Creaming and breaking of liquid emulsions: a free boundary problem

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We could assume $\varrho_1 < \varrho_2$ (although the opposite inequality is also possible): this means that we think of the guest as *oil* and of the host as *water*. This is also called sometimes an *oil-in-water* emulsion, as opposite to the other case called instead a *water-in-oil* emulsion.

Everyday life examples

(a) **water-in-oil**: in butter and margarine, a continuous lipid phase surrounds droplets of water

(b) **oil-in- water**: mayonnaise is stabilized with egg yolk lecithin.

Chemical substances that stabilize an emulsion are called **emulsifiers**, or surfactants. Example are proteins, phospholipids, and similar low-molecular-weight substances.

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Other examples of industrially important emulsions: **many medical preparations, cosmetics, some complex foods, most paints, the photosensitive side of a photographic film, a cutting fluid for metalworking**.

Liquid emulsions and their static stability

Emulsions are part of a more general class of two-phase systems of matter called **colloids**. Although the terms colloid and emulsion are sometimes used interchangeably, the word emulsion tends to imply that the dispersed and continuous phases are both liquids. **We refer only to liquid emulsions**.

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Unfortunatley, unless some chemical surfactant is added to the emulsion, homogenized systems show a more or less strong tendency to separate with time! **The emulsion breaks down**.

Sources of instability

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Ostwald ripening is a further process which causes an emulsion to separate.

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A very simple example of an unstable emulsion is homemade salad dressing made by mixing oil and vinegar. The two phases separate quickly unless shaken repeatedly. **The phases separate when smaller droplets recombine to form larger ones by *coalescence***.

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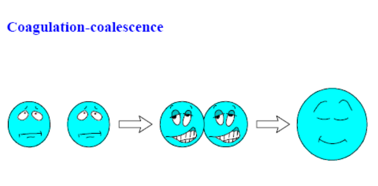
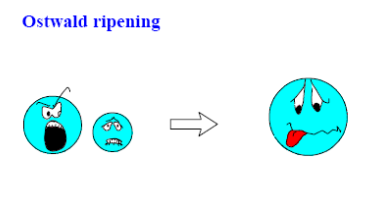
Fluid emulsions may also suffer from **creaming**: a layer of densely packed bubbles forms at the top (or bottom) of the container where the volume fraction of the continuous phase is negligible and droplets are prectically at rest.

Visually...

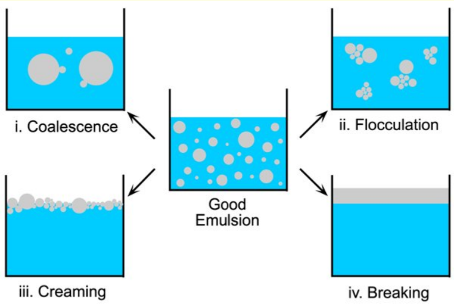


(c) **coalescence**



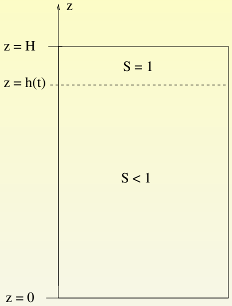
(d) **Ostwald ripening**

Schematic picture



See `http://www.pharmpedia.com/Emulsion` for a deeper discussion about problems related to preparation and stability vs. time of emulsions for example in pharmacology and food industry.

Geometry



The emulsion fills completely a (cylindrical) container with height H and cross sectional area $\mathscr{A}$.

The axis z of the cylinder is parallel to the gravity vector and directed upward.

Notations

We introduce the **oil volume fraction** S and the **volumetric distribution of oil drops** n per unit volume.

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The whole problem will be treated in **one spatial dimension**, that is we allow n to depend, besides on the volume v of the drops and on time t, only on the spatial coordinate z. Thus $S = S(z,t)$ and $n = n(z,v,t)$.

