Multiple fragmentation of liquid droplets in agitated dispersions

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Abstract

A new approach is presented to investigate the multiple breakage of liquid droplets in agitated dispersions. The model includes, besides coalescence and breakage, also the volume scattering as leading mechanism of the evolution. The key role played, as far as the global existence of the solution, by a positive threshold size below which drops are stable versus fragmentation and coalescence is clearly emphasized.

1. Introduction

In [2,3] we presented a new model for the dynamics of dispersions in an agitated vessel in which the break–up of droplets with volume above the critical size is introduced in a quite natural way trough a new effect that we called *volume scattering*. This effect consists in the coalescence of two droplets into a single unit with total volume above the maximum value $v_{\rm m}$ allowed for the system (which in turn depends on side parameters, like the rotational velocity of the impeller, its geometry, the chemical and physical nature of the two liquids and so on) followed by an immediate rupture into *two* smaller droplets with volume below the critical size (the model was confined to *binary* breakage).

Another novelty in [2,3] was the presence of an efficiency factor (depending on the number of droplets and the total inter–facial area) in the evolution equation,

whose role is to offer a more precise control of the whole dynamics than the one achievable by just modeling the interaction kernels.

However fragmentation appears to be instantaneously *multiple* in most experiments and this implies that also the scattering term should be modified in our model in order to allow the fragmentation of an unstable droplet in, possibly, more than two pieces. In [4] we already presented a possible approach to this kind of generalization and in [5] we worked out explicitly a case in which only a ternary mode of rupture was taken into account. In any case the breakage rate in the above quoted papers was taken to be bounded. Here we will present not only a more natural approach to the problem of dealing with multiple breakage but also remove the hypothesis of a bounded breakage rate.

It must be said that multiple breakage has been considered also by other Authors (see [1,8,7]).

However our approach is rather different. Indeed in [8] the structure of the breakage term in the evolution equation is taken to be the following:

$$L_{\mathrm{b}} f(v,t) = \int_v^{v_{\mathrm{m}}} \alpha(w)\mu(w)\beta(w,v)f(w,t)\ \mathrm{d}w - \alpha(v)f(v,t). \tag{1.1}$$

Here $f\,(v,t)$ (v = volume, t = time) denotes the distribution function of droplet size (per unit volume of dispersion), and

1. $\alpha(w)$ is the breakage frequency of a particle with size w
2. $\mu(w)$ the number of droplets with the same parent w
3. $\beta(w,v)\ \mathrm{d}w$ is the probability that a parent w generates a daughter of size v, independently of the breakage mode.

The whole process is seen as a composite of a large number of independent random processes. If the Central Limit Theorem applies, then $\beta(w,v)$ can be approximated by a normal distribution with mean $w^\star = w/\mu(w) \in (0, w/2)$ and a standard variation to be conveniently chosen.

A similar point of view is taken in the papers of [1,7]: according to these Authors

$$L_{\mathrm{b}} f(v,t) = \int_v^{+\infty} \gamma(w,v)(w)f(w,t)\ \mathrm{d}w - \frac{f(v,t)}{v}\int_0^v \omega(v,w)w\ \mathrm{d}w, \tag{1.2}$$

but no assumption about the normality of β is made a priori. In this case $\gamma(w,v)$ ($0 \le v \le w < +\infty$) is the multiple fragmentation kernel. Although seemingly different, these two formulations can be proved to be equivalent (see [7]) (except for the finite upper bound of the integral in (1.1)). Moreover the case of binary breakage can be incorporated into this formulation by means of a suitable change of variables (and by using the symmetry of the kernel characteristic of the binary case, see [7] again).

From our point of view the above models for multiple breakage have some disadvantages: first of all none of them seems to take into account the randomness of the breakage mode since the kernel *is the same for all modes*. Moreover the fact that one of these models can be proved to be equivalent to that suited for the binary breakage case, appears to be a strong indication that the physics of the process has been somehow oversimplified. Finally if no information on the breakage mode is specified (as in (1.1) and (1.2)) it is still possible to know the total number of droplets but it appears impossible to count just those due to breakage.

The above models share the common underlying philosophy of capturing a global information about breakage, in view of the difficulty of analyzing the single modes. Here we prefer to follow a more direct approach, evaluating the contribution of each event to the rate of change of the distribution function.

2. Evolution equation

As in [2,3] we write the evolution equation for f as follows

$$\frac{\partial f}{\partial t} = \phi(t)\left(L_{\mathrm{c}} f + L_{\mathrm{b}} f + L_{\mathrm{s}} f\right) \ . \tag{2.1}$$

where (see [2,3] for example) $\phi(t) = \Psi\left[\mathcal{N}(t), \mathcal{S}(t)\right]$ is the *efficiency factor* and

$$\mathcal{N}(t) = \int_0^{v_{\mathrm{m}}} f(v,t)\,\mathrm{d}v \ , \qquad \mathcal{S}(t) = \int_0^{v_{\mathrm{m}}} v^{(2/3)} f(v,t)\,\mathrm{d}v \ ,$$

represents, respectively, the instantaneous total *number of droplet* and *interfacial area* per unit volume of dispersion.

The operators at the r.h.s. of (2.1) have a rather complex structure: L_{c} is the *coalescence* operator and depends on a coalescence kernel τ_{c} which is a known function of the sizes of the two colliding droplets; L_{b} is the *breakage* operator summing up the contributions of the various rupture modes (binary, ternary, etc.), having defined, for each breakage mode, its *frequency* α_{k} and the *probability density* β_{k} of its outcome. Finally L_{s} is the *scattering* operator and the kernel of the k-th mode is just the product of β_{k} and τ_{c}. We recall that L_{s} has represented the main novelty of our model in the current literature about drops dynamics since we first proposed it in [2,3]. Its role is to justify the instability of droplets resulting from coalescence and with volume above the threshold value v_{m} without invoking any extra condition besides the true physics involved: indeed L_{s} is nothing but a suitable combination of the two main factors driving the dynamics of droplets, namely coalescence and breakage.

