Minicourse content

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- Degradating Bingham fluids

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  - the free boundary problem
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  - driving interactions
  - existence–uniqueness

Minicourse in Industrial Mathematics

Segundo Enquentro Italo–Argentino

First lecture: Dynamics of degradating slurries in a pipe

F. Rosso

`www.math.unifi.it/˜rosso/Baires/minicorso.pdf`

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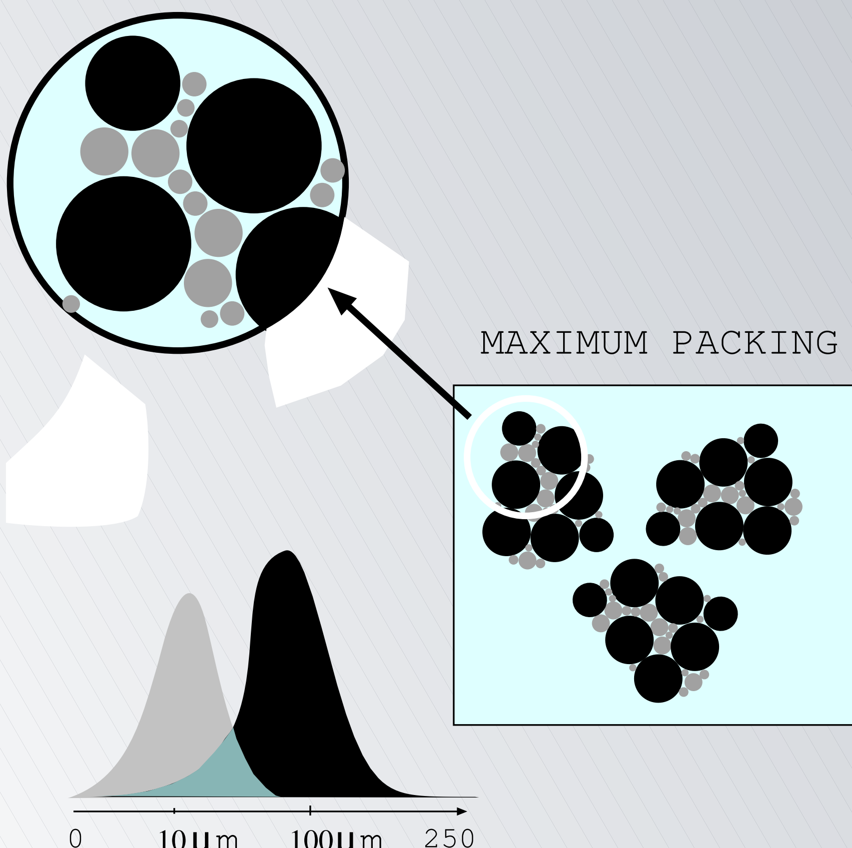
What is a CWS?

A Coal Water Slurry (CWS) is a mixture of coal (up to 70% in weight), water (up to 29%) and suitable fluidizing agents (about 1%). Coal particles are micronized with a top size of about 250 μm and a bimodal size distribution centered at 10 and 100 μm for optimal (maximum) packing. An industrial CWS is totally stable at rest (therefore it can be stocked for long periods of time) and burned without needing a preliminary dehydration.



CWS structure

Industrial interest

These characteristics make CWS an important alternative source of energy for the production of power in electric plants. Indeed an industrial CWS is totally stable *at rest* as far as both rheological properties and sedimentation are concerned. Thus the product can be stocked for a very long period of time (years) without any significant alteration. During the oil crises of past decades the CWS technology has been deeply investigated and industrially operated in some areas of the World where the abundance of coal and the geomorphology suggested the possibility of pipelining under pressure the mixture from the production site to the electric plant.

Main problems

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- *Rheological degradation*: it's a long-time effect due to shear. The apparent viscosity reaches extremely high values and the product becomes so viscous to be no longer pumpable in a pipe and thus useless. This problem embodies various subproblems: we will mention only some of them

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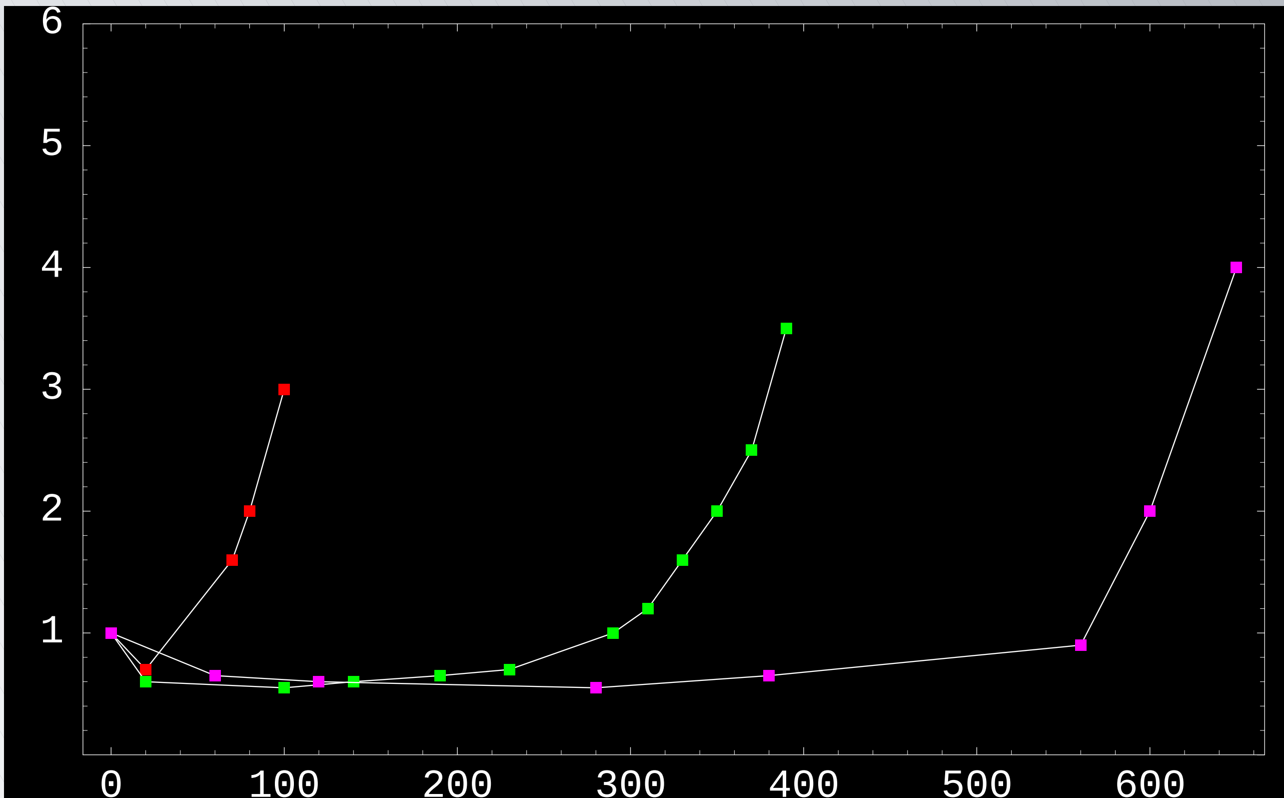
- *Sedimentation*: it's also a long-time effect also due to shear which is responsible of a *breaking-up the internal structure of the mixture* allowing manufacturing impurities (sand, ashes, iron oxides, . . .) which are not stabilized by chemical additives to settle on the bottom of the pipeline. A sedimentation bed grows up on the bottom of the first kilometers of a pipeline eventually compromising the optimal discharge unless the pumping operation is stopped and the conduct is cleaned up [THIS WILL BE THE OBJECT OF THE NEXT LECTURE]

What is degradation?