Concentration and drop number

Clearly

$$S(z,t) = \int_0^{v_{\max}} v n(z,v,t)\,\mathrm{d}\,v. \qquad (1)$$

S is nondimensional, while $[n] = L^{-6}$. The **drop density** is defined by

$$N(z,t) = \int_0^{v_{\max}} n(z,v,t)\,\mathrm{d}\,v, \qquad ([N] = L^{-3}) \qquad (2)$$

Moreover, since $S(z,t)\mathscr{A}\,\mathrm{d}\,z$ is the amount of oil in the elementary horizontal layer centered in z at time t, then

$$\Phi(t) = \int_0^H S(z,t)\mathscr{A}\,\mathrm{d}\,z = \int_0^H \int_0^{v_{\max}} v n(z,v,t)\mathscr{A}\,\mathrm{d}\,v\,\mathrm{d}\,z \qquad (3)$$

provides at $t = 0$ the so-called **holdup** that is the total volume of oil dispersed in water. For $t > 0$ it includes also the separated phase.

Volume conservation

We notice that any meaningful mathematical model has to guarantee that both the boundedness of S and volume conservation (4).

Obviously, since we are not considering any chemical reaction, we should have

$$\Phi(t) = \text{constant} = \Phi(0). \tag{4}$$

We shall see that this is actually true (under simple and physically obvious conditions).

Drop volume upper bound

Without any local stress applied to the fluid, there is no active mechanism to control the growth of drops driven by coalescence besides the constraint of total volume of dispersed phase.

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From this point of view the mathematics is definitely easier (we deal with a finite interval instead of an infinite one)

Balance equations

The following balance equation incorporates the effects of buoyancy, diffusion and coalescence.

$$\partial_t n(z,v,t) + \partial_z j(z,v,t) = L_c[S;n] \tag{5}$$

where

$$j(z,v,t) := V(S(z,t),v,t)n(z,v,t) - D(S(z,t),v)\partial_z n(z,v,t) \tag{6}$$

is the *flux* and V, D are the **upward velocity** of a drop of volume v due to the Archimedean force and the **diffusivity** coefficient of drops in the bulk, respectively and $L_c[S;n]$ is the **coalescence operator**.

Coalescence operator

$$L_c[S;n] = \frac{1}{2}\int_0^v n(z,v-w,t)n(z,w,t)Q(S(z,t);v-w,w)\,\mathrm{d}\,w \\ -n(z,v,t)\int_0^{v_{\max}-v} n(z,w,t)Q(S(z,t);v,w)\,\mathrm{d}\,w \tag{7}$$

Physical meaning: (7) represents the difference between the production rate of drops of volume v due to coalescence of a drop of volume $v-w$ with one of volume w and the loss rate of drops of volume v due to coalescence with other drops in the allowable range.

The coalescence kernel Q is a function with suitable regularity properties that will be specified later.

Some general aspects and remarks

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- Notice that v has the role of a parameter in (5). Some Authors let it vary in a discrete set.
- The coalescence operator L_c, for Q symmetric and not depending explicitly on S, is known to satisfy the **fundamental volume preserving relation**

$$\int_0^{v_{\max}} v L_c[n] \, \mathrm{d}\, v = 0. \tag{8}$$

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- Bearing (1) and (8) into account, integrate (5) with respect to v over $(0, v_{\max})$. It follows that

$$\partial_t S + \partial_z J(z,t) = 0. \tag{9}$$

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- It is not generally true that equation (9) reduces to a pure equation for S. However, **if both V and D do not depend on v, we immediately get the simplified form**

$$\partial_t S - D(S)\, \partial_{zz} S$$
$$+ S\, V'(S,t)\, \partial_z S + V(S,t)\, \partial_z S - D'(S)\, (\partial_z S)^2 = 0 \tag{10}$$

The free boundary

We may locate the free boundary $z = h(t)$ through the condition $\lim_{z \to h^+(t)} S(z,t) = S_{\text{crit}}$ where $S_{\text{crit}} \approx 1$ corresponds, effectively, to the saturation of oil at the onset of the "cream". From the practical point of view we can take $S_{\text{crit}} = 1$, without generating any important perturbation.