Natural size limitations among droplets impose particular care when the integration domains of the various terms on the r.h.s. of (2.1) are specified. To be precise we put

$$
\begin{aligned}
L_{\mathrm{c}} f(v,t) = & \int_0^{v/2} \tau_{\mathrm{c}}(w, v-w) f(w,t) f(v-w,t)\, \mathrm{d}w \\
& - f(v,t) \int_0^{v_{\mathrm{m}}-v} \tau_{\mathrm{c}}(w,v) f(w,t)\, \mathrm{d}w, \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
L_{\mathrm{b}} f(v,t) = & \int_v^{v_{\mathrm{m}}} \alpha_2(s)\beta_2(s,v) f(s,t)\, \mathrm{d}s \\
& + \sum_{k=3}^{N} \int_v^{v_{\mathrm{m}}} \alpha_{\mathrm{k}}(s) f(s,t)\, \mathrm{d}s \\
& \times \int_{D_{\mathrm{k}}(s,v)} \beta_{\mathrm{k}}\left(s, u_1, \ldots, u_{\mathrm{k}-2}, s-v-U_{\mathrm{k}-2}\right)\, \mathrm{d}\sigma_{\mathrm{k}-2} \\
& - \sum_{k=2}^{N} \alpha_{\mathrm{k}}(v) f(v,t), \qquad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
L_{\mathrm{s}} f(v,t) = & \int_{v_{\mathrm{m}}}^{v_{\mathrm{m}}+v} \lambda_2(s)\beta_2(s,s-v)\, \mathrm{d}s \int_{s-v_{\mathrm{m}}}^{s/2} \tau_{\mathrm{c}}(s-w,w) f(w,t) f(s-w,t)\, \mathrm{d}w \\
& + \sum_{k=3}^{N} \left[\int_{v_{\mathrm{m}}}^{v_{\mathrm{m}}+v} \lambda_{\mathrm{k}}(s)\, \mathrm{d}s \int_{s-v_{\mathrm{m}}}^{s/2} \tau_{\mathrm{c}}(s-w,w) f(w,t) f(s-w,t)\, \mathrm{d}w \right. \\
& \left. \times \int_{D_{\mathrm{k}}(s,v)} \beta_{\mathrm{k}}\left(s, u_1, \ldots, u_{\mathrm{k}-2}, s-v-U_{\mathrm{k}-2}\right)\, \mathrm{d}\sigma_{\mathrm{k}-2} \right] \\
& - f(v,t) \int_{v_{\mathrm{m}}}^{v_{\mathrm{m}}+v} \tau_{\mathrm{c}}(v, s-v) f(s-v,t)\, \mathrm{d}s, \qquad (2.4)
\end{aligned}
$$

where $\mathrm{d}\sigma_{\mathrm{k}-2} = \mathrm{d}u_1 \cdots \mathrm{d}u_{\mathrm{k}-2}$ denotes the measure element in $\mathbb{R}^{k-2}$. Notice that

$$
\int_{v_{\mathrm{m}}}^{v_{\mathrm{m}}+v} \tau_{\mathrm{c}}(v, s-v) f(s-v,t)\, \mathrm{d}s = \int_{v_{\mathrm{m}}-v}^{v_{\mathrm{m}}} \tau_{\mathrm{c}}(w,v) f(w,t)\, \mathrm{d}w, \qquad (2.5)
$$

so that the last term in (2.4) is nothing but the continuation of the last term in (2.2). The symbols appearing above have the following meaning:

- $\alpha_{\mathrm{k}}(s)$ is the *breakage rate* of droplets with volume $s \in (0, v_{\mathrm{m}})$ in k pieces.
- $\lambda_{\mathrm{k}}(s)$ is a suitable *weight* (to be chosen conveniently) measuring the chance of the parent droplet $s \in (v_{\mathrm{m}}, 2v_{\mathrm{m}})$ to break exactly in k pieces within the scattering process; we assume $\sum_{k=2}^{N} \lambda_{\mathrm{k}}(s) = 1$ in any case.
- $U_{\mathrm{n}} = \sum_{h=1}^{n} u_{\mathrm{h}}$.

– $\beta_{\mathrm{k}}\left(s, u_1, \ldots, u_{\mathrm{k}-1}\right)$ is the breakage probability density of drops with volume $s \in (0, 2v_{\mathrm{m}})$ to generate by breakage k fragments with prescribed volumes u_{j} ($j = 1, \ldots, k-1$) in increasing order, $0 < u_1 \leq u_2 \leq \ldots \leq u_{\mathrm{k}-1}$.
– $\tau_{\mathrm{c}}(v, w)$ is the *coalescence kernel*, that is proportional to the probability that two colliding droplets of respective volumes v and w coalesce to form a unique droplet of volume $v + w$.