Relative apparent viscosity at $10\ s^{-1}$ vs. specific cumulative energy (kJ/kg) for a polish CWS. The different marks identify mixtures with 0.5%, 0.75%, and 1.00% of dispersed additive

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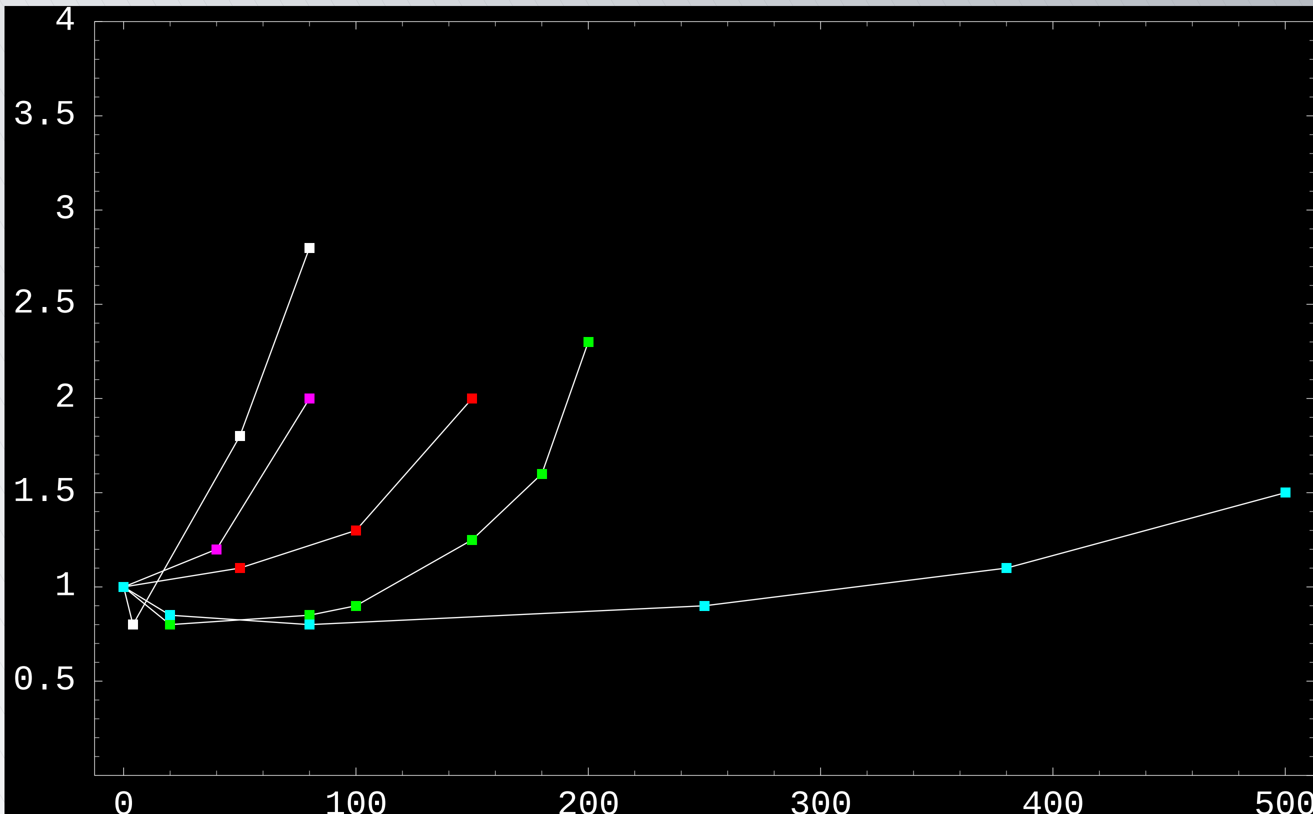
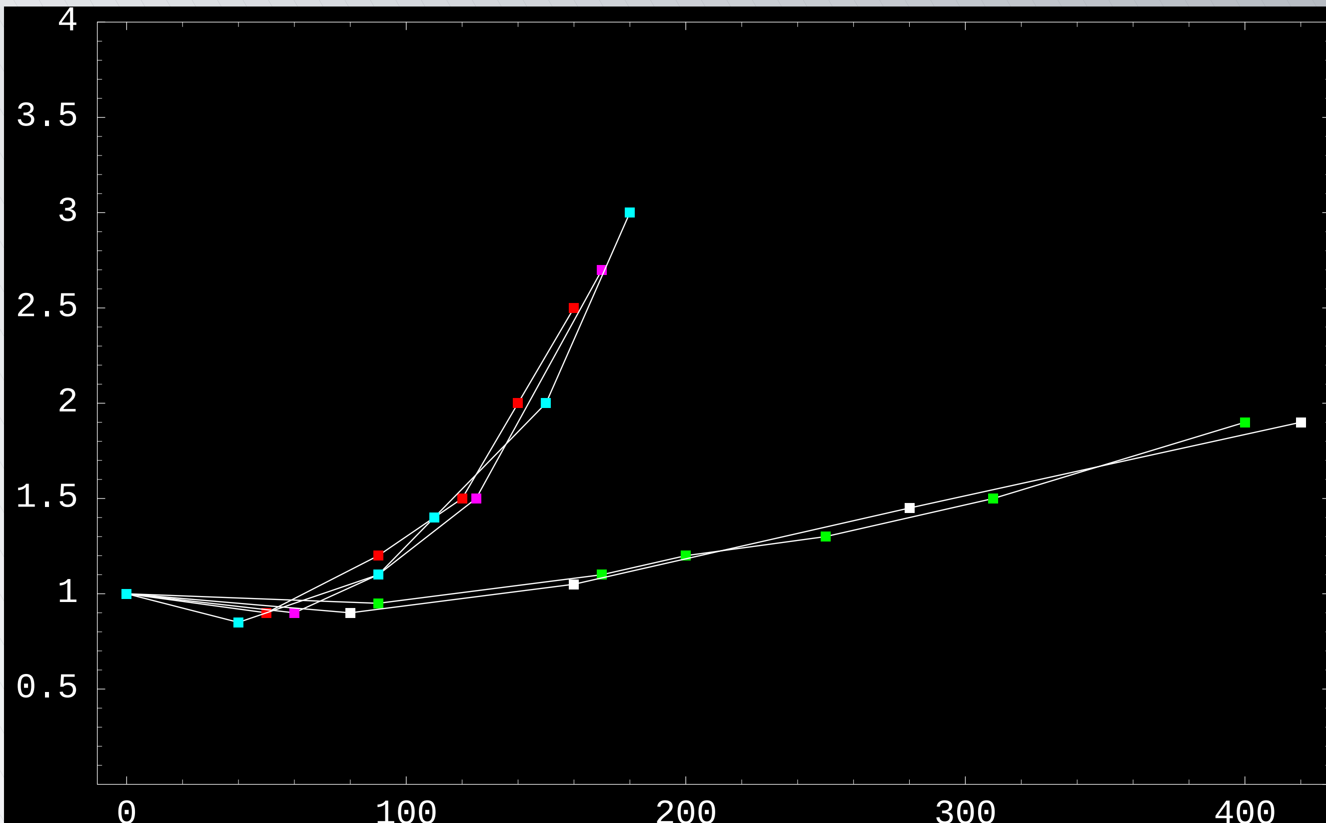


Relative apparent viscosity at $10\ s^{-1}$ vs. time for two different CWS at various shear rates. The white and green marks identify a type of mixture (Colombian CWS) at two different shear rates ($20\ s^{-1}$ and $50\ s^{-1}$); the other marks identify another mixture (Russian CWS) at three different shear rates ($20\ s^{-1}$, $50\ s^{-1}$, and $80\ s^{-1}$)

What is degradation?



Relative apparent viscosity at $10\ s^{-1}$ vs. specific cumulative energy using the same data of the previous plot. All marks related to the same type of mixture arrange themselves on a unique curve *regardless of the operated shear rate*

Additive dynamics

The additive dynamics is one keypoint to understand rheological degradation. The whole process can be modelled via population dynamics. Let us define A to be the % of additive available in water, B the % of additive adsorbed by non–ionized sites on coal particles, Y the concentration of ions adsorbed on coal particles, I the concentration of ions in water, $\bar{B}$ the maximum quantity of dispersant adsorbable on coal particles, and D the % of “inert” additive adsorbed on coal particles.