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$$\dot{h}\left[S_{\text{crit}} - S(h(t),t)\right] = -J(h(t),t)$$

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$$\left[\partial_z n(z,v,t)\right]_{z=h(t)} = 0.$$

The natural **condition at the bottom surface** $z = 0$ is $j(0,v,t) = 0$, $t > 0$.

Mathematical model

$$\partial_t n(z,v,t) + \partial_z j(z,v,t) = L_c[S;n] \tag{11}$$

$$n(z,v,0) = n_0(z,v) \tag{12}$$

$$S(z,0) = \int_0^{v_{\max}} v n_0(z,v)\,\mathrm{d}\,v = S_0(z) < S_{\text{crit}} \tag{13}$$

$$j(0,v,t) = 0 \tag{14}$$

$$[\partial_z n(z,v,t)]_{z=h(t)} = 0 \tag{15}$$

$$\dot{h}\ [S_{\text{crit}} - S(h(t),t)] = -J(h(t),t) \tag{16}$$

Condition $[\partial_z n(z,v,t)]_{z=h(t)} = 0$ implies that (16) simply reduces to

$$\dot{h}\ [S_{\text{crit}} - S(h(t),t)] = -\int_0^{v_{\max}} vV(S(h(t),t),v)n(h(t),v,t)\,\mathrm{d}\,v. \tag{17}$$

Volume conservation

Lemma 1

Let us suppose that $Q(S(z,t),v,w)$ is a non-negative smooth function, symmetric w.r.t. the pair (v,w). Then

$$\int_0^H \left(\int_0^{v_{\max}} v L_c[S;n] \,\mathrm{d}\, v \right) \mathrm{d}\, z = 0, \quad \text{for all } t \geq 0. \tag{18}$$

Lemma 2

Let us assume $S(H,t) = 1$, and $D(1,v) = V(1,v) = 0$. Then

$$\int_0^H \left(\int_0^{v_{\max}} v \partial_z j(z,v,t) \,\mathrm{d}\, v \right) \mathrm{d}\, z = 0, \quad \text{for all } t \geq 0.$$

Theorem 3

Under the hypotheses of Lemmas 1 and 2 we have that $\Phi(t) =$constant (i.e. volume is conserved).

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Phi(t) = \int_0^H \int_0^{v_{\max}} v\partial_t n(z,v,t)\,\mathrm{d}\,v\,\mathrm{d}\,z$$

$$= \int_0^H \left(\int_0^{v_{\max}} vL_c[S;n]\,\mathrm{d}\,v \right) \mathrm{d}\,z - \int_0^H \left(\int_0^{v_{\max}} v\partial_z j(z,v,t)\,\mathrm{d}\,v \right) \mathrm{d}\,z$$

□

Positivity of n (and S)

Consider the following *modified* version of equation (11):

$$\partial_t n(z,v,t) + \partial_z j(z,v,t) = L_c^+[S,n] \tag{19}$$

where $L_c^+[S,n]$ is the following modified version of the coalescence operator

$$\frac{1}{2}\int_0^v n^+(z,v-w,t)n^+(z,w,t)Q(S(z,t);v-w,w)\,\mathrm{d}\,w$$
$$-n(z,v,t)\int_0^{v_{\max}-v} |n(z,w,t)|Q(S(z,t);v,w)\,\mathrm{d}\,w$$

and n^+ (n^-) denotes the positive (negative) part of n $(= n^+ + n^-)$.

Positivity of n (and S)

(19), coupled with (12)-(15) and (16) replaced with

$$\frac{\mathrm{d}\, h(t)}{\mathrm{d}\, t}\left[S_{\text{crit}} - S(h(t), t)\right] = -\int_0^{v_{\max}} v \left(n^+ V\right)_{z=h(t)} \mathrm{d}\, v \qquad (20)$$

has **non-negative solutions** provided that

$$\begin{cases} n(z, v, 0) \geq 0 \\ V \leq M < +\infty \\ D \geq m > 0 \end{cases} \qquad (21)$$

where m, M are given constants (notice that the hypothesis V bounded is perfectly consistent with the fact that drop volume cannot exceed a finite value).