The definitions of the functions $\beta_{\mathrm{k}}\left(s, u_1, \ldots, u_{\mathrm{k}-1}\right)$ and of the domains $D_{\mathrm{k}}(s, v)$ need several preliminaries: we devote the following Section just to this topic. As far as the regularity properties of the functions appearing in the kernels of $L_{\mathrm{c}}, L_{\mathrm{b}}, L_{\mathrm{s}}$ and the efficiency factor Ψ, we assume the following

(H1) Ψ is strictly positive, Lipschitz continuous and bounded in $\mathbb{R}^2$. We also assume $\inf_{\mathbb{R}^2} \Psi = \widehat{\Psi} > 0$.
(H2) τ_{c} is non–negative, symmetric and continuously differentiable in $[0, v_{\mathrm{m}}] \times [0, v_{\mathrm{m}}]$.
(H3) For $k = 2, \ldots, N$, α_{k} is non–negative, continuously differentiable and non decreasing in $[0, v_{\mathrm{m}})$, unbounded as v tends to v_{m}. We also assume $\sum_{k=2}^{N} \alpha_{\mathrm{k}} > 0$ for all $v \in (v_{\mathrm{crit}}, v_{\mathrm{m}})$ and
  a) $\alpha_{\mathrm{k}} \simeq (v_{\mathrm{m}} - v)^{-\mu}$ with $\mu \in (0, 1)$ in a left neighbourhood of $v = v_{\mathrm{m}}$,
  b) $\alpha_{\mathrm{k}} \simeq (v - v_{\mathrm{crit}})_+^{\delta}$ with $\delta > 0$ in a right neighbourhood of $v = v_{\mathrm{crit}}$,
  where $v_{\mathrm{crit}} \in [0, v_{\mathrm{m}})$ is a given threshold and $(\bullet)_+ := \max\{\bullet, 0\}$.
(H4) For $k = 2, \ldots, N$, functions λ_{k} are continuous in $[v_{\mathrm{m}}, 2v_{\mathrm{m}}]$.

A widely used (empirical) law, called *Weber relation*, assumes

$$v_{\mathrm{crit}} = 10^{-4} \pi \left(\sigma/\varrho\right)^{9/5} \left(\omega^2 D^{4/3}\right)^{-9/5},$$

where σ and ϱ are, respectively, the surface tension and the density of the dispersed phase, ω is the angular velocity of the impeller and D is the impeller diameter. Therefore it is quite reasonable to think of v_{crit} as a very small but not vanishing value. Indeed $v_{\mathrm{crit}} \to 0$ only if either $\omega \to +\infty$, i. e. by spending an infinite amount of energy, or $\sigma \to 0$ which is a rather unphysical situation. It is worth noticing that – while the local existence in time of the unique solution to the Cauchy problem for equation (2.1) can be achieved regardless of being v_{crit} equal to zero or not – to prove the global existence *we are forced to assume* $v_{\mathrm{crit}} > 0$. This fact is strictly related to the very physics of the problem.

3. Functions β_{k} and domains D_{k}

Function β_2 is such that

$$\beta_2(s,u) = \beta_2(s, s-u), \qquad \beta_2(s,u) = 0, \quad \text{if } s \le u \tag{3.1}$$

In other words, for each $s \in (0, 2v_{\rm m}]$, we can take β_2 as assigned only in $[0, s/2]$. Now, for a given $k \ge 3$, let us first consider the case $s \in (0, v_{\rm m}]$ and define the set of $\mathbb{R}^{k-1}$

$$T_{\rm k,1}(s) = \{(u_1, \ldots, u_{\rm k-1}) \mid 0 < u_1 \le \ldots \le u_{\rm k-1} \le s - U_{\rm k-1} \le v_{\rm m}\}. \tag{3.2}$$

Clearly $\widetilde{u} = s - U_{\rm k-1}$ identifies one of the k daughters of s and $T_{\rm k,1}$ is characterized by the circumstance of $\widetilde{u}$ being the *largest* daughter. Function $\beta_{\rm k}$ is assigned on $T_{\rm k,1}$ in such a way that

$$\int_{T_{\rm k,1}(s)} \beta_{\rm k}\left(s, u_1, \ldots, u_{\rm k-1}\right)\, {\rm d}\sigma_{\rm k-1} = 1. \tag{3.3}$$

For a fixed $s \in (0, v_{\rm m}]$, define $T_{\rm k,j}(s)$ to be the set of points of $(u_1, \ldots, u_{\rm k-1}) \in \mathbb{R}^{k-1}$ such that

$$\begin{array}{l} \{0 < u_1 \le \ldots \le u_{\rm k-j} \le s - U_{\rm k-1} \le u_{\rm k-j+1} \le \ldots \le u_{\rm k-1} \le v_{\rm m}\}, \\ j = 2, \ldots, k, \end{array} \tag{3.4}$$

where, by definition, $u_0 = 0$. If $s \le v_{\rm m}$, the last inequality in (3.2) and (3.4) is obviously redundant. We notice that the $j-$th domain $T_{\rm k,j}$ is characterized by the circumstance $\widetilde{u}$ has an intermediate size between $u_{\rm k-j}$ and $u_{\rm k-j+1}$. We now consider, again for a fixed s, the maps

$$\begin{cases} C_{\rm j} : (\xi_1, \ldots, \xi_{\rm k-1}) \mapsto (u_1, \ldots, u_{\rm k-1}), \\ j = 1, \ldots, k-1, \end{cases} \tag{3.5}$$

defined by

$$\begin{cases} u_1 = \xi_1, \ldots, u_{\rm k-j-1} = \xi_{\rm k-j-1}, \\ u_{\rm k-j} = s - \displaystyle\sum_{i=1}^{k-1} \xi_{\rm i}, \\ u_{\rm k-j+1} = \xi_{\rm k-j+1}, \ldots, u_{\rm k-1} = \xi_{\rm k-1}. \end{cases} \tag{3.6}$$

It is not difficult to check the maps (3.6) "re–locate" the *residual drop* $\widetilde{u}$ with respect to the ordered set of the other daughters and that

$$\begin{cases} C_{\rm j}\left(T_{\rm k,j}(s)\right) = T_{\rm k,j+1}(s), \\ C_{\rm j}\left(T_{\rm k,j+1}(s)\right) = T_{\rm k,j}(s) \end{cases} \tag{3.7}$$

so that $C_{\rm j} = C_{\rm j}^{-1}$.

Evidently the residual drop is the largest one in $T_{\mathrm{k},1}$ and the smallest one in $T_{\mathrm{k,k}}$.