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- While $A \to B$ is reversible, $I \to Y$ and $B \to D$ are not
- Irreversible transitions are activated only by internal dissipation due to shear

Additive dynamics

These remarks suggest the following set of equations for the unknowns A, B, Y, D with initial conditions $A(0) = A_0$, $B(0) = B_0$, $I(0) = I_0$, $f(W)$ function of the dissipated power and B_∞ asymptotic value of B. Constants μ_1, μ_2 are the rate of adsorption and desorption respectively. Clearly $I(t) + Y(t) = I_0$ and $A + B = D$

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\begin{aligned}
\dot{A} &= -\mu_1 A(\bar{B} - B) + \mu_2 B && (\mu_1, \mu_2 > 0 \quad \text{constants}) \\
\dot{B} &= \mu_1 A(\bar{B} - B) - \mu_2 B && \\
\dot{D} &= -\lambda Y D && (\lambda > 0 \quad \text{constant}) \\
\dot{Y} &= \alpha_1 (\bar{B} - B - Y)(I_0 - Y) - \alpha_2 Y && (\alpha_1, \alpha_2 > 0 \quad \text{constants}) \\
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Once function B is determined, the CWS mixture can modeled as a *Bingham fluid* where the characteristic rheological parameters are functions of B. Experimental data *in a batch reactor* fit very well the model (in this case all parameters depend only on time, not on spatial coordinates)

CWS as a Bingham fluid

However *in a pipeline* the spatial dependence of rheological parameters cannot be neglected and the problem is much more complicated.
This leads to the problem of analyzing degradation in a pipe loop.
From the rheological point of view a CWS shows all peculiarities of a *Bingham fluid*.
Within laminar regime (which is the standard operating condition), this means that there exists a yield stress τ_0 entering the relationship between the stress τ and the shear rate $\dot{\gamma}$

$$(\tau - \tau_0)_+ = \eta_B \dot{\gamma},$$

where $(\cdot)_+$ denotes the positive part and η_B=constant is the plastic viscosity. The main effect of this constitutive law is that the region in which $\tau < \tau_0$ undergoes no deformation, while the Navier–Stokes equation governs the flow in the complementary region.

Model equations

For an axisymmetric incompressible flow in a cylinder $0 < r < R$ driven by a sufficiently large pressure gradient $-G$, we have an inner rigid core $0 < r < s(t)$, while for $s(t) < r < R$ and $t > 0$ the velocity satisfies the equation

$$\rho \partial_t v = G - \frac{1}{r} \partial_r (r\tau)$$

ρ being the CWS density, with the usual no–slip condition at the rigid wall $v(R,t) = 0$. The function $s(t)$ is a free–boundary which has to satisfy the following conditions

$$\partial_r v \Big|_{r=s(t)} = 0$$ (absence of strain rate at the boundary)

$$\rho \partial_t v \Big|_{r=s(t)} = G - \frac{2\tau_0}{s(t)}$$ (momentum balance of a unit length portion of the rigid core)

Finding the right time scale

There is experimental evidence that the observed rheological degradation is mainly due to the increase of τ_0, the parameter η_B remaining virtually constant. Moreover the microscopic model shows rather clearly that the rate of change of basic rheological parameters is proportional to the dissipated power. This suggests the following law for the local variation of τ_0

$$\partial_t \tau_0 = \alpha \tau \left| \partial_r v \right| = \alpha \tau \frac{1}{\eta_B} (\tau - \tau_0)_+$$

where $\alpha > 0$ is a given non–negative dimensionless constant.
The full free boundary problem, supplemented with initial conditions for v, s and τ_0 looks exceedingly complex.

However the variation of τ_0 is very slow in the natural time scale, that is $\alpha \ll 1$. This amounts to say that the degradation time scale is much larger than the loop circulation time and the problem can be approached through a quasi–steady approximation.

Quasi–steady approximation

The main variables in the axisymmetric geometry are $\tau(r,t)$ (the shear stress), η_B=constant (the plastic viscosity), $\tau_0(r,t)$ (the yield stress), $v(r,t)$ (the velocity), r (the radius), $G(t)$ (the pressure gradient), $s(t)$ (the free surface).The mathematical model for this case (quasi–steady approximation) is given by the following equations (dimensionless form)

$$\begin{cases} (\tau-\tau_0)_+ = (\zeta/4)|\partial_r v|, & r\in(0,1), t>0 \\ G(t) = (1/r)\partial_r(r\tau) & r\in(0,1), t>0 \\ \partial_r v\,|_{r=s(t)} = 0 & t>0 \\ s(t) = 2\tau_0(s(t),t)/G(t) & t>0 \\ \partial_t\tau_0 = \tau|\partial_r v|, & r\in(0,1), t>0 \end{cases}$$

Initial and boundary conditions

The boundary condition is $v(1,t) = 0$ and the initial conditions are

$$\begin{cases} \tau_0(r,0) = 1, & 0 \le r \le 1, \\ v(r,0) = v_0(r), & 0 \le r \le 1, \\ s(0) = s_0 = 2/\zeta, & \end{cases} \tag{0}$$

where $\zeta = G(0) > 2$ and $v_0(r)$ is given by

$$\begin{cases} v_0(r) = 1 - (r)^2 - 2s_0(1-r), & s_0 := (2/\zeta) \le r \le 1, \\ v_0(r) = (1-s_0)^2, & 0 \le r \le s_0. \end{cases} \tag{-1}$$

Solving the quasi–steady problem

This system shows *unexpected phenomena*: the free boundary (separating the sheared and the unsheared regions) can touch the pipe wall or *a new free boundary can grow on the wall and meet the original one in a finite time*. Both cases lead to the blockage of the pipeline. However this does not happen in real cases since, because of the significant difference between the degradation time scale and the characteristic pipelining time, the time needed for a CWS to be pipelined between two pumping stations is much shorter then the time required for its rheological degradation to start.

Main theorem

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Moreover, each free boundary of the problem has the same regularity as the function G

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- Let (s, τ, v) be a solution of the problem. If $\dot{G} \geq 0$, the difference $Y(r,t) = \tau(r,t) - \tau_0(r,t)$ will remain positive in the domain $s(t) < r < 1$, $t > 0$. This can be obtained by analyzing the equation for Y:

$$\partial_t Y + (2r/\zeta)G(t)Y = (r/2)\dot{G}(t), \qquad Y \geq 0$$

with the initial condition $Y(r,0) := Y_0(r) = \zeta(r/2) - 1$. This is easy since it is a linear ODE!

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- Integration gives

$$Y(r,t) = \begin{cases} \exp[-rF(t)]\left\{\frac{r}{2}\int_0^t \exp[rF(\vartheta)]\dot{G}(\vartheta)\,\mathrm{d}\vartheta + Y_0(r)\right\}, & r \in [s_0,1], t > 0 \\ \exp[-rF(t-t_0(r))]\frac{r}{2}\int_{t_0(r)}^t \exp[rF(\vartheta)]\dot{G}(\vartheta)\,\mathrm{d}\vartheta, & r \in [2/G(t), s_0], t > t_0(r) \end{cases}$$

with $F(t) = \dfrac{2}{\zeta}\displaystyle\int_0^t G(\vartheta)\,\mathrm{d}\vartheta, \qquad t_0(r) = \sup\{t : G(t) = 2/r\}$, for $r > 2/G(t)$

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  2. The necessary condition for a finite time extinction or appearance of a new free boundary is that there exists a solution t^* such that
     $$\int_0^{t^*} \exp[F(\vartheta)]\dot{G}(\vartheta)\,\mathrm{d}\vartheta + \zeta = 2.$$
     If t^* is meant as the $\inf$ of these instants and it also turns out that $\int_0^{t^*} F(\vartheta)\exp[F(\vartheta)]\dot{G}(\vartheta)\,\mathrm{d}\vartheta \geq -2$ then t^* is the extinction time, i.e. $s(t^*) = 1$, otherwise we have $s(t^*) < 1$ and a new free boundary will be present for $t > \tilde{t}$, being $\tilde{t} = \inf\{t : t > t^*, \dot{G}(t) < 0\}$. The new free boundary bounds a rigid region $s_1(t) < r < 1$, provided that $\dot{G} \neq 0$ identically for $t > t^*$.

