Minicourse in Industrial Mathematics

Segundo Enquentro Italo–Argentino

Second lecture: Dynamics of sediments during a CWS pipelining

F. Rosso

`www.math.unifi.it/˜ rosso/Baires/minicorso.pdf`

`fabio.rosso@math.unifi.it`

Math. Dept. "U.Dini" Univ. of Firenze – 13th December 2002

Hydrotransport: an old idea

Besides intuitive, the idea of transporting solids using a carrier fluid is rather old. Indeed according with Apollodorus and Pausanias (1^{st} century B.C.) this transportation technique, nowadays known with the name of *Hydrotransport*, was first utilized by Herakles, who in one day cleaned up the thirty years' accumulated filth left by thousands of cattle in King Augeas' stables by diverting two rivers to form an open-channel hydrotransport system. Despite of the mythological nature of this tale, known as the fifth Herakles' task, in principle any kind of solid can be moved from a place to another hundreds of miles away using a liquid as transportation tool.

Hydrotransport today

The modern stages of slurry pipeline technology dates back, more or less, to forty years ago. A “slurry”essentially is a suspension of solid particles in a carrier fluid; the interest within these suspensions is that, by using an appropriate technology, they can be pipelined very far away from their production site. Generally the required technology is rather specialized depending on the chemical and physical nature of the suspended particles. The underlying idea is that hydrotransport may be, in many cases, an attractive alternative to other modes of transport (tracks, ships and so on). It also has several advantages like a moderate environmental impact, a relatively little infrastructure work needed and possible low operation and maintenance costs.

Turbulent or laminar regime?

For a long time, especially at the early stage of the development of this technology, it was generally thought that the only operational regime to prevent particle settling was the turbulent one; indeed the primary duty when designing a slurry pipeline is to ensure that it will not block because of sediments accumulation on pipe lower wall. However also the cost of maintaining a turbulent regime has to be considered. For this reason recent studies on slurries are addressed towards the possibility to control settling remaining within the laminar regime (which requires less pumping power and is therefore less expensive). If the sediment build–up process could be modeled accurately, then designers would be able to predict the conditions that lead to blockage, and thereby design systems so that the possibility of a blockage is avoided.

Pipelining a concentrated CWS

Recall my first lecture: we consider a CWS and this belongs to the category of *concentrated* slurries since consists of about 70% (by weight) of ground coal such that the size distribution has two peaks around $10\ \mu m$ and $100\ \mu m$. The remaining 30% is water with a small percentage (0.5%) of chemical additive needed to fluidize the suspension. This fluid is perfectly stable *at rest* even after years, i.e. particle concentration remains everywhere constant in time. This stability is completely to be ascribed to the action of the chemical agent: indeed additive molecules, being highly polar, coat the coal particles with positive charges so that mutual repulsive forces prevent natural sedimentation.

Constancy of rheological parameters

The rheological behaviour of concentrated CWS at low shear rates can be reasonably described by the Bingham model, that is (in laminar flows with simple geometry)

$$(\tau - \tau_0)_+ = \eta_B \dot{\gamma}, \tag{1}$$

where τ, τ_0, η_B, $\dot{\gamma}$ denote the shear stress, the yield stress, the plastic viscosity and the shear rate respectively, and $(\bullet)_+$ means the positive part of $(\bullet)$. Of course more complicated models can be used but (1) is sufficient for our purposes.

Recall that shear induces degradation of rheological properties of a CWS; however the time scale of this phenomenon is many orders of magnitude larger than that of sedimentation, so that if we focus on the latter problem, the time dependency of rheological parameters can be neglected. Therefore we can consider a CWS as a time-independent Bingham fluid.

Sedimentation of impurities

The solid fraction does not generally consist only of *pure* coal and even after a suitable treatment (*beneficiation*), ground coal from the mill hardly contains less than about 6% of other micronized minerals and steel residues due to beneficiation itself.

Impurities have generally a size comparable with or higher than that of the top size of coal particles; the inner structure of a CWS provided by the residual adhesion forces among the coal particles (which is responsible for the yield stress) is sufficient to prevent settling at rest. This is no longer true when a CWS experiences a shear rate, as if it is stirred in a viscometer of pumped in a pipeline. Indeed while the tendency to settle of the pure coal particles is prevented by the action of the chemical stabilizer, this is not true for impurities which, having a different chemical structure, do not react with the additive.