Then we can complete the definition of the maps (3.5)–(3.6) also for $j = k$, by assuming that $C_{\mathrm{k}}\left(T_{\mathrm{k,k}}(s)\right) = T_{\mathrm{k},1}(s)$ which makes the family of maps C_{j} *cyclic* among the domains $T_{\mathrm{k,j}}(s)$. Because of (3.7) it turns out that $C_{\mathrm{k}} = C_1 \circ C_2 \circ \ldots \circ C_{\mathrm{k}-1}$ The main reason for introducing the maps C_{j} is to extend the probability density over all domains $T_{\mathrm{k,j}}$. The procedure is the following. Indeed we can prove that, for all $k \geq 3$ all open domains $\overset{\circ}{T}_{\mathrm{k,j}}$ are mutually disjoint and that $\bigcap_{j=1}^{k} T_{\mathrm{k,j}}$ reduces to a single point which can be identified with the event $u_1 = u_2 = \ldots = u_{\mathrm{k}-1} = \frac{s}{k}$. that is "all droplets have the same volume". Now, by means of the maps C_{j} we extend β_{k} from $T_{\mathrm{k},1}$ to $T_{\mathrm{k},2}$, from $T_{\mathrm{k},2}$ to $T_{\mathrm{k},3}$ and so on, up to $T_{\mathrm{k,k}}$. In other words we put $\widetilde{\beta}_{\mathrm{k}}\left(s, u_1, \ldots, u_{\mathrm{k}-1}\right)$ equal to

$$\begin{cases} \beta_{\mathrm{k}}\left(s, u_1, \ldots, u_{\mathrm{k}-1}\right), & \text{in } T_{\mathrm{k},1}(s), \\ \beta_{\mathrm{k}} \circ C_1\left(u_1, \ldots, u_{\mathrm{k}-1}\right), & \text{in } T_{\mathrm{k},2}(s), \\ \vdots & \vdots \\ \beta_{\mathrm{k}} \circ C_1 \circ C_2 \circ \ldots \circ C_{\mathrm{k}-1}\left(u_1, \ldots, u_{\mathrm{k}-1}\right), & \text{in } T_{\mathrm{k,k}}(s). \end{cases} \tag{3.8}$$

Because of the properties of the maps C_{j}, we have $\int_{T_{\mathrm{k,j}}(s)} \widetilde{\beta}_k \ \mathrm{d}\sigma_{\mathrm{k}-1} = 1$ for all $j = 1, \ldots, k$; if we define $T_{\mathrm{k}} = \bigcup_{j=1}^{k} T_{\mathrm{k,j}}$ and recall that $\overset{\circ}{T}_{\mathrm{k,j}} \cap \overset{\circ}{T}_{\mathrm{k,i}} = \varnothing$ for $i \neq j$, we also have $\int_{T_{\mathrm{k}}(s)} \widetilde{\beta}_k \ \mathrm{d}\sigma_{\mathrm{k}-1} = k$. We now put $D_{\mathrm{k}}(s, v) = T_{\mathrm{k}}(s) \cap \{U_{\mathrm{k}-1} = s - v\}$ (see figure 3.1 for the case $k = 3$). Thus in all the $T_{\mathrm{k,j}}$ contributing to $D_{\mathrm{k}}(s, v)$, the volume v is the one of the "residual drop". Notice that $T_{\mathrm{k}}(s)$ is $(k-1)-$dimensional polytope, so that $D_{\mathrm{k}}(s, v)$ is a $(k-2)-$dimensional set. From now on we drop the "tilde" above β_{k} in (3.8), i.e. we identify β_{k} with its extension over $T_{\mathrm{k}}(s)$. Since we allow s in the interval $(0, 2v_{\mathrm{m}}]$, function β_{k} is defined in the $k-$dimensional polytope $\mathcal{T} \in \mathbb{R}^k$

$$\left\{s \in (0, 2v_{\mathrm{m}}], 0 < u_1 \leq u_2 \leq \ldots \leq u_{\mathrm{k}-1} \leq u_{\mathrm{k}}, U_{\mathrm{k}} = s\right\}. \tag{3.9}$$

The domain $T_{\mathrm{k}}(s)$ is nothing but the intersection of $\mathcal{T}$ with the plane $s =$constant. We now extend the definitions (3.2) and (3.4) of the domains $T_{\mathrm{k,j}}(s)$ to the case $s \in (v_{\mathrm{m}}, 2v_{\mathrm{m}}]$. In this case the last inequality appearing in the definitions (3.2) and (3.4) (which, in the case $s \in (0, v_{\mathrm{m}}]$, is automatically satisfied) plays an effective role. We also extend the assumption (3.3): we put

$$\int_{T_{\mathrm{k},1}(s)} \beta_{\mathrm{k}}\left(s, u_1, \ldots, u_{\mathrm{k}-1}\right) \ \mathrm{d}\sigma_{\mathrm{k}-1} = 1, \tag{3.10}$$

regardless of the size of s in $(0, 2v_{\mathrm{m}}]$. The maps C_{j} then allow to extend β_{k} over the whole set $T_{\mathrm{k}}(s)$ also for $s \in (v_{\mathrm{m}}, 2v_{\mathrm{m}})$.