Qualitative properties of the solution

The velocity of the flow is given by

$$v(r,t) = \frac{4}{\zeta} \int_r^1 [Y(u,t)]_+ \, \mathrm{d}u$$

Thus we can easily predict whether the system will come to rest

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- If G is bounded, the velocity field tends to zero as $t \to +\infty$ uniformly in $(0,1)$
- In the following class of decreasing functions

$$G(t) = \zeta - \varepsilon(1 - \exp(-\beta t))$$

for any $\varepsilon \in (0, \zeta - 2)$ we can select $\beta > 0$ in such a way that there is no finite stopping time

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  (b) the two free boundaries may stay separate for ever and the velocity field never vanishes identically

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- If $\dot{G} \geq 0$ the flow has no free boundaries. If $\dot{G} \leq 0$ and t^* exists such that

$$\int_0^{t^*} \exp[F(\vartheta)]\dot{G}(\vartheta)\,\mathrm{d}\vartheta + 2 = 0$$

then a rigid layer is formed ar $r = 1$ for $t > \bar{t}$ defined by

$$\bar{t} = \inf\{t : t > T^*, \dot{G}(t) \leq 0\}.$$

An inner rigid core is never present

Initially Newtonian CWS

Some CWS show no yield stress before stress–induced degeneration. For these we can prove that

- If $\dot{G} \geq 0$ the flow has no free boundaries. If $\dot{G} \leq 0$ and t^* exists such that

$$\int_0^{t^*} \exp[F(\vartheta)]\dot{G}(\vartheta)\,\mathrm{d}\vartheta + 2 = 0$$

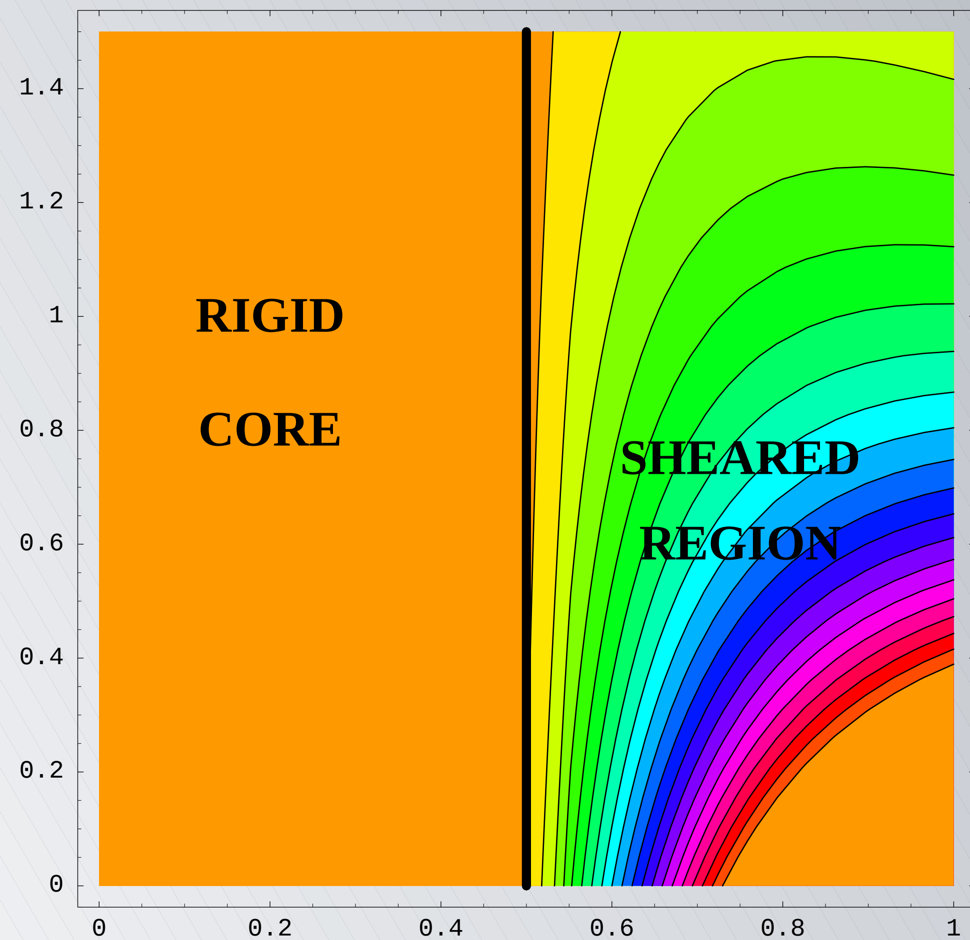
then a rigid layer is formed ar $r = 1$ for $t > \bar{t}$ defined by

$$\bar{t} = \inf\{t : t > T^*, \dot{G}(t) \leq 0\}.$$

An inner rigid core is never present

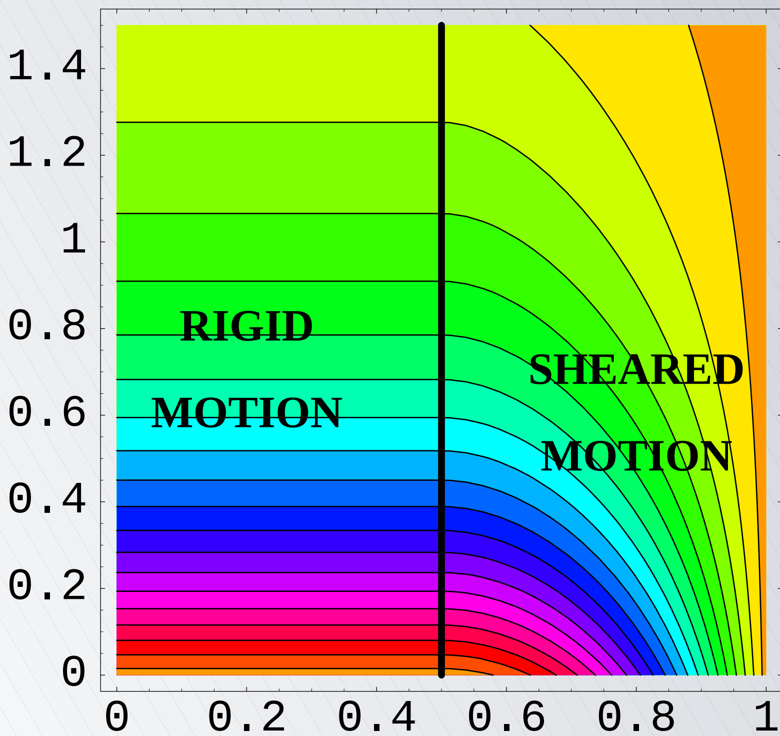
- A necessary and sufficient condition for finite time extinction of the flow is that G vanishes. The extinction time is the first zero of G

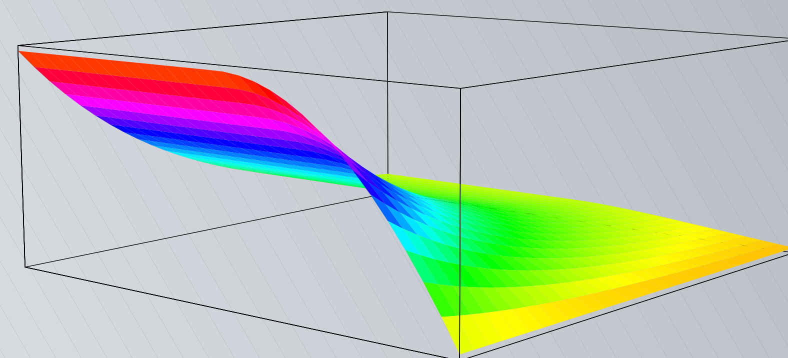
Solutions



Stationary free boundary. In this case $G(t) = \zeta$ constantly so that $s(t) = 2/\zeta$. The figure refers to $\zeta = 4$ and shows the level curves of Y. The thick line rapresents the free boundary

