of Applied Mathematics*, Wiley-Interscience, New York, 1989.

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