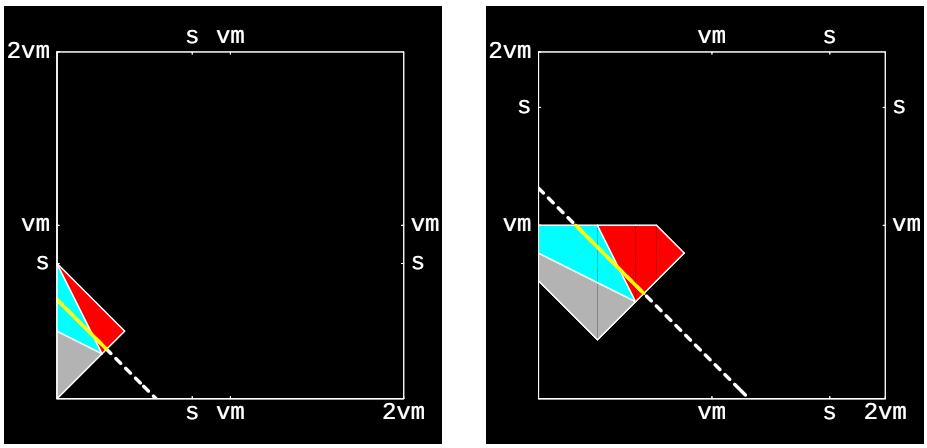

Fig. 3.1: The domain $D_3(s,v)$ when s is smaller (left) than v_m and when s is larger (right) than v_m. The domain $D_3(s,v)$ is represented by the (yellow) intersection of the dashed line with the $T_3(s)$ domain. In both figures $T_{3,1}$ is gray-colored, $T_{3,2}$ is cyan-colored, $T_{3,3}$ is red-colored

4. Further preliminary results

First of all we complete the set of hypotheses (H) by adding the following

(H5) (*Normalization*). When $s \in (0, v_\mathrm{m}]$, β_2 is normalized as follows

$$\int_0^{s/2} \beta_2(s,v)\, \mathrm{d}v = 1, \tag{4.1}$$

while, for $s \in (v_\mathrm{m}, 2v_\mathrm{m}]$

$$\int_{s-v_\mathrm{m}}^{s/2} \beta_2(s,u)\, \mathrm{d}u = 1. \tag{4.2}$$

For $k = 3, \dots, N$, we recall that we have set

$$\int_{T_{\mathrm{k,h}}(s)} \beta_\mathrm{k}\left(s, u_1, \dots, u_{\mathrm{k}-1}\right)\, \mathrm{d}\sigma_{\mathrm{k}-1} = 1, \qquad \forall h = 1, \dots, k. \tag{4.3}$$

(H6) (*Regularity*).

(i) $\beta_2(s,v)$ is continuous in $(0, 2v_\mathrm{m}) \times (0, s)$ and continuously differentiable in $(0, 2v_\mathrm{m})$ for all $s \in (0, 2v_\mathrm{m})$ and in addition

(a) $\alpha_2(s) \left| \dfrac{\partial \beta_2(s,\xi)}{\partial \xi} \right|_{\xi = s-v} \in L^1(v, v_\mathrm{m})$ for all $s \in [v, v_\mathrm{m}]$ and for all $v \in (0, v_\mathrm{m}]$;

(b) $\left|\dfrac{\partial}{\partial\xi}\beta_2(s,\xi)\right|_{\xi=s-v} \leq M_1$ for a suitable positive constant M_1, for all $s \in [v_\mathrm{m}, v_\mathrm{m}+v]$ and all $v \in (0, v_\mathrm{m}]$.

(ii) For $k = 3, \ldots, N$, we assume β_k continuous in $\overset{\circ}{T}$ and continuously differentiable in $\overset{\circ}{T}_\mathrm{k}(s)$ for all $s \in (0, 2v_\mathrm{m})$. Moreover

(c) for all $k \geq 2$, β_k vanishes if the size of the smallest daughter goes to zero; in particular

$$\lim_{v\to 0} \beta_\mathrm{k}\Big|_{D_\mathrm{k}(s,v)} = 0,$$

for all $k \geq 3$.

(d) $\left(\displaystyle\int_{D_\mathrm{k}(s,v)} \beta_\mathrm{k}\left(s, u_1, \ldots, u_{\mathrm{k}-2}, s-v-U_{\mathrm{k}-2}\right)\, \mathrm{d}\sigma_{\mathrm{k}-2}\right)\Big|_{s=v_\mathrm{m}+v} < M_2$ for a suitable positive constant M_2, for all $v \in (0, v_\mathrm{m}]$.

(e) $\alpha_\mathrm{k}(s) \displaystyle\int_{D_\mathrm{k}(s,v)} \frac{\partial\beta_\mathrm{k}}{\partial u_{\mathrm{k}-1}}\left(s, u_1, \ldots, u_{\mathrm{k}-2}, s-v-U_{\mathrm{k}-2}\right)\, \mathrm{d}\sigma_{\mathrm{k}-2} \in L^1(v, v_\mathrm{m})$ for all $v \in (0, v_\mathrm{m})$.

(f) $\left(\dfrac{\partial\beta_\mathrm{k}}{\partial u_{\mathrm{k}-1}}\left(s, u_1, \ldots, u_{\mathrm{k}-1}\right)\right)\Big|_{u_{\mathrm{k}-1}=s-v-U_{\mathrm{k}-2}} \in L_1\left[D_\mathrm{k}(s,v)\right]$ for all $s \in (v, v_\mathrm{m}+v)$ and for all $v \in (0, v_\mathrm{m})$

(g) $\left(\beta_\mathrm{k}\left(s, u_1, \ldots, u_{\mathrm{k}-1}\right)\right)|_{u_{\mathrm{k}-1}=s-v-U_{\mathrm{k}-2}} \in L_1\left[\partial D_\mathrm{k}(s,v)\right]$ for all $s \in (v, v_\mathrm{m}+v)$ and $v \in (0, v_\mathrm{m})$