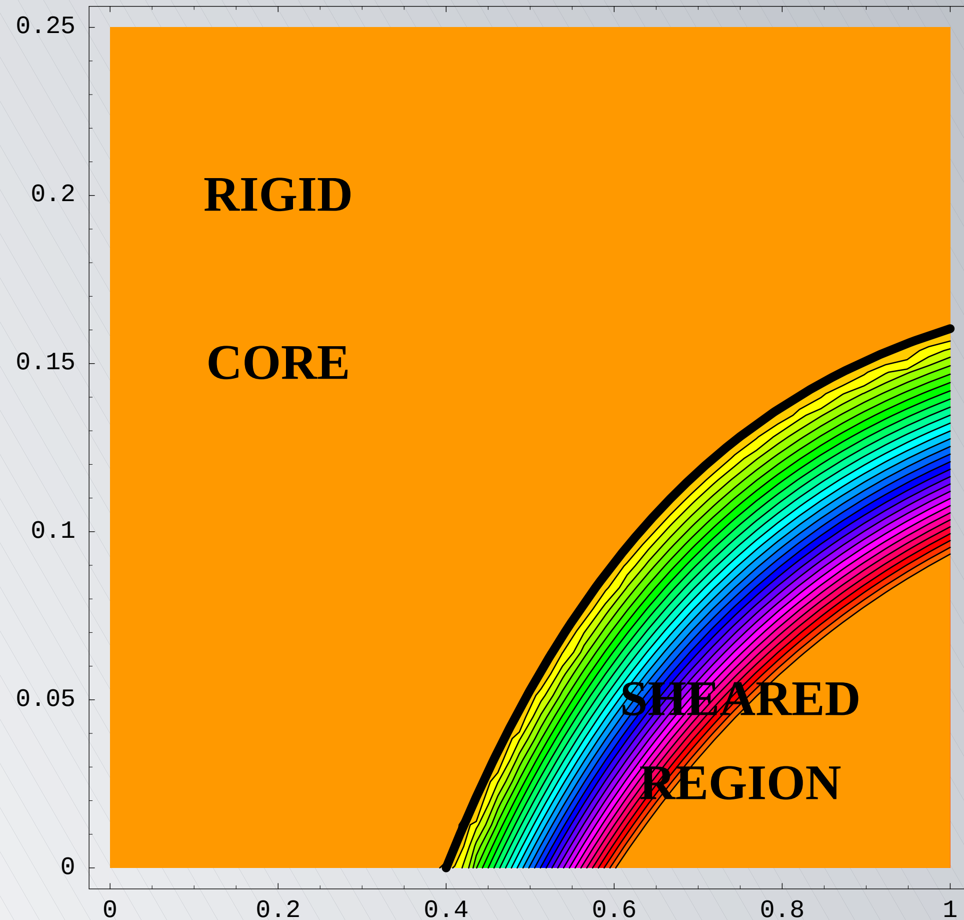
Solutions





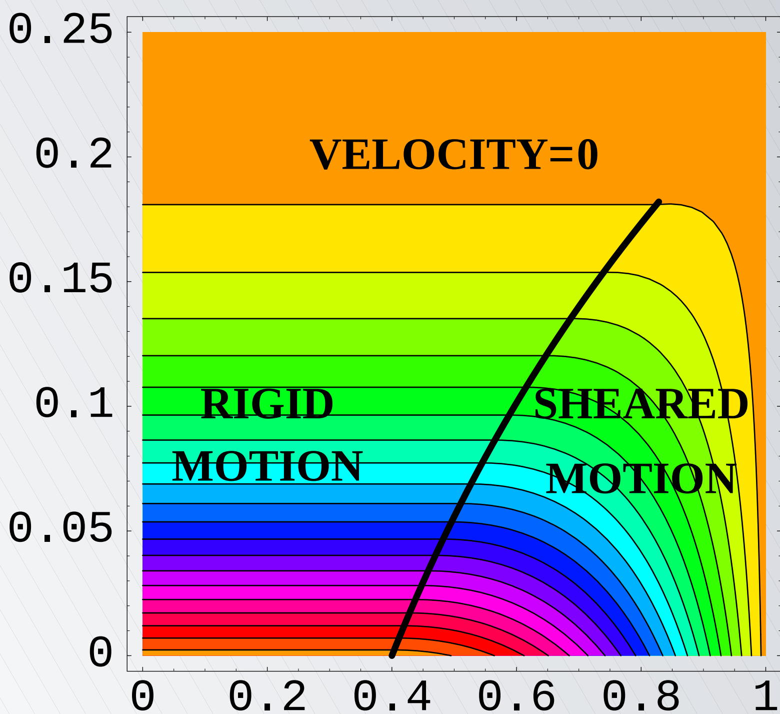
Plot of the velocity field and its level lines for the previous figure

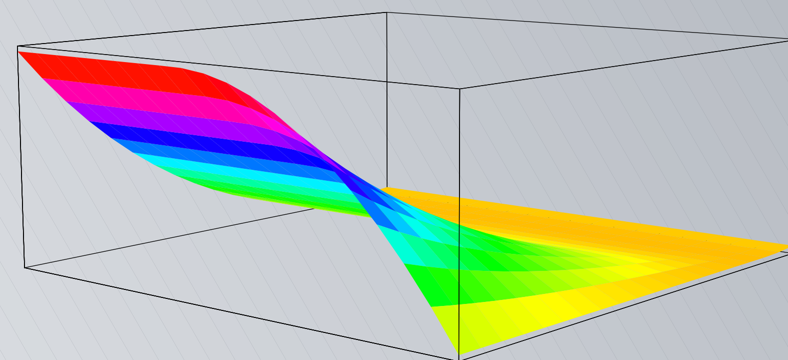
Solutions



One free boundary only, hitting the outer wall in a finite time. In this case $G(t) = 5\exp(-4t)$. The thick line rapresents the free boundary