Positivity of n (and S)

Evidently solutions of the original initial boundary value problem and the modified one identify over non-negative solutions. Therefore, provided a uniqueness theorem holds, a non-negative initial data guarantees the positiveness of the regular solution of problem (11)-(15).

Theorem 4

Under hypotheses (21) and boundary conditions (12)-(15), all regular solutions of the original free-boundary problem are non-negative.

Sketch of the proof

Multiply (11) by n^- and integrate over

$$\Omega_{\hat{t}} := \left\{(z,t) \mid z \in (0,h(t)), t \in (0,\hat{t})\right\}$$

to get

$$\frac{1}{2}\int_{\Omega_{\hat{t}}} \partial_t (n^-)^2 \,\mathrm{d}\,\boldsymbol{\sigma} + \int_{\Omega_{\hat{t}}} \partial_z (n^- j)\,\mathrm{d}\,\boldsymbol{\sigma} - \int_{\Omega_{\hat{t}}} j\partial_z (n^-)\,\mathrm{d}\,\boldsymbol{\sigma}$$
$$= \int_{\Omega_{\hat{t}}} n^- L_c^+[S;n], \qquad (\mathrm{d}\,\boldsymbol{\sigma} := \mathrm{d}\,z\,\mathrm{d}\,t).$$

Then use $n^- L^+[S,n] \leq 0$ and $n(z,v,0) = n_0(z,v) \geq 0$ be obtain

$$\frac{1}{2}\int_0^{h(\hat{t})} (n^-)^2 \,\mathrm{d}\,z - \int_0^{\hat{t}} (n^-(h(t)))^2 h'(t)\,\mathrm{d}\,t$$
$$+ \int_{\Omega_{\hat{t}}} \partial_z (n^- j)\,\mathrm{d}\,\boldsymbol{\sigma} - \int_{\Omega_{\hat{t}}} j\partial_z (n^-)\,\mathrm{d}\,\boldsymbol{\sigma} \leq 0$$

Sketch of the proof

Then use boundary conditions (20) to prove that

$$-\int_0^{\hat{t}} (n^-(h(t)))^2 \left(-\frac{1}{S_{\text{crit}} - S(h(t),t)} \int_0^{v_{\max}} v \left(n^+ V\right)_{z=h(t)} \mathrm{d}\, v \right) \mathrm{d}\, t$$

is non-negative. The third term is non-negative too since, because of (14), it reduces to

$$\int_0^{\hat{t}} \left(V (n^-)^2\right)_{z=h(t)} \mathrm{d}\, t$$

Therefore we remain with the following

$$\frac{1}{2} \int_0^{h(\hat{t})} (n^-)^2 \,\mathrm{d}\, z \leq \int_{\Omega_{\hat{t}}} j \partial_z (n^-) \,\mathrm{d}\, \boldsymbol{\sigma} = \int_{\Omega_{\hat{t}}} (Vn - D\partial_z n)\, \partial_z (n^-) \,\mathrm{d}\, \boldsymbol{\sigma}$$

Sketch of the proof

Hypotheses (21) and

$$n\partial_z(n^-) = n^-\partial_z(n^-), \qquad \partial_z n\partial_z(n^-) = \partial_z n^-\partial_z(n^-)$$

imply

$$\frac{1}{2}\int_0^{h(\hat{t})}(n^-)^2\,\mathrm{d}\,z + m\int_{\Omega_{\hat{t}}}\left(\partial_z(n^-)\right)^2\mathrm{d}\,\boldsymbol{\sigma} \leq M\int_{\Omega_{\hat{t}}} n^-\partial_z(n^-)\,\mathrm{d}\,\boldsymbol{\sigma}$$