Necessity of a mathematical model

Since the settling of impurities cannot be controlled, a sedimentation bed will start growing at the bottom of the pipe when the CWS is pumped at long distances. In the pipe flow the reduction of the hydraulic diameter due to the sediment build–up leads either to increase the pressure gradient (assuming that we can) in order to maintain constant the discharge or to decrease the latter if we can not increase the former. Both cases are either unsafe or economically disadvantgeous. Therefore what is needed is a model to predict the evolution of the growing bed in order to choose the optimal discharge and to plan the periodical (unavoidable) shut-down of the pumps and cleaning of the first portion of the pipeline (the only one interested to the settling phenomenon).

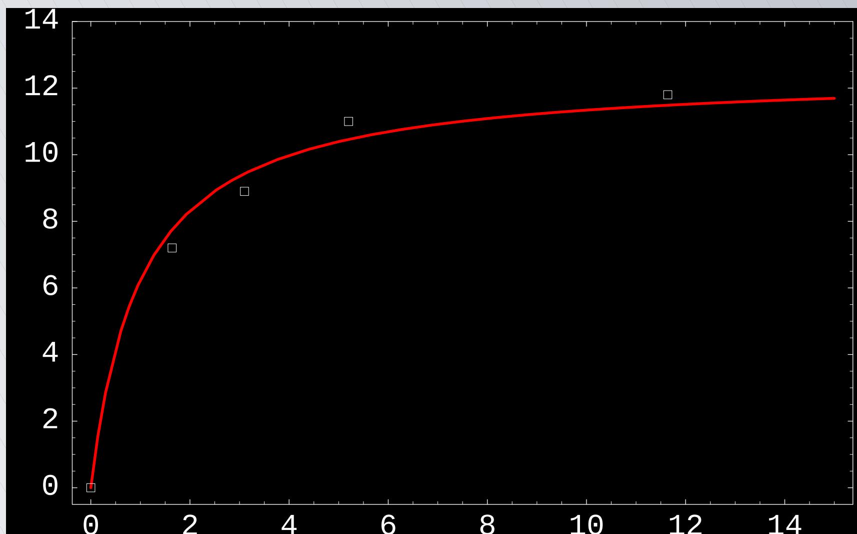
Sedimentation velocity

The classical Stokes law is doesn't hold any longer and is replaced by $v_s(\dot{\gamma}, r) = \alpha(\dot{\gamma}) r^2$

where r is the particle "virtual radius"

Sedimentation velocity

The classical Stokes law is doesn't hold any longer and is replaced by $v_s(\dot{\gamma}, r) = \alpha(\dot{\gamma}) r^2$ where r is the particle "virtual radius"



Behaviour of the sedimentation coefficient α versus shear in a Bingham fluid. The proportionality coefficient of the "square radius"–law depends on the shear rate. Note that $v_0(0) = 0$!

Mathematical model

If $\frac{\mathrm{d}p}{\mathrm{d}x}$ denotes the constant pressure gradient (< 0) and $V_x(r)$ the velocity profile of the main flow, $\dot{\gamma} = \dot{\gamma}(r) = \left|\frac{\mathrm{d}V_x}{\mathrm{d}r}\right|$ is the shear rate; since we modelled a CWS as a Bingham fluid, we still assume $(\tau - \tau_0)_+ = \eta_B \dot{\gamma}$. By coupling the Navier-Stokes equations with boundary conditions, one gets

$$V_x(r) = \begin{cases} -\dfrac{R^2}{4\eta_B}\dfrac{\mathrm{d}p}{\mathrm{d}x}\left(1 - \dfrac{r^2}{R^2}\right) - \dfrac{\tau_0 R}{\eta_B}\left(1 - \dfrac{r}{R}\right), & \text{for} \quad r \geq \hat{R}, \\ -\dfrac{1}{4\eta_B}\dfrac{\mathrm{d}p}{\mathrm{d}x}(R - \hat{R}), & \text{for} \quad r \leq \hat{R}, \end{cases}$$

where $r = \hat{R}$ is the interface bounding the rigid core.