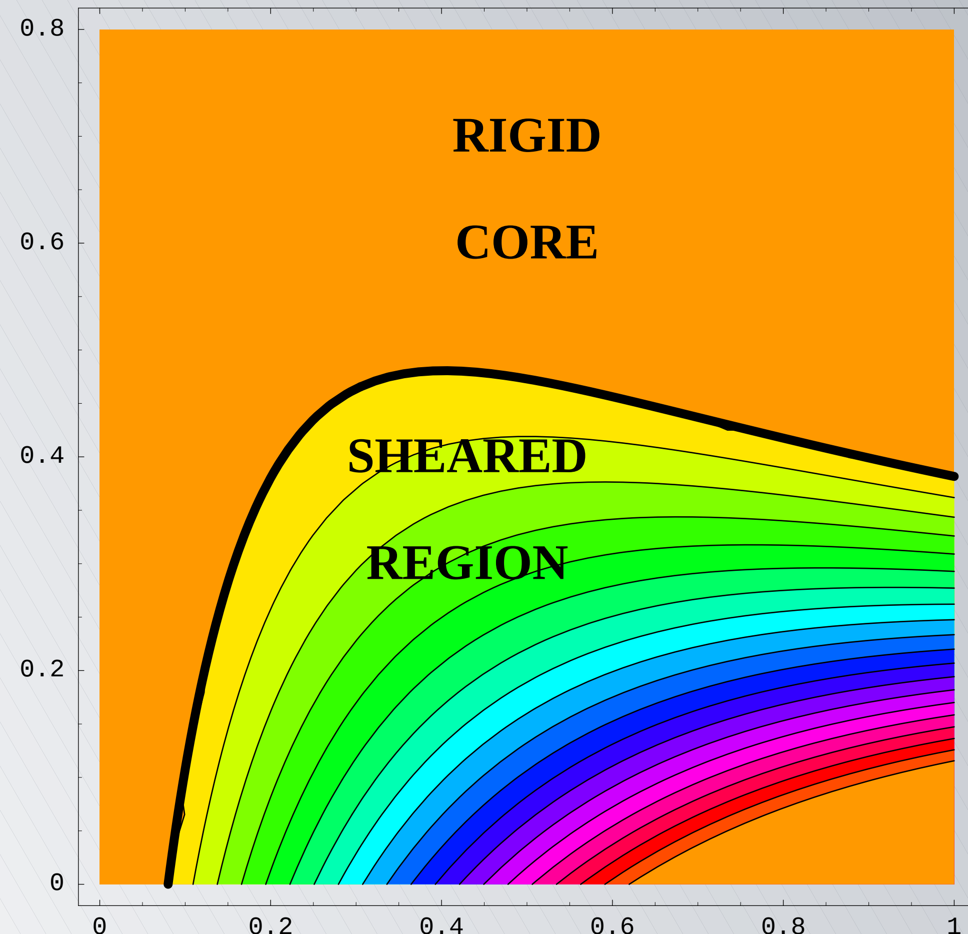
Solutions





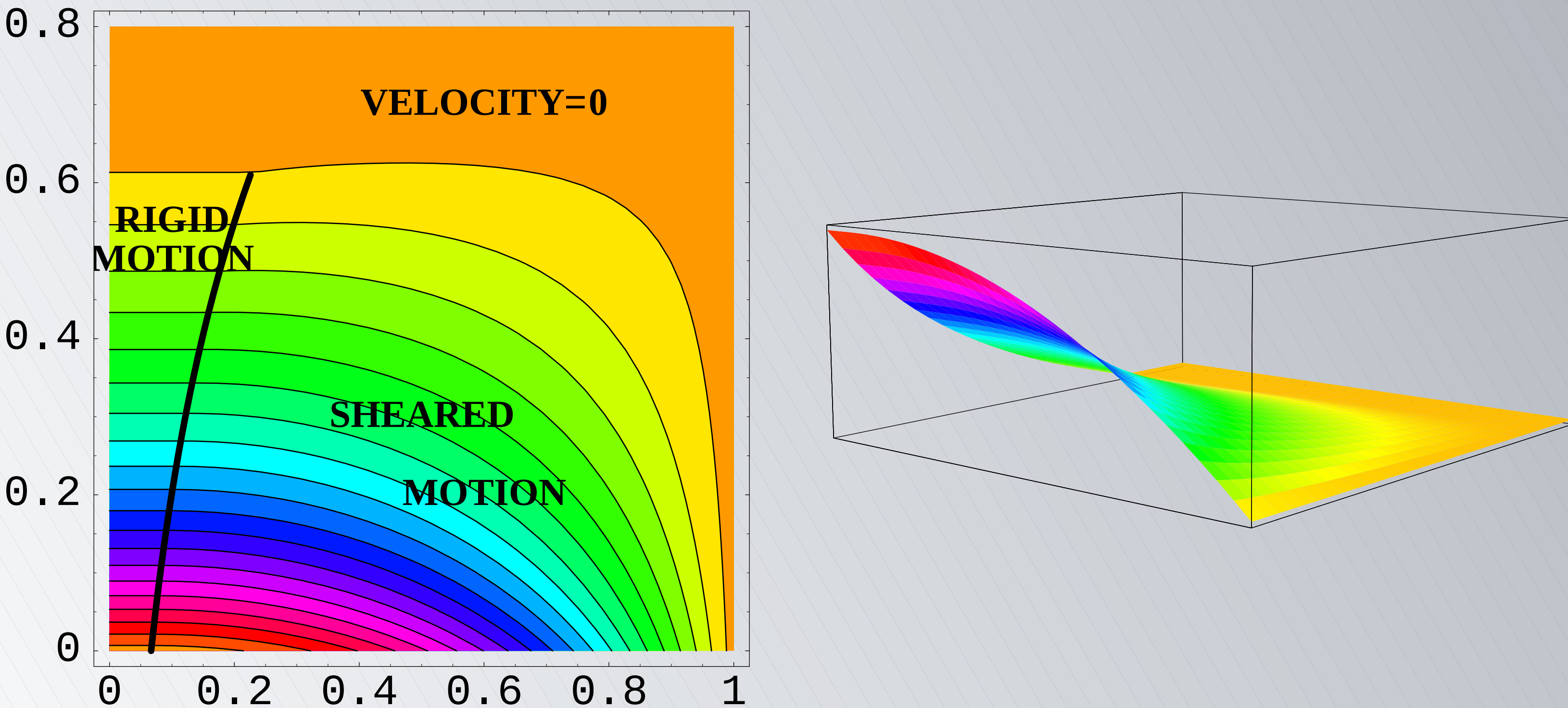
Plot of the velocity field and its level lines for the previous figure

Solutions



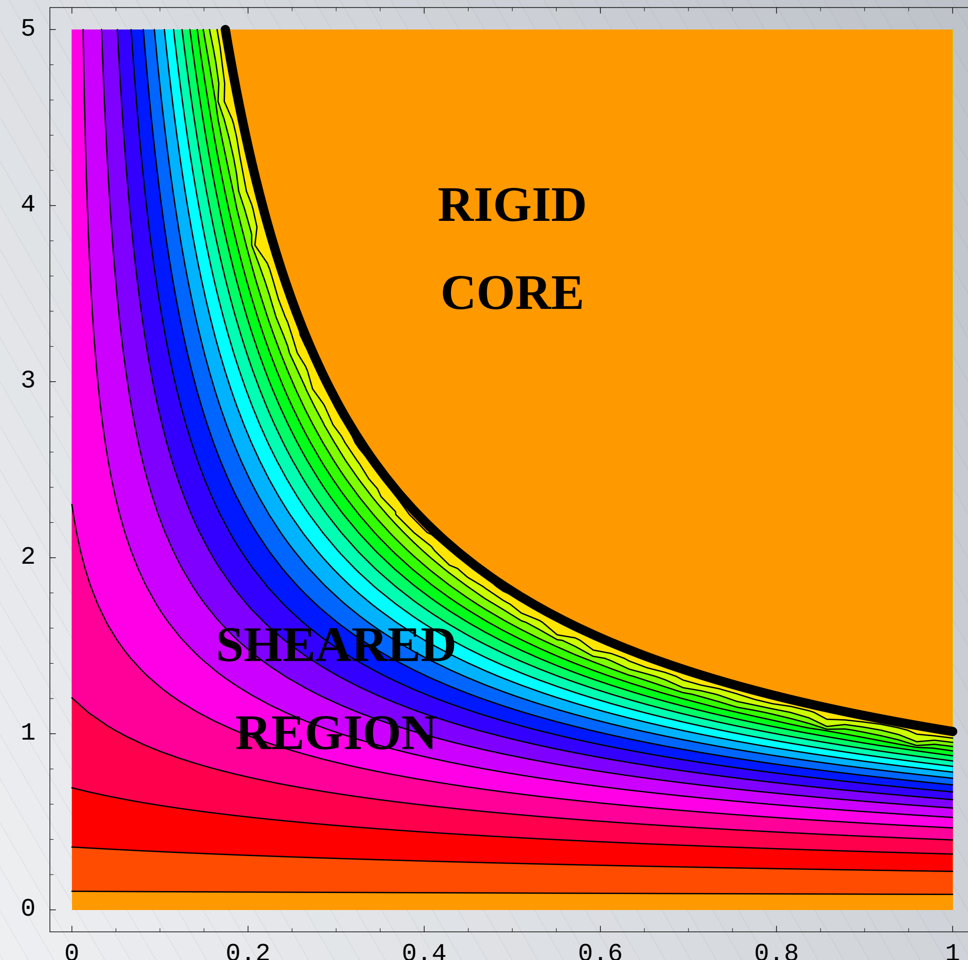
Appearance of a second free boundary. Here we take $G(t) = 30\exp(-2t)$. The two free boundaries meet with infinite speed.