Finally use classical Young's to get, for arbitrary $\varepsilon > 0$ and suitable$C(M,\varepsilon) > 0$

$$\int_0^{h(\hat{t})}(n^-)^2\,\mathrm{d}\,z \leq C(M,\varepsilon)\int_{\Omega_{\hat{t}}}(n^-)^2\,\mathrm{d}\,\boldsymbol{\sigma}$$

Sketch of the proof

Being

$$y(t) := \int_0^{h(t)} (n^-)^2 \,\mathrm{d}\, z,$$

we have

$$y(\hat{t}) \leq C(M, \varepsilon) \int_0^{\hat{t}} y(\tau) \,\mathrm{d}\, \tau$$

for any $\hat{t} > 0$, which in turn implies that n^- vanishes a.e. in $\Omega_{\hat{t}}$.

On the behaviour of small drops

At rest no breackage is possible to contrast coalescence. Therefore one expects that if the dispersion has no drops below a given threshold $v_{\min}$ at time $t = 0$, then no drops will ever be observed in that range for $t > 0$. This of course is false if the Ostwald ripening is taken into account, which however is not our case.

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Theorem 5

Let n be a regular solution of problem (11)-(15) such that $n(z, v, 0)$ is equal to zero for all $v \in (0, v_{\text{min}})$ and for all $z \in [0, h(0)]$. Let also Q, D, V be smooth bounded functions. Then $n(z, v, t)$ remains equal to zero for $v \in (0, v_{\text{min}})$ for all $z \in [0, h(t)]$ and for all $t > 0$.

Sketch of the proof

Multiply equation (5) by n and integrate over $\Omega_{\hat{t}}$. Boundary conditions (14), (15) and the hypothesis $v \in (0, v_{\min})$ imply that

$$\begin{aligned} \int_{\Omega_{\hat{t}}} n\partial_z j \,\mathrm{d}\boldsymbol{\sigma} &= -\int_{\Omega_{\hat{t}}} j\partial_z n \,\mathrm{d}\boldsymbol{\sigma} + \int_0^{\hat{t}} (nj)\big|_0^{h(t)} \\ &= -\int_{\Omega_{\hat{t}}} j\partial_z n \,\mathrm{d}\boldsymbol{\sigma} + \int_0^{\hat{t}} (nj)_{z=h(t)} \,\mathrm{d}t \\ &= -\int_{\Omega_{\hat{t}}} j\partial_z n \,\mathrm{d}\boldsymbol{\sigma} + \int_0^{\hat{t}} (Vn^2)_{z=h(t)} \,\mathrm{d}t; \end{aligned} \tag{22}$$

therefore (being $n_0(z,v) = 0$ for $v \in (0, v_{\min})$) we get

$$\frac{1}{2}\int_{\Omega} n^2(z,v,\hat{t}) \,\mathrm{d}z + \int_0^{\hat{t}} \left(Vn^2\right)_{z=h(t)} \mathrm{d}t - \int_{\Omega_{\hat{t}}} Vn\partial_z n \,\mathrm{d}\boldsymbol{\sigma}$$
$$+ \int_{\Omega_{\hat{t}}} D\left(\partial_z n\right)^2 \mathrm{d}\boldsymbol{\sigma} = \int_{\Omega_{\hat{t}}} nL_c[S;n] \,\mathrm{d}\boldsymbol{\sigma}$$

(23)

Sketch of the proof

Dropping the second term on the l.h.s. and a standard use of Young's inequality then produces

$$\begin{aligned}&\frac{1}{2}\int_{\Omega} n^2(z,v,\hat{t})\,\mathrm{d}\,z + \left(m - \frac{M}{2}\varepsilon\right)\int_{\Omega_{\hat{t}}} (\partial_z n)^2\,\mathrm{d}\,\boldsymbol{\sigma}\\ &\leq \frac{M}{2\varepsilon}\int_{\Omega_{\hat{t}}} n^2\,\mathrm{d}\,\boldsymbol{\sigma} + \int_{\Omega_{\hat{t}}} nL_c[S;n]\,\mathrm{d}\,\boldsymbol{\sigma}.\end{aligned} \tag{24}$$

The coalescence source term can be estimated as follows

$$\int_{\Omega_{\hat{t}}} nL_c[S;n]\,\mathrm{d}\,\boldsymbol{\sigma} \leq \frac{K}{2}\int_{\Omega_{\hat{t}}} n\left(\int_0^v n^2\,\mathrm{d}\,w\right)\mathrm{d}\,\boldsymbol{\sigma}$$

where K is an upper bound for Q.