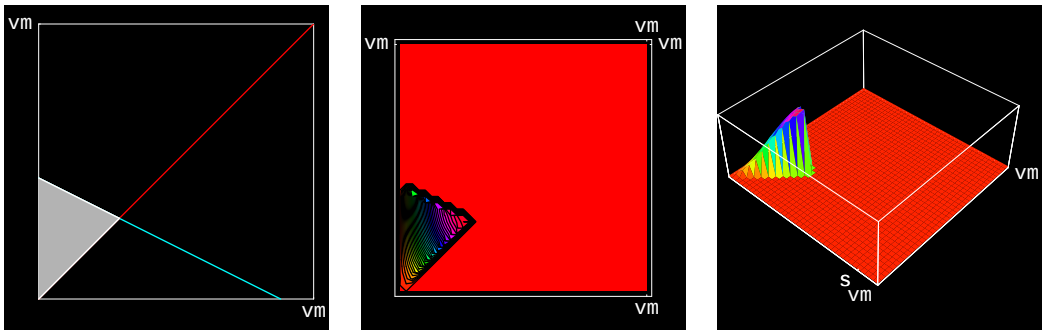

Fig. 4.1: The domain $T_{3,1}(s)$ (left), the contour level plot (center) and graph (right) of an example of function β_3. In this case s is smaller than v_m

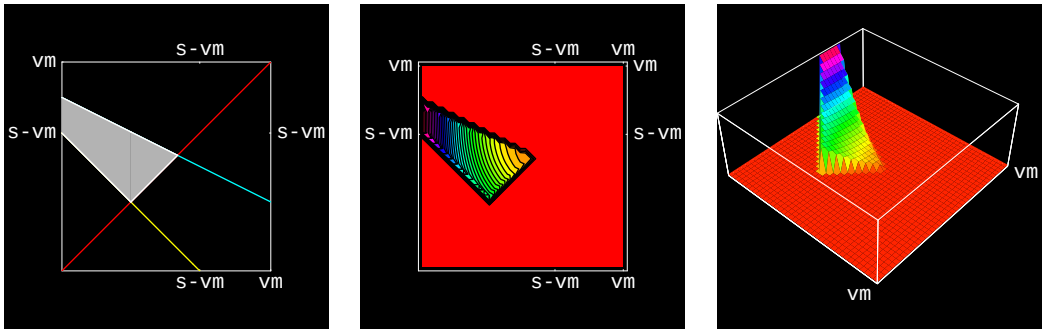

Fig. 4.2: The domain $T_{3,1}(s)$ (left), the contour level plot (center) and graph (right) of an example of function β_3. In this case s is greater than v_{m}

In an extended version of this paper we will show that all the hypotheses we made are consistent by showing, in particular, an example of β and α function in the case that both binary and ternary fragmentation modes may occur. Here we confine to show just a figure of β_3 (see figures 4.1 and 4.2).

Concerning the initial data we assume that

$$\begin{aligned} &f_{\mathrm{o}}(v) \text{ is continuously differentiable in } [0, v_{\mathrm{m}}], \\ &f_{\mathrm{o}}(v) \text{ is non–negative in } [0, v_{\mathrm{m}}], \\ &f_{\mathrm{o}}(0) = f_{\mathrm{o}}(v_{\mathrm{m}}) = 0. \end{aligned} \tag{4.4}$$

As in [3] we look for a solution– in a suitable class of regular functions f to be specified later – to both the *original Cauchy problem*

$$\begin{cases} \dfrac{\partial f}{\partial t} = \phi(t)(L_{\mathrm{c}} f + L_{\mathrm{b}} f + L_{\mathrm{s}} f), \\ f(v,0) = f_{\mathrm{o}}(v), \end{cases} \tag{4.5}$$

and the so–called *modified Cauchy problem*

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = \phi(t)(L_{\mathrm{c}}^{+} \psi + L_{\mathrm{b}}^{+} \psi + L_{\mathrm{s}}^{+} \psi), \\ \psi(v,0) = f_{\mathrm{o}}(v), \end{cases} \tag{4.6}$$

where the L^{+}–operators are defined as follows

$$L_{\mathrm{c}}^{+}\psi(v,t) = \int_0^{v/2} \tau_{\mathrm{c}}(w, v-w)\psi_{+}(w,t)\psi_{+}(v-w,t)\,\mathrm{d}w$$
$$- \psi(v,t)\int_0^{v_{\mathrm{m}}-v} \tau_{\mathrm{c}}(w,v)|\psi(w,t)|\,\mathrm{d}w, \tag{4.7}$$

$$L_{\mathrm{b}}^{+}\psi(v,t) = \int_v^{v_{\mathrm{m}}} \alpha_2(s)\beta_2(s,v)\psi_{+}(s,t)\,\mathrm{d}s$$
$$+ \sum_{k=3}^{N}\int_v^{v_{\mathrm{m}}} \alpha_{\mathrm{k}}(s)\psi_{+}(s,t)\,\mathrm{d}s \int_{D_{\mathrm{k}}(s,v)} \beta_{\mathrm{k}}\left(s,u_1,\ldots,u_{\mathrm{k}-2},s-v-U_{\mathrm{k}-2}\right)\,\mathrm{d}\sigma_{\mathrm{k}-2}$$
$$- \sum_{k=2}^{N}\alpha_{\mathrm{k}}(v)\psi(v,t), \tag{4.8}$$

$$L_{\mathrm{s}}^{+}\psi(v,t) = \int_{v_{\mathrm{m}}}^{v_{\mathrm{m}}+v} \lambda_2(s)\beta_2(s,s-v)\,\mathrm{d}s \int_{s-v_{\mathrm{m}}}^{s/2} \tau_{\mathrm{c}}(s-w,w)\psi_{+}(w,t)\psi_{+}(s-w,t)\,\mathrm{d}w$$
$$+ \sum_{k=3}^{N}\left[\int_{v_{\mathrm{m}}}^{v_{\mathrm{m}}+v} \lambda_{\mathrm{k}}(s)\,\mathrm{d}s \int_{s-v_{\mathrm{m}}}^{s/2} \tau_{\mathrm{c}}(s-w,w)\psi_{+}(w,t)\psi_{+}(s-w,t)\,\mathrm{d}w\right.$$
$$\left.\times \int_{D_{\mathrm{k}}(s,v)} \beta_{\mathrm{k}}\left(s,u_1,\ldots,u_{\mathrm{k}-2},s-v-U_{\mathrm{k}-2}\right)\,\mathrm{d}\sigma_{\mathrm{k}-2}\right]$$
$$- \psi(v,t)\int_{v_{\mathrm{m}}-v}^{v_{\mathrm{m}}} \tau_{\mathrm{c}}(v,w)|\psi(w,t)|\,\mathrm{d}w, \tag{4.9}$$

where, in writing L_{s}^{+}, we made use of (2.5).