Mathematical model

The momentum balance equation implies $\tau = \frac{r}{2}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|$, so that

$$\hat{R} = \frac{2\tau_0}{\left|\mathrm{d}p/\,\mathrm{d}x\right|};$$

moreover

$$\dot{\gamma}(r) = \frac{1}{2\eta_B}\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right|(r-\hat{R})_+.$$

A key assumption of the present model is that the sediment thickness h is sufficiently small compared to the pipe radius R so that the flow geometry will never be significantly affected by the sedimentation bed growing at the bottom of the pipe. Bearing in mind that for values of h close to 2% of the pipe diameter a precautional shut down and cleaning operation is mandatory, the above assumption does not sound as a limitation.

Where particles falls?

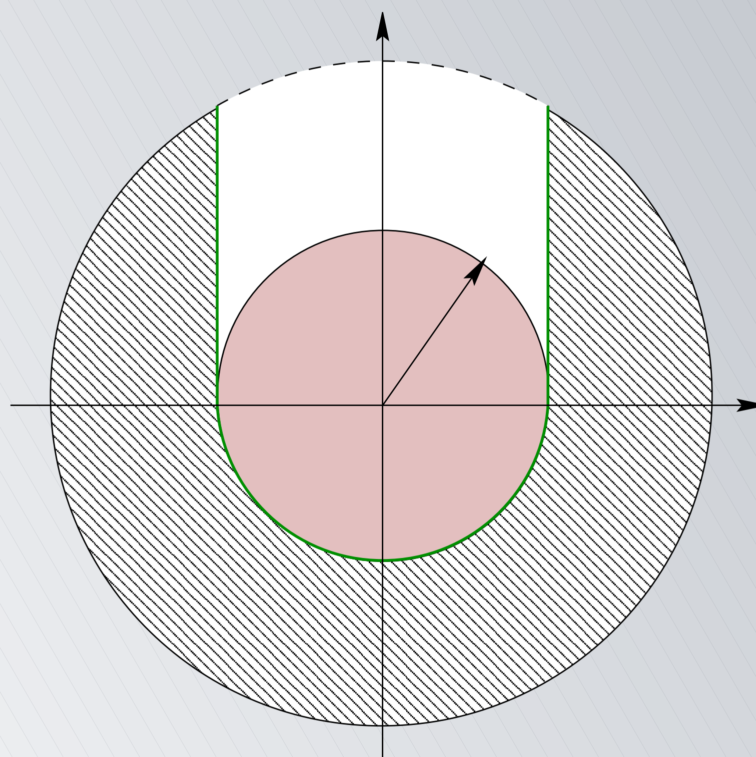
The trajectory of a particle $P = P(\delta, \rho_s, [0, y_0, z_0])$ of radius δ, density ρ_s entering the pipe at the initial position $(0, y_0, z_0)$ is fully described by the system

$$\begin{cases} \dot{x} = & V_x(r), \\ \dot{z} = & -v_s(\delta, r), \\ r = & \sqrt{z^2 + y_0^2}. \end{cases}$$

We *assume* the settling velocity v_s to be $v_s(\delta, r) = \alpha(\dot{\gamma}(r))\delta^2$. where the function $\alpha(\dot{\gamma})$ is determined experimentally.

Contributing regions

It is quite evident from the above system that particle motion remains confined in the vertical plane $y = y_0$. Moreover experiments definitely show that $\alpha(0) = 0$ and $\dot{\alpha} \geq 0$. As a consequence *only particles incoming the pipe through the gray–shaded region shown in figure do contribute to the growing sediment*.



Transverse cross–section of the pipe with contributing and noncontributing regions to the