Solutions



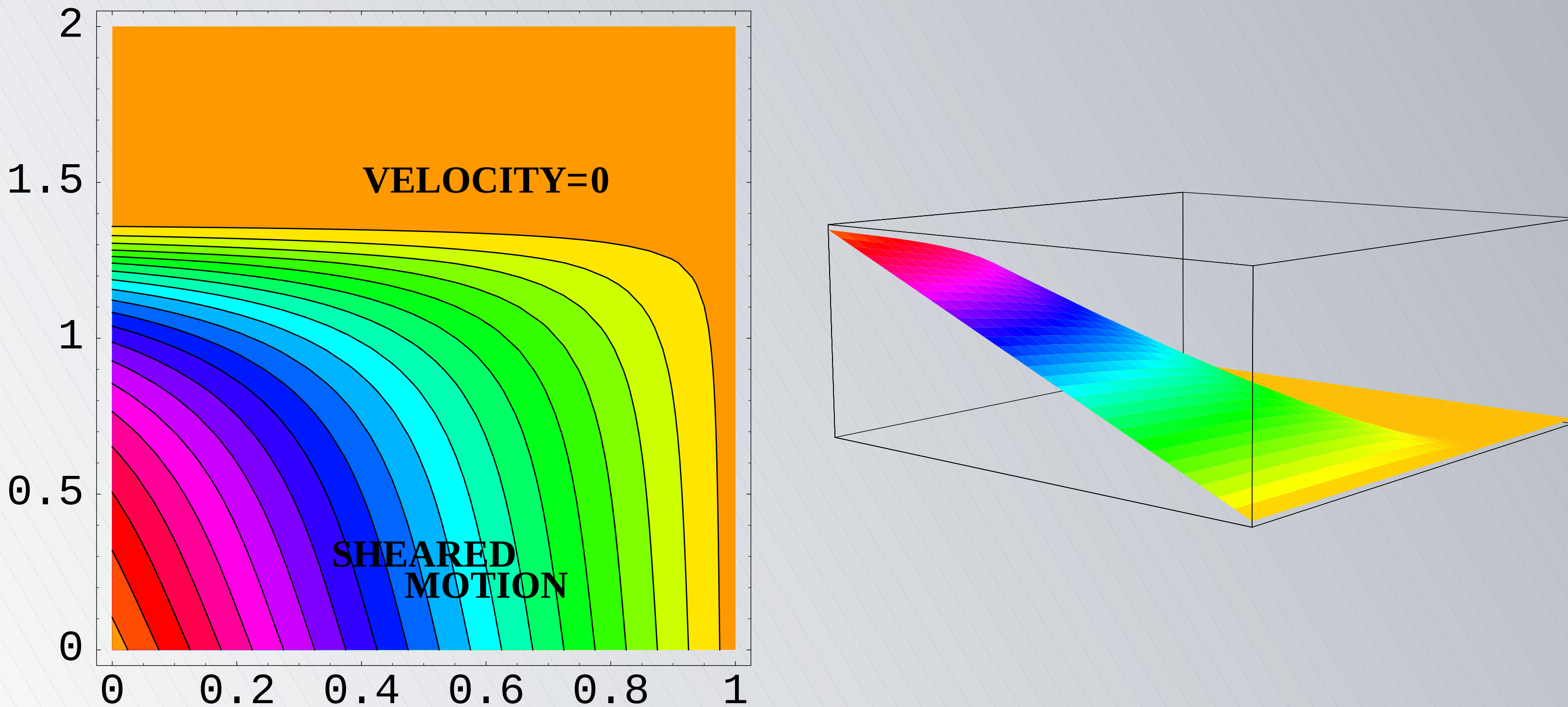
Plot of the velocity field and its level lines for the previous figure

Solutions



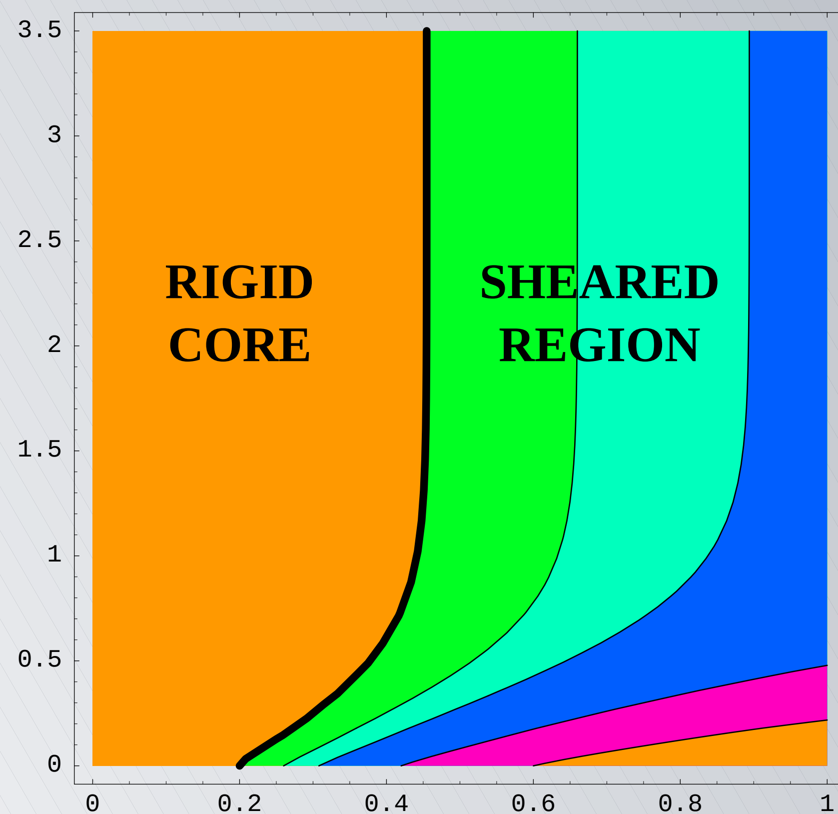
Initially Newtonian CWS. In this case $\tau_0(0) = 0$ and $G(t) = 0.5\exp(-t) + 1.5$. A rigid shell is formed at the outer boundary at time $t^* > 0$

Solutions



Plot of the velocity field and its level lines for the previous figure

Solutions



The inner core grows as $t \to +\infty$ but the free boundary never hits the outer wall

Another interesting problem

When degradation starts, how long can we pipeline a CWS until the viscosity is so high that the is a serious rirk to break the pumping system? In other words:
since a constant flow rate cannot be maintained forever, how long is the the critical interval of time between the beginning of the degradation process and the moment in which the yield stress is, say, two or three times the initial one?
This is important from the industrial point of view, since if the critical interval is very short (say few hours) safety reasons suggest to stop the system very well in advance.

Fortunately the interval can be estimated in a couple of days which makes the risk of overloading the pumping system very low.

The constant flow rate problem

Recall that the velocity field is given by

$$v(r,t) = \frac{4}{\zeta}\int_r^1 Y_+(u,t)\,\mathrm{d}u;$$

Equivalent forms of the volumetric flow rate are listed below

$$Q(t) = 2\pi\int_0^1 rv(r,t)\,\mathrm{d}r = \frac{4\pi}{\zeta}\int_0^1 r^2 Y_+(r,t)\,\mathrm{d}r = -\int_{s(t)}^1 r^2\frac{\partial v}{\partial r}(r,t)\,\mathrm{d}r$$

To keep $Q(t) = Q(0)$ we need a growing pressure gradient

Theorem *Let $\{v,\tau,\tau_0,s\}$ be a smooth solution of the free boundary value problem corresponding to a smooth pressure gradient $-G(t)$ over the interval $[0,T)$ and such that $Q(t) = Q(0)$ for all $t\in[0,T)$. Then G is strictly increasing on $[0,T)$. Moreover if $\lim_{t\to T} G(t) < +\infty$ the T cannot be equal to $+\infty$.*

Generalized Buckingham equation

It is not difficult to see that for a Bingham fluid with constant rheological parameters (τ_0 and η_B) the request $Q(t) = Q(0)$ implies that G must solve a forth degree algebraic equation.

This is no longer true in our case: when τ_0 and η_B depend on time, condition $Q(t) = Q(0)$ implies that G must solve a complicated nonlinear integral equation that we called Generalized Buckingham equation.

Let's see how this equations looks like:

$$\begin{cases} G & = \widetilde{\mathscr{M}}\left(\int_0^t [\mathscr{N}_1(G,\widetilde{G}) + \mathscr{N}_2(G)]\, du\right) \\ \widetilde{G} \circ G & = \text{Id} \;\; (\text{Id is the identity map in } R) \end{cases}$$

where $\widetilde{\mathscr{M}}$ is the inverse function of $\mathscr{M}(x) = \dfrac{\zeta}{16}\left[\log\left(\dfrac{x}{\zeta}\right) + 4(x^{-4} - \zeta^{-4})\right]$ for $x \in [\zeta, \infty)$

(recall that $\zeta > 2$) and

Generalized Buckingham equation

$$\mathcal{N}_1[G,\widetilde{G}](u) = \frac{1}{2}\int_{s(u)}^{s_o} r^4 e^{-rF(u-t_o(r))}\{G(u)e^{rF(u)}$$

$$- \frac{2}{r}e^{rF(t_o(r))} - \frac{2}{\zeta}\int_{t_o(r)}^{u} rG^2(s)e^{rF(s)}\,\mathrm{d}s\}\,\mathrm{d}r,$$

$$\mathcal{N}_2[G](u) = G(u)\frac{(1-s_o^5)}{10} - \int_{s_0}^{1} r^3 e^{-rF(u)}\,\mathrm{d}r$$

$$- \frac{1}{\zeta}\int_0^u\int_{s_o}^{1} r^5 G^2(s)e^{-r[F(u)-F(s)]}\,\mathrm{d}r\,\mathrm{d}s.$$

Solving the Buckingham equation

Let $C^k([0,T])$ denote the Banach space of functions continuous up to their k-th derivative over $[0,T]$, for a fixed finite $T>0$, equipped with the usual norm $||.||_k$. For given $M_1, M_2 > 0$ and $K > 0$ such that

- $K < g(\zeta)$ where

$$g(\zeta) = \frac{4\,(-2+\zeta)\,\left(4+4\,\zeta+3\,\zeta^2+2\,\zeta^3\right)}{5\,(2+\zeta)\,(4+\zeta^2)} > 0,$$

- $\zeta < M_1$,

It is easy to check that $\dot{G}(0) = g(\zeta)$ for all $\zeta > 2$. We define

$$\mathscr{X} = \{ f \in C^1([0,T]) /$$
$$||f||_0 \leq M_1, ||\dot{f}||_0 \leq M_2, \dot{f} \geq K, f(0) = \zeta, \dot{f}(0) = g(\zeta)\}$$

It is evident that $\mathscr{X}$ is a closed convex subset of $C^1([0,T])$.