5. Physical consistency of the model

It is worth noticing that in the exceptional case of binary ruptures only (that is $\alpha_{\mathrm{k}} = \lambda_{\mathrm{k}} = 0$ for all $k \geq 3$) the model we propose identifies with the one presented in [2,3].

We now list the main results that we can prove.

Lemma 5.1. *(*Positiveness*) Under assumptions from* (H1) *to* (H6)*, all bounded solutions to problem (4.6) are non–negative.*

Theorem 5.1. *(*Volume conservation*) Let $f_{\mathrm{o}}(v)$ be a (continuous) initial data for $f(v,t)$. Then, if $f(v,t)$ is a regular solution to equation (2.1), we have*

$$\int_0^{v_{\mathrm{m}}} v f(v,t)\,\mathrm{d}v = \int_0^{v_{\mathrm{m}}} v f_{\mathrm{o}}(v)\,\mathrm{d}v. \tag{5.1}$$

Theorem 5.2. *(*Uniqueness*) Under assumptions from (H1) to (H6), problem (4.5) has at most one bounded solution.*

We now prove that problem (4.6) has a local bounded solution provided that the initial data go to zero sufficiently fast as v goes to v_{m}. Because of positivity, all solutions to problem (4.6) with initial data $f_{\mathrm{o}}(v)$ also satisfy problem (4.5) with the same data. Moreover, because of the uniqueness Theorem, to achieve the existence of solutions to problem (4.5), it suffices to prove it for problem (4.6).

Theorem 5.3. *(*Local existence*) Assume that hypotheses from (H1) to (H6) be satisfied and that $f_{\mathrm{o}}(v)$ satisfy both (4.4) and*

$$||\alpha'_{\mathrm{k}}(v) f_{\mathrm{o}}(v)|| < +\infty, \qquad \forall k = 2, \ldots, N. \tag{5.2}$$

Then problem (4.6) has at least one Lipschitz continuous solution in $[0, v_{\mathrm{m}}] \times [0, T)$ for a suitable finite $T > 0$.

All proofs are rather long and need several side lemmas. For this reason they will be published elsewhere.

6. The key point for proving global existence

The reason for which *we need to assume* $v_{\mathrm{crit}} > 0$ find its justification in the very physics of the problem. Indeed it is well known that droplets of arbitrary small size cannot be obtained through breakage from larger droplets unless we spend an infinite power (infinite rotational speed of the impeller). The same request is needed for τ_{c}: indeed it is also very difficult to make two very small droplets coalesce (see figures 6.1 and 6.2 taken from [6]) because of the large energy needed to drain and break the interposed enveloping protective film. Similarly, if we look at the probability to get a very small droplet v as the final product of either a breakage or a scattering event, since very small droplets are very unlike to appear as a rupture event of a larger parent.

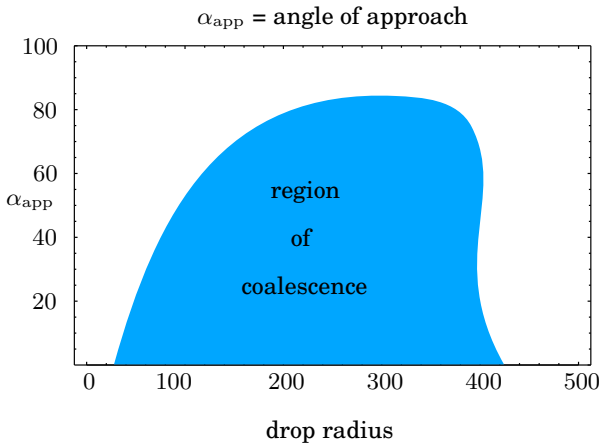

Fig. 6.1: Coalescence region for drops of *equal size*: very large and very small droplets do not coalesce regardless of the mutual angle of approach ($\alpha_{app} = 0^\circ$ means "head-on collision", $\alpha_{app} = 90^\circ$ means "grazing droplets").

This circumstance suggests to assume, in addition to all previous hypotheses made for α_k, also the following

$$\alpha_k \equiv 0, \qquad \forall k \geq 2, \quad \forall v \in [0, v_{crit}], \tag{6.1}$$

where $0 < v_{crit} \ll v_m$ is a (small) threshold value (see Remark 2 at page 5). The same request is needed for τ_c: indeed it is also very difficult to make two very small droplets coalesce (see figures 6.1 and 6.2 taken from [6]) because of the large energy needed to drain and break the interposed separating film[1]

Therefore we also assume

$$\tau_c \equiv 0, \qquad \text{in} \quad [0, v_{crit}] \times [0, v_{crit}]. \tag{6.2}$$

Similarly, considering there is no chance to get a droplet of subcritical size as the final product of either a breakage or a scattering event, we need to impose

$$\beta_k \equiv 0, \qquad \text{if} \quad v \in [0, v_{crit}]. \tag{6.3}$$

[1] One should eventually distinguish between a lower threshold value $v_{crit}^{(b)}$ below which a drop is unbreakable and a similar value $v_{crit}^{(c)}$ under which a pair of droplets are unable to coalesce into a unique object. This further complication has not been considered here although the mathematics involved does not change very much.