Settling particle tajectories

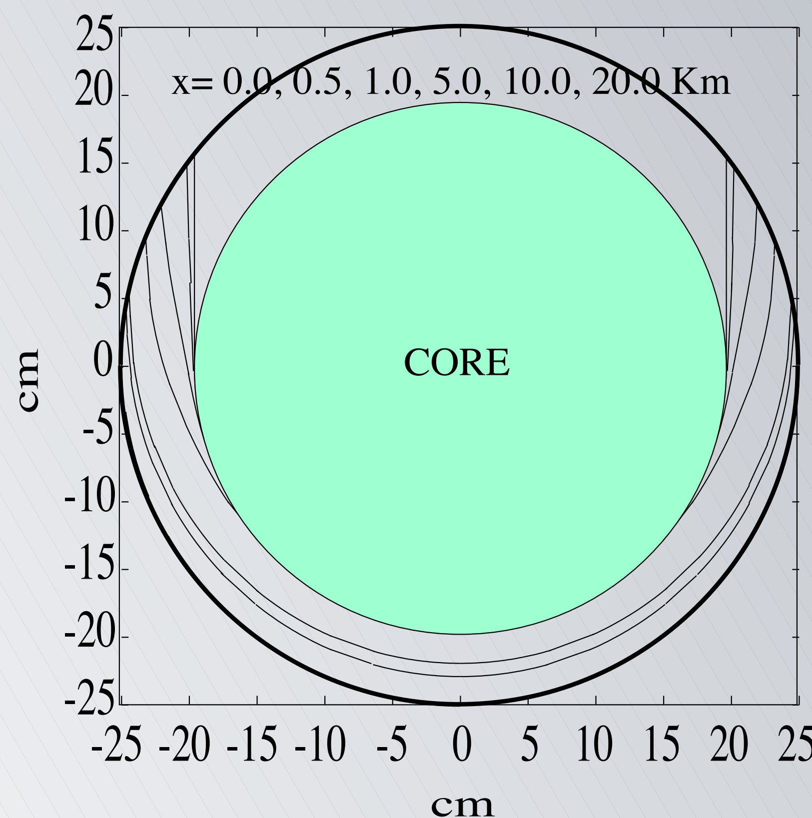
Any particle P incoming through the gray-shaded region of the initial cross-section $\{x=0\}$ ends its trajectory at the point $(x^\star, y_0, z^\star)$ defined via

$$\begin{cases} z^\star = & -\sqrt{R^2 - y_0^2}, \\ x^\star = & \displaystyle\int_z^{z_0} \frac{V_x(\tilde{r})}{v_s(\tilde{r},\delta)}\, d\tilde{z}, \\ & \text{with } \tilde{r} = \sqrt{\tilde{z}^2 + y_0^2}. \end{cases}$$

From the above equation it can be easily proved that, for any fixed $x^\star$, the locus $\Gamma(x^\star, \delta)$ of points on $\{x=0\}$ formed by those particles with radius δ, ending their trajectories on the pipe wall at $x = x^\star$, is actually *a graph* which we denote by

$$z_0 = \mathscr{Z}(y_0; x^\star, \delta).$$

U–shaped curves



Some $\Gamma(x^\star,\delta)$ *graphs for* $Q = 250\ m^3/h$, $\delta = 0.0067\ cm$.*The slopes of the U–shaped curves increase with* $x^\star$. *The steepest U–shaped curve corresponds to* $x^\star = +\infty$

Evaluating the settling rate

The settling rate per unit length at distance x from the initial cross-section due to particle with radii between δ and $\delta + d\delta$ is given by

$$S(x;\delta)d\delta = \frac{4}{3}\pi\delta^3\rho_s N(\delta)d\delta \int_{-y_M}^{+y_M} v_s(\sqrt{y_0^2 + (\mathscr{Z})^2}, \delta)dy_0,$$

where $\mathscr{Z} = \mathscr{Z}(y_0; x, \delta)$, $N(\delta)d\delta$ is the number of settling particles with radii between δ and $\delta + d\delta$ per unit volume of mixture ($\mathscr{P} + CWS$), and the endpoints $\pm y_M$ of $\Gamma(x, \delta)$ are implicitly defined by