Main result

Theorem *For a sufficiently small* $T(M_1, M_2, K, \zeta)$*, there exists one and only one solution* $G(t)$ *of the generalized Buckingham equation in* $\mathscr{X}$*.*

The proof is based on a fixed point argument applied to $\mathscr{F}$: we first prove that $\mathscr{F}$ maps $\mathscr{X}$ into itself and then that $\mathscr{F}$ is a contraction mapping provided that T is sufficiently small.

Lower estimate of the critical time

A numerical algorithm to calculate the solution $G(t)$ predicted by Theorem 2 is not easy, due mainly to the complicated structure of the term $\mathscr{N}_1$ which involves $\widetilde{G}$ in a nasty way. Here we confine ourselves to determine a *supersolution* $G^\star(t)$, that is a function which bounds from above the unique solution of the original integral equation. This supersolution does not exists for all times either but the upper bound $T^\star$ of its interval of existence gives an estimate *from below* of the analogous value of T relatively to the true solution $G(t)$. We shall find out that the dimensional value of $T^\star$ is of order $2\ days$ and this is greatly significant from a physical point of view, particularly, when compared with some field data. Indeed in a commercial pipeline, the mean velocity is usually of order $1\ m/sec$; therefore the fluid can be pumped for more than $150\ Km$ before reaching a critical value of the pumping apparatus.

Lower estimate of the critical time

First of all notice that

$$|\mathcal{N}_1+\mathcal{N}_2| \le (1/10)G(t)[1+s_o^5(e^{s_oF(t)}-1)] := \mathcal{H}[G(t),F(t)].$$

Moreover we have $\widetilde{\mathcal{M}} \le \widetilde{\mathcal{M}}^\dagger$, where we defined $\mathcal{M}^\dagger = \frac{\zeta}{16}[\log(\frac{x}{\zeta}) - \frac{4}{\zeta^4}]$.

Consider now the equation

$$G^\star = \widetilde{\mathcal{M}}^\dagger \left(\int_0^t \mathcal{H}[G^\star(u),F^\star(u)]\,\mathrm{d}u \right),$$

where $F^\star(t) = (2/\zeta)\int_0^t G^\star(u)\,\mathrm{d}u$. Since $\widetilde{\mathcal{M}}^\dagger(x) = \zeta\exp((4/\zeta^4)+(16x/\zeta))$,
we immediately get that the functions $G^\star, F^\star$ obey the following initial value problem

$$\begin{cases} \dot{G}^\star = 8s_oG^\star\mathcal{H}(G^\star,F^\star), \\ \dot{F}^\star = s_oG^\star, \\ G^\star(0) = \zeta\exp(\frac{4}{\zeta^4}), \\ F^\star(0) = 0. \end{cases}$$

Numerics

We solved the differential system numerically for various values of $s_o = \frac{2}{\zeta}$. The table below shows some values of the critical time of existence of $G^\star$. Numerical calculations show clearly that the critical time of existence $T^\star$ of $G^\star$ increases with ζ; there is also a robust evidence that a finite limit $\hat{T} = \lim_{\zeta \to +\infty} T^\star(\zeta)$ exists. The asymptotic value of $\hat{T}$ appears to be $\approx .625$.

ζ	$T^\star$
3	.531998
4	.600767
5	.618233
6	.622724
8	.624565
10	.624865
20	.62499
50	.624992

Why $\hat{T} \approx .625$?

The asymptotic value of T is consistent with the limit case in which the fluid is initially Newtonian, that is the rigid core is absent at $t = 0$ (i.e. $s_o = 2/\zeta = 0$) but may develop later at some $t > 0$. Indeed the expression of $Y(r,t)$ is in this case

$$Y(r,t) = \exp[-rF_N(t)]\frac{r}{2}\left\{\int_0^t \exp[rF_N(\theta)]\dot{G}(\theta)\,\mathrm{d}\theta + 2\right\}.$$

Hence $Y > 0$ everywhere if $\dot{G} \geq 0$. Thus instead of full Buckingham equation we have now the following

$$G(t) = 2 + \frac{4}{5}\int_0^t G^2(s)\,\mathrm{d}s$$

$$-4\int_0^t G(s)\int_0^1 r^5\left\{\int_0^s G^2(u)e^{r[F_N(u)-F_N(s)]}\,\mathrm{d}u\right\}\,\mathrm{d}r\,\mathrm{d}s$$

where now

$$F_N(t) = \int_0^t G(\xi)d\xi.$$

A remarkable result

Sincd in the *initially Newtonian case* the generalize Buckingham equation does not involve $\widetilde{G}$ (the inverse of G), a supersolution as well its critical time can be calculated *exactly*: indeed the r.h.s. of the integral equation can be estimated by simply canceling out the second integral term. Thus a supersolution can be obtained from the equation

$$G^\star = 2 + (4/5)\int_0^t (G^\star(u))^2 \, \mathrm{d}u$$

whose solution is $G^\star(t) = 10(5-8t)^{-1}$ with critical time $T^\star = 5/8 = .625$
All this considered, it is not surprising that $\hat{T}$ might tend to $5/8$ as s_o tends to zero (i.e. as ζ tends to infinity), for all supersolutions. Indeed we can prove the following
Theorem *Let $T^\star(\zeta)$ be the critical time of existence of a supersolution $G^\star(t)$. Then*

$$\lim_{\zeta\to\infty} T^\star(\zeta) = 5/8$$

and

$$\frac{\mathrm{d}T^\star}{\mathrm{d}\zeta} > 0.$$

Conclusions

The adimensional values of $T^\star$ shown before can be converted in dimensional units. Typical reference field values are

$$R \approx 25\,cm, \qquad V_o \approx 80\,cm \times s^{-1}, \qquad \alpha \approx 10^{-6} \qquad \widetilde{\tau}_o \approx 5\,Pa, \qquad ,\Gamma_o \approx .2\,Pa \times cm^{-1}$$

Using these values we get

$G^\star(0)\ (Bar \times Km^{-1})$	$T^\star\ (Hour)$
.5	37.6
.6	44.3
.8	50
.9	51
1	51.5
1.2	51.9
1.6	52.04
2	52.07
4	52.08
10	52.08

Safety operation regime

To make an example let us consider $1\,Bar \times Km^{-1}$ to be an industrially significant value of $G(0)$. According to the model, once degradation has begun, a flow velocity of $.8\,m \times sec^{-1}$ can be maintained for other $31\,hours$ (or, equivalently, for $\approx 90\,Km$ of pipeline) by increasing the pressure gradient up to a value of $2.5\,Bar \times Km^{-1}$ which is still a reasonable value of $G(t)$. The corresponding flow rate is $Q_o = \pi R^2 V_o \approx 565\,m^3 h^{-1}$. The values we found are definitely those of industrial interest. Indeed field data provided by Snamprogetti and based on the operating pipeline Belovo-Novosibirsk, indicate a flow rate $Q \approx 565\,m^3/h$ when the driving pressure gradient G is about $1\,Bar/km$ and a Russian CWS with $\approx 62\%$ of solid concentration is used.