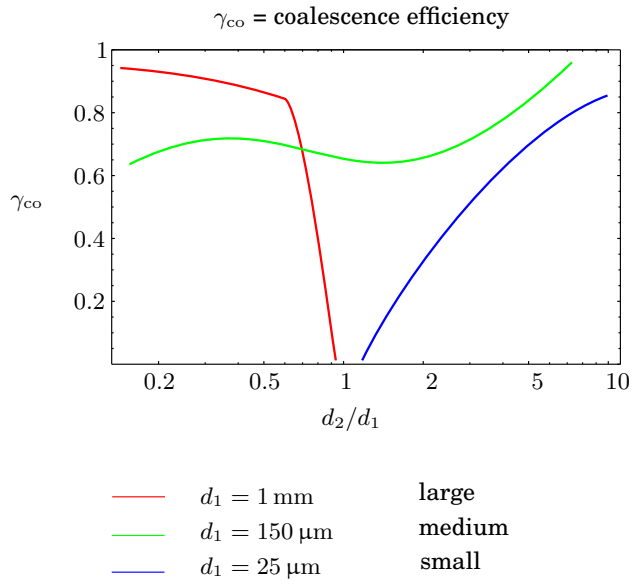

Fig. 6.2: Coalescence efficiency *vs.* droplets ratio

As a consequence the only physical mechanism remaining active for $v \in [0, v_{\text{crit}}]$ is the loss of small droplets due to coalescence with ones of ordinary size (see figures 6.1 and 6.2 again).

The additional assumptions (6.1), (6.2) and (6.3) have an immediate consequence on the behaviour of f in a right neighbourhood of $v = 0$. Indeed from (2.2), (2.3), (2.4), we get that

$$v \in [0, v_{\text{crit}}] \Rightarrow \begin{cases} L_{\text{c}} f(v,t) = -f(v,t) \displaystyle\int_{v_{\text{crit}}}^{v_{\text{m}}-v} \tau_{\text{c}}(w,v) f(w,t)\, \mathrm{d}w, \\ L_{\text{b}} f(v,t) = 0, \\ L_{\text{s}} f(v,t) = -f(v,t) \displaystyle\int_{v_{\text{m}}-v}^{v_{\text{m}}} \tau_{\text{c}}(w,v) f(w,t)\, \mathrm{d}w, \end{cases} \tag{6.4}$$

so that

$$\frac{\partial f}{\partial t} = -\phi(t) f(v,t) \int_{v_{\text{crit}}}^{v_{\text{m}}} \tau_{\text{c}}(w,v) f(w,t)\, \mathrm{d}w < 0, \qquad \forall v \in [0, v_{\text{crit}}]. \tag{6.5}$$

Consequently

$$f(v,t) \le f_{\mathrm{o}}(v), \qquad (v,t) \in [0, v_{\mathrm{crit}}] \times [0,T]. \tag{6.6}$$

Relation (6.6) implies that, because of the conservation of volume (Theorem 5.1), also the number of droplets cannot go to infinity because of a possible non–integrable singularity of f near $v = 0$. Indeed from

$$\int_0^{v_{\mathrm{crit}}} v f(v,t)\, \mathrm{d}v + \int_{v_{\mathrm{crit}}}^{v_{\mathrm{m}}} v f(v,t)\, \mathrm{d}v \equiv \mathcal{V}(t) = \mathcal{V}(0), \tag{6.7}$$

we obtain

$$v_{\mathrm{crit}} \int_{v_{\mathrm{crit}}}^{v_{\mathrm{m}}} f(v,t)\, \mathrm{d}v \le \mathcal{V}(0), \tag{6.8}$$

and also

$$\int_0^{v_{\mathrm{crit}}} v f(v,t)\, \mathrm{d}v \le \int_0^{v_{\mathrm{crit}}} v f_{\mathrm{o}}(v)\, \mathrm{d}v \le v_{\mathrm{crit}} \int_0^{v_{\mathrm{crit}}} f_{\mathrm{o}}(v)\, \mathrm{d}v. \tag{6.9}$$

Therefore

$$\begin{aligned} \mathcal{N}(t) &= \int_0^{v_{\mathrm{m}}} f(v,t)\, \mathrm{d}v = \int_0^{v_{\mathrm{crit}}} f(v,t)\, \mathrm{d}v + \int_{v_{\mathrm{crit}}}^{v_{\mathrm{m}}} f(v,t)\, \mathrm{d}v \\ &\le \int_0^{v_{\mathrm{crit}}} f_{\mathrm{o}}(v)\, \mathrm{d}v + \frac{\mathcal{V}(0)}{v_{\mathrm{crit}}} \le \mathcal{N}(0) + \frac{\mathcal{V}(0)}{v_{\mathrm{crit}}}. \end{aligned} \tag{6.10}$$

Estimate (6.10) for $\mathcal{N}$ is *a priori*, global and independent of any bound for f in the local time of existence.

Proposition 6.1. *Let f be a bounded solution to problem (4.5); then, under assumptions from* (H1) *to* (H6)*, all products*

$$\alpha_{\mathrm{k}}(v) f(v,t), \qquad k = 2, \dots, N \tag{6.11}$$

are bounded for all $v \in [0, v_{\mathrm{m}}]$.

Now we can go back to (2.1): from (6.10), hypotheses (H1) to (H6) and Lemma 6.1 we easily get that

$$\left(\frac{\partial f}{\partial t}\right)_+ \le C\left(1 + \{f\}_t\right), \tag{6.12}$$

where C does not depend on f. We can thus proceed as in [2] to prove that (6.12) implies the global existence of f. In conclusion we have

Theorem 6.1. *(*GLOBAL EXISTENCE*) If the hypotheses of Theorem 5.3 are completed with (6.1), (6.2) and (6.3), then the solution to problem (4.5) given by Theorem 5.3 can be extended through $t = T$ over any finite time interval with the same regularity properties.*

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