Sketch of the proof

Thus, for a sufficiently small $\varepsilon > 0$, (24) rewrites

$$\frac{1}{2}\int_{\Omega} n^2(z,v,\hat{t})\,\mathrm{d}\,z \le \frac{M}{2\varepsilon}\int_{\Omega_{\hat{t}}} n^2\,\mathrm{d}\,\boldsymbol{\sigma} + \frac{K}{4}\int_{\Omega_{\hat{t}}} n^2\,\mathrm{d}\,\boldsymbol{\sigma} + \frac{K}{4}\int_{\Omega_{\hat{t}}}\left(\int_0^v n^2(z,w,t)\,\mathrm{d}\,w\right)^2\mathrm{d}\,\boldsymbol{\sigma} \tag{25}$$

The last integral on the r.h.s. can be increased substituting v with $v_{\min}$. Then if we set

$$y(\hat{t}) := \int_{\Omega}\left(\int_0^{v_{\min}} n^2(z,v,\hat{t})\,\mathrm{d}\,v\right)\mathrm{d}\,z$$

and integrate (25) over $(0, v_{\min})$ we get

Sketch of the proof

$$y(\hat{t}) \le \left(\frac{M}{2\varepsilon} + \frac{K}{4}\right) \int_0^{\hat{t}} y(t)\,\mathrm{d}\,t + \frac{K}{4} v_{\min} \int_0^{\hat{t}} y^2(t)\,\mathrm{d}\,t. \qquad (26)$$

Notice now that, because of the hypotheses, we have $y(0) \equiv 0$ and that $y(\hat{t}) \le s(\hat{t})$ for all $\hat{t} > 0$ where $s(\hat{t})$ solves the integral equation

$$s(\hat{t}) = \left(\frac{M}{2\varepsilon} + \frac{K}{4}\right) \int_0^{\hat{t}} s(t)\,\mathrm{d}\,t + \frac{K}{4} v_{\min} \int_0^{\hat{t}} s^2(t)\,\mathrm{d}\,t. \qquad (27)$$

Also notice that $s(0) = 0$; then (27) has $s(t) \equiv 0$ as unique solution. In conclusion $y(\hat{t}) \le 0$ for all $\hat{t} > 0$, i.e. n vanishes identically for all z, t and $v \in (0, v_{\min})$.

A priori estimates for S

Theorem 6

Maximum principle and Hopf's boundary point lemma imply that

$$|S| \le \max S(z,0) < S_{\text{crit}}, \qquad 0 \ge \frac{\mathrm{d}\, h}{\mathrm{d}\, t} \ge -M \tag{28}$$

Theorem 7

For D and V independent of v, the following *energy-type* estimate holds for S

$$\int_0^{h(t)} S^2(z,t)\,\mathrm{d}\, z + \int_{\Omega_t} (\partial_z S)^2\,\mathrm{d}\,\boldsymbol{\sigma} \le C_1(M,\varepsilon,S_{\text{crit}},T) \tag{29}$$

Some Hölder spaces

We denote with $||f||$ the standard supremum norm for f continuous in $\overline{\Omega}_T$, with $C(\overline{\Omega}_T)$ the corresponding Banach space, with $|f|^{(\alpha)}$ the norm

$$||f|| + \sup_{x' \neq x''} \frac{|f(x',t) - f(x'',t)|}{|x' - x''|^{\alpha}} + \sup_{t' \neq t''} \frac{|f(x,t') - f(x,t'')|}{|t' - t''|^{\alpha/2}}$$

for $f \in C(\overline{\Omega}_T)$, $\alpha \in (0,1)$, and with $H^{\alpha,\alpha/2}(\overline{\mathscr{S}})$ the corresponding Banach space with $|f|^{(\alpha)}$ bounded.