$$y_M^2(x,\delta) + \mathscr{Z}^2(x^\star, y_M(x,\delta)) = R^2.$$

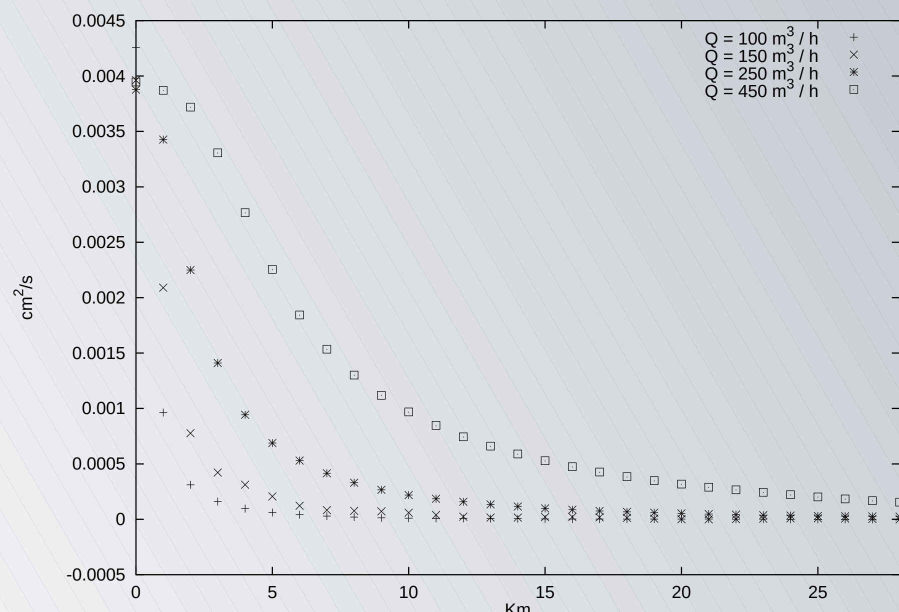
Since N vanishes outside $[\delta_{min}, \delta_{max}]$, the overall settling rate (per unit length of the pipeline) will be given by

$$S_T(x) = \int_{\delta_m}^{\delta_M} S(x;\delta)d\delta.$$

The settling rate S_T

Analysis shows that S_T is a rapidly decreasing function of x and is practically zero if x is sufficiently large.

In our simulations with a known population of sand particles ($\delta_{min} = 0.0035\, cm$, $\delta_{max} = 0.0113\, cm$, $\rho_s = 2.67\, g/cm^3$) and with a CWS with known rheological characteristic parameters ($\tau_0 = 8.89\, P$, $\eta_B = 0.16\, Ps$) we found $S_T \simeq 0$ at $x = 10\, Km$ for a flow rate Q of $\simeq 100\, m^3 h^{-1}$ and at $x = 60\, Km$ for a flow rate Q of $\simeq 450\, m^3 h^{-1}$ (pipe radius $R = 25\, cm$).



Function $S_T(x)$ at various flow rates.

Which cross–sectional bed profile?

Function S_T provides the source term balancing the rate of change of the cross-sectional area $a(x,t)$ of the bed.

Which cross–sectional bed profile?

Function S_T provides the source term balancing the rate of change of the cross-sectional area $a(x,t)$ of the bed.

A precise and complete description of the actual dynamics of the bed appears to be rather complex. Indeed there are infinitely many different profiles $h(\phi,x,t)$ which correspond to the same function $a(x,t)$ and there is no “natural”equation to describe the evolution of h.

Which cross–sectional bed profile?

Function S_T provides the source term balancing the rate of change of the cross-sectional area $a(x,t)$ of the bed.

A precise and complete description of the actual dynamics of the bed appears to be rather complex. Indeed there are infinitely many different profiles $h(\phi,x,t)$ which correspond to the same function $a(x,t)$ and there is no "natural"equation to describe the evolution of h. *However such level of sophistication is absolutely not needed in our case, since for the particular nature of the problem, experimental observations cannot be that accurate*. Indeed the best equipment available (a gamma-densimeter) provides only a reasonable measure of the average thickness of the bed as a function of time where the monitoring device is placed.

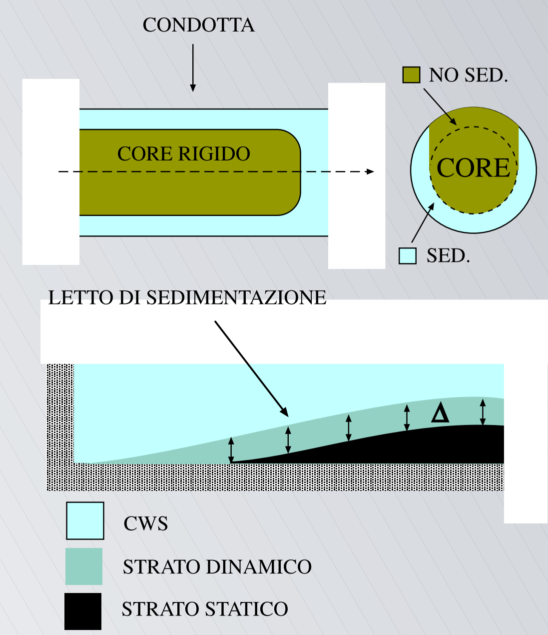
Other driving mechanisms?

The only clear experimental indication is that pure settling *cannot be the only driving mechanism in the dynamics of the bed*.

Indeed, were this the case, the bed should grow continuously, filling the pipe in a finite time (and this has never been observed)

Transport of sediment by the main flow

Pure settling (fully described by $S_T(x)$) needs to be coupled with a *transport* action, which consists in *a partial mass removal* in the horizontal direction due to the action of the main flow. The idea is that, during a first stage, the whole bed flows in the pipe; then, when the bed has reached a *critical thickness* Δ, a *static* layer begins to grow just below the dynamic one. The value of Δ depends on the main volumetric flow rate Q.



The settling and transport actions driving the dynamics of the bed; when the dynamic layer has reached the critical thickness Δ, the static sublayer begins to develop below it.

Further hypotheses

Further hypotheses

(i) the radial thickness $h(\phi,x,t)$ of the bed is proportional to its cross-sectional area $a(x,t)$ via $h(\phi,x,t) = C(\phi)a(x,t)$;

Further hypotheses

(i) the radial thickness $h(\phi,x,t)$ of the bed is proportional to its cross-sectional area $a(x,t)$ via $h(\phi,x,t) = C(\phi)a(x,t)$;

(ii) the cross-sectional area of the bed remains always small enough so that the flow geometry of the main flow is not significantly modified; in other words $a(x,t) \ll \pi R^2$.

Further hypotheses

(i) the radial thickness $h(\phi, x, t)$ of the bed is proportional to its cross-sectional area $a(x,t)$ via $h(\phi, x, t) = C(\phi)a(x,t)$;

(ii) the cross-sectional area of the bed remains always small enough so that the flow geometry of the main flow is not significantly modified; in other words $a(x,t) \ll \pi R^2$.

(iii) The cross-sectional profile is *sufficiently regular* and *physically consistent* in the sense that $C(\phi)$ must satisfy the following constraints:

1. $C(0) = C(\pi) = 0$,
2. $C'(\phi) > 0$, for all $\phi \in [0, \pi/2)$,
3. $C(\phi) = C(\pi - \phi)$, for all $\phi \in [0, \pi/2)$,
4. $C'(\phi) \leq \left[\dfrac{R}{a(x,t)} - C(\phi)\right] \cot(\phi)$, for all $\phi \in (0, \pi/2)$,

Further hypotheses

(iv) The volumetric flow rate in the x direction *due to the moving layer of the bed* is described by the function

$$q(x,t) = 2\int_0^{\frac{\pi}{2}} \tilde{q}(x,t,\phi)\, d\phi,$$

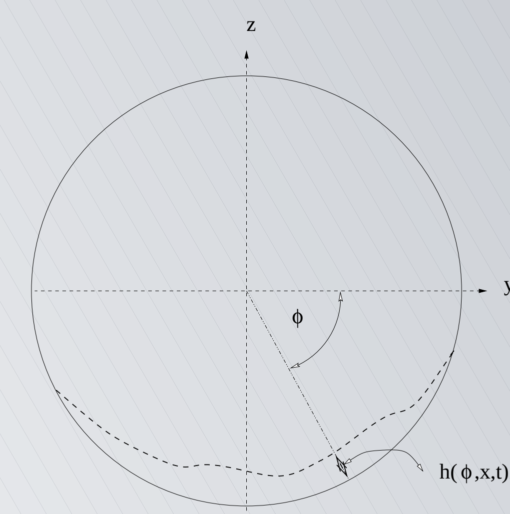
where

$$\tilde{q}(x,t,\phi) = \begin{cases} \lambda_1 R h(x,t,\phi), & \text{for } h(x,t,\phi) < \Delta, \\ \lambda_2(a(x,t)) R\Delta, & \text{for } h(x,t,\phi) \geq \Delta, \end{cases}$$

in such a way that $\tilde{q}(x,t,\phi)d\phi$ is just the volume of sediment passing through a section of width $d\phi$ per unit time consistently with the hypotheses $h \ll R$ and $\Delta \ll R$. The parameter λ_1 as well as the function $\lambda_2(a)$ have to be chosen so that $\tilde{q}(x,t,\phi)$ be continuous.