Some Hölder spaces

The spaces $H^{1+\alpha,1/2+\alpha/2}(\overline{\Omega}_T)$ and $H^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$ are defined accordingly with bounded norms

$$|f|^{(1+\alpha)} := ||f|| + |\partial_x f|^{(\alpha)}, \qquad |f|^{(2+\alpha)} := ||f|| + |\partial_t f|^{(\alpha)} + |\partial_{xx} f|^{(\alpha)}$$

respectively. For continuous functions g of a single real variable over an open interval $I \subset \mathbb{R}$ we define the Hölder spaces $H^{\alpha/2}(\overline{I})$, $H^{1+\alpha/2}(\overline{I})$ etc. and their related norms quite similarly.

A priori estimates for S in Hölder spaces

We can prove the following result:

Theorem 8

For D and V independent of v, we get

$$|S|^{(\alpha)} \leq C, \qquad \left|\frac{\mathrm{d}\, h}{\mathrm{d}\, t}\right|^{(\alpha/2)} \leq C$$

in $\overline{\Omega}_T$ and $[0, T]$ respectively, for a suitable positive constant C, an arbitrary $T > 0$ and a suitable $\alpha \in (0, 1)$.

The proof is based on the rectifying transformation $y = z/h(t)$ and suitable estimates by Ladyzenskaja et al. for quasi-linear parabolic equations

A priori estimates for S in Hölder spaces

By means of the Kirchoff's transformation $\widehat{S} := \int_0^S D(\eta)\, \mathrm{d}\, \eta$ applied to (10) and the rectifying transformation plus a classical bootstrap argument we can also prove

Theorem 9

For V and S independent of v we have

$$|h|^{(2+\beta/2)} \leq C \tag{30}$$

$$|S|^{(2+\beta)} \leq C \tag{31}$$

in $[0, T]$ and $\overline{\Omega}_T$ respectively, where β is the order of Hölder continuity of the second derivative of the initial data.

A priori estimates for n

It can be proved that

$$\int_0^{v_{\max}} L_c[S;n]\,\mathrm{d}\,v \leq 0 \tag{32}$$

Then N (the local number of drops per unit volume) obeys

$$\partial_t N - [D(S)]\,\partial_{zz} N + \left[V(S) - D'(S)\partial_z S\right]\partial_z N + \left[V'(S)\partial_z S\right] N \leq 0 \tag{33}$$

If we take initial values $n(v,z,0)$ and $N(z,0)$ that are bounded, then

$$0 \leq N \leq C\exp(\lambda t) \leq C\exp(\lambda T) := G(T) \quad \text{in} \quad \Omega_t \tag{34}$$

for C and λ sufficiently large.

A priori estimates for n

For bounded Q and because of (34), $L_c n \leq G(T) \max_{v\in(0,v_{\max})} n(z,v,t)$. This implies that the function

$$n^*(t) := \max_{\substack{z\in(0,h(0))\\ v\in(0,v_{\max})}} n(v,z,0) \exp[(G(T)+\mu)t]$$

is a supersolution of equation (5) provided that the upper bound M of Q is sufficiently large and μ is chosen conveniently. In conclusion for any solution of (5) we have

$$0 \leq n(z,v,t) \leq \max_{\substack{z\in(0,h(0))\\ v\in(0,v_{\max})}} n(v,z,0) \exp[(G(T)+\mu)T] \tag{35}$$

Higher order estimates for n

With a combined and subtle use of the rectifying transformation and some results by Ladyzenskaja et al. on quasi-linear parabolic equations we can prove the following further estimates

- $|n|^{(2+\gamma)} \leq C(v)$ where γ is the Hölder order of continuity of $\partial_{zz} n^*(z, v, 0)$.
- $\partial_v n \in L^\infty$
- $|\partial_v n| \leq C(v)$

All estimates obtained so far, joined with a local existence theorem (still to be proved), do imply the global existence and uniqueness of a solution S, h to the original problem.