Significant class of bed profiles

Obviously $C(\phi)$ takes its maximum at $\phi = \pi/2$; therefore (4) is satisfied, for example, if $C'(\phi) \leq \left[\frac{R}{a(x,t)} - C(\pi/2)\right] \cot(\phi)$ provided that $a(x,t) < R/C(\pi/2)$. Condition ((iii)–4) can be easily interpreted: the z–coordinate of a point on the bed profile is given by $z(\phi,x,t) = -(R - C(\phi)a(x,t))\sin\phi$. Provided that $a(x,t)$ is sufficiently small, condition ((iii)i-4) is equivalent to saying that $\partial z/\partial\phi < 0$ for all $\phi \in [0,\pi/2)$.



Function $h(\phi,x,t)$ describes the radial thickness of the bed; ϕ is the azimuthal coordinate.

Assumptions on h are justified by the fact that in absence of experimental information *it is convenient to choose working hypotheses which are both simple and meaningful*.

q as a function of a

It can be proved that $q(x,t)$ is as an explicit function of $a(x,t)$: in particular we have

$$q'(a) = \begin{cases} \lambda_1, & \text{for } a < a_0, \\ \pi R\Delta\{\frac{\lambda_1}{a_0} G(\frac{2}{\pi}\widehat{\phi}(a)) & \\ \quad +\widehat{\lambda}_1(1-\frac{2}{\pi}\widehat{\phi}(a)) - (\frac{2}{\pi}\widehat{\lambda}_1(a-a_0)\widehat{\phi}'(a)\}, & \text{for } a \geq a_0, \end{cases}$$

It is not difficult to check that $q'(a) > 0$; thus $q(a)$ is invertible in $[0, q_\infty)$ being $q_\infty = \lim_{a\to\infty} q$. If $\lim_{a\to a_0^+}(a-a_0)\frac{d}{da}g^{-1}(a_0/a) = 0$, then $q'(a)$ is also continuous for all $a > 0$. For future purpose it is also useful to notice that

$$q''(a) = \begin{cases} 0, & \text{for } a < a_0, \\ \pi R\Delta[(\frac{\lambda_1}{a} - 2\widehat{\lambda}_1)\widehat{\phi}'(a) - \widehat{\lambda}_1(a-a_0)\widehat{\phi}''(a)], & \text{for } a \geq a_0. \end{cases}$$

Sedimentation dynamics in a pipe

Summarizing ze have

- Model variables: r pipe radial coordinate, x longitudinal axis of the pipe, t time, $\dot{\gamma}(r)$ (known function of τ, τ_0, η_B)

Sedimentation dynamics in a pipe

Summarizing ze have

- Simplifying assumptions: the ratio “sediment thickness/pipe radius” is rather small, rheological parameters remain constant (the degradation time scale is much greater than the pipelining time), the sediment bed is partially transported away by the main flux, the geometry of the sediment cross–section is “essentially known”, the mass flux of sediment per unit time $q(x,t)$ through a cross–section at distance x is a known function of $a(x,t)$, the area of the sediment cross–section at distance x and time t.

Sedimentation dynamics in a pipe

Summarizing ze have

- Consequences: the total sediment rate $S_T(x)$ per unit length of the pipe at distance x from the origin can be explicitely evalutated as a function of the settling particle concentration and the convective velocity of the main flux

Sedimentation dynamics in a pipe

Summarizing ze have

- Evolution equation for $a(x,t)$

$$\frac{\partial a}{\partial t} + q'(a)\frac{\partial a}{\partial x} = S_T(x)$$

with the initial–boundary conditions $a(x,0) = a(0,t) = 0$

Solutions

- best regime:
- attention regime:

Solutions

- **best regime:** if the maximum thickness of sediments remains lower than a threshold Δ (which depends on the discharge) then $q = \lambda a$ (trivial case) $\Rightarrow$ sediments never sticks to the bottom of the pipeline
- **attention regime:**