Existence and uniqueness

We use the technique of constructing a suitable contraction mapping. Define the space

$$\mathscr{H}_T^\alpha := H^{1+\alpha,(1+\alpha)/2}\left(\overline{R}_T\right) \times H^{1+\alpha/2}\left([0,T]\right),$$

$||(\psi,\chi)||_T^{(\alpha)} = |\psi|_{\overline{R}_T}^{(1+\alpha)} + |\chi|_{[0,T]}^{(1+\alpha/2)}$ its norm and denote with $\mathscr{E}$ the set of solutions of equation (36) with the relevant initial-boundary conditions (37). Finally define

$$\mathscr{U}_T^\alpha := \left\{(\overline{\omega},\overline{h}) \in \mathscr{H}_T^\alpha \text{ s.t. } ||(\overline{\omega},\overline{h})||_T^{(\alpha)} \leq M, \overline{\omega} \in \mathscr{E}\right\}$$

Existence and uniqueness

Now apply again the rectifying transformation $y = z/h(t)$ and set $\omega := \widehat{S}(yh(t), t)$: equation for S trasforms into

$$\partial_t \omega - \left[\frac{D}{h^2}\right] \partial_{yy}\omega + \left[\frac{V}{h} + \frac{V'\mathscr{F}}{h} - \frac{yh'}{h}\right] \partial_y \omega = 0 \qquad (36)$$

We remark explicitly that all coefficients in equation (36) have bounded Hölder norms and will therefore considered as known. The initial-boundary conditions to be appended to (36) are

$$\begin{aligned} &\omega = \int_0^{S_0(yh(0))} D(\eta)\,\mathrm{d}\,\eta && \text{for} \quad t = 0 \\ &\partial_y \omega = 0 && \text{for} \quad y = 1 \\ &-\partial_y \omega + hV(\mathscr{F}(\omega))\mathscr{F}(\omega) = 0 && \text{for} \quad y = 0. \end{aligned} \qquad (37)$$

Existence and uniqueness

We now refer to equation (36): define the map

$$\mathscr{T} : (\overline{\omega}, \overline{h}) \mapsto (\widehat{\omega}, \widehat{h})$$

where $\widehat{\omega}$ verifies equation (36) with the terms in square brackets recalculated using the pair $(\overline{\omega}, \overline{h})$ and $\widehat{h}$ satisfies equation

$$\widehat{h}' = V(\widehat{S}) \frac{\widehat{S}}{S_{\text{crit}} - \widehat{S}(\overline{h}(t), t)} \tag{38}$$

Theorem 10

Provided that $T \ll 1$ and $M \gg 1$, the operator $\mathscr{T}$ maps $\mathscr{U}_T^\alpha$ into itself, i. e. $\mathscr{T}(\mathscr{U}_T^\alpha) \subseteq \mathscr{U}_T^\alpha$. Moreover for a suitable choice of parameters, $\mathscr{T}$ is a contraction in $\mathscr{U}_T^\alpha$

Existence and uniqueness

The local existence theorem for n is still based upon the construction of a suitable contraction mapping defined over the space

$$\mathscr{K} := \{n^*(y,v,t) \text{ s.t. } n \in L^\infty(A_T), n^*(y,v,0) = n_0^*(y,v),$$
$$||n||_{n \in L^\infty(A_T)} \leq M \}$$

where $A_T := [0,1] \times [0, v_{\max}] \times [0,T]$. Consider the map

$$\mathfrak{F} : \overline{n} \mapsto \widehat{n}$$

where $\widehat{n}$ solves equation (5) with L_c at the r.h.s. evaluated in $\overline{n}$.

Theorem 11

For a suitable choice of parameters, $\mathfrak{F}$ is a contraction in $\mathscr{K}$.