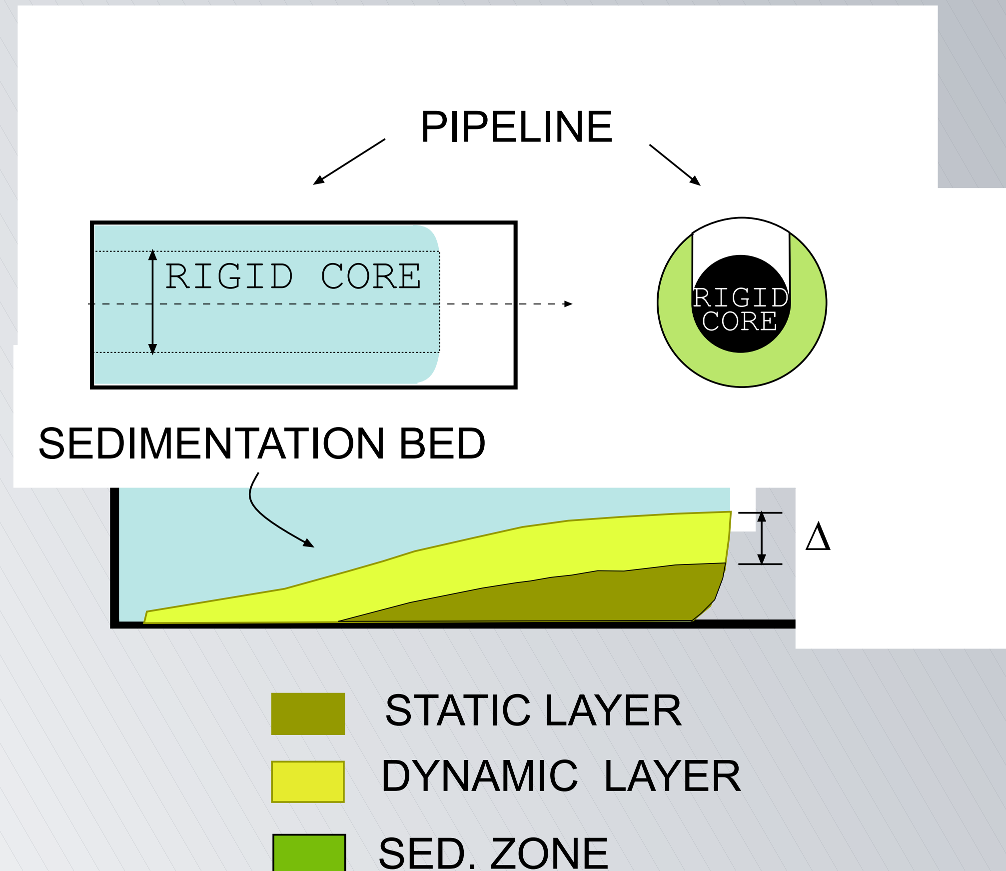
Solutions

- **best regime:** if the maximum thickness of sediments remains lower than a threshold Δ (which depends on the discharge) then $q = \lambda a$ (trivial case) $\Rightarrow$ sediments never sticks to the bottom of the pipeline
- **attention regime:** if the maximum thickness grows above Δ, then q depends nonlinearly in a complicated way by a. In this case two sediment layers form, a lower one which sticks at the bottom, and an upper one of thickness Δ which is transported away by the flux

Solutions

- **best regime:** if the maximum thickness of sediments remains lower than a threshold Δ (which depends on the discharge) then $q = \lambda a$ (trivial case) $\Rightarrow$ sediments never sticks to the bottom of the pipeline
- **attention regime:** if the maximum thickness grows above Δ, then q depends nonlinearly in a complicated way by a. In this case two sediment layers form, a lower one which sticks at the bottom, and an upper one of thickness Δ which is transported away by the flux
- In all cases of physical interest the solution exists globally

Sedimentation bed profiles



dynamics of the bed

Bed profile at various time: behind the maximum the bed remains stationary

Sedimentation bed profiles

Q (m^3/h)	100	150	250	450
Δ (cm)	0.7	1.0	1.8	3.2

Estimated values of Δ as a function of Q

Sedimentation bed profiles

Q (m^3/h)	Δ (cm)	Gap (cm)	t_{cr}(days)	x_{cr} (Km)
100	0.7	3.5	0.4	0.48
150	1.0	4.2	0.5	1.24
250	1.8	5.4	∞	≥ 100
450	3.2	7.1	∞	≥ 100

Critical time and distances needed by the static sediment to reach 2% (=1cm) of the pipe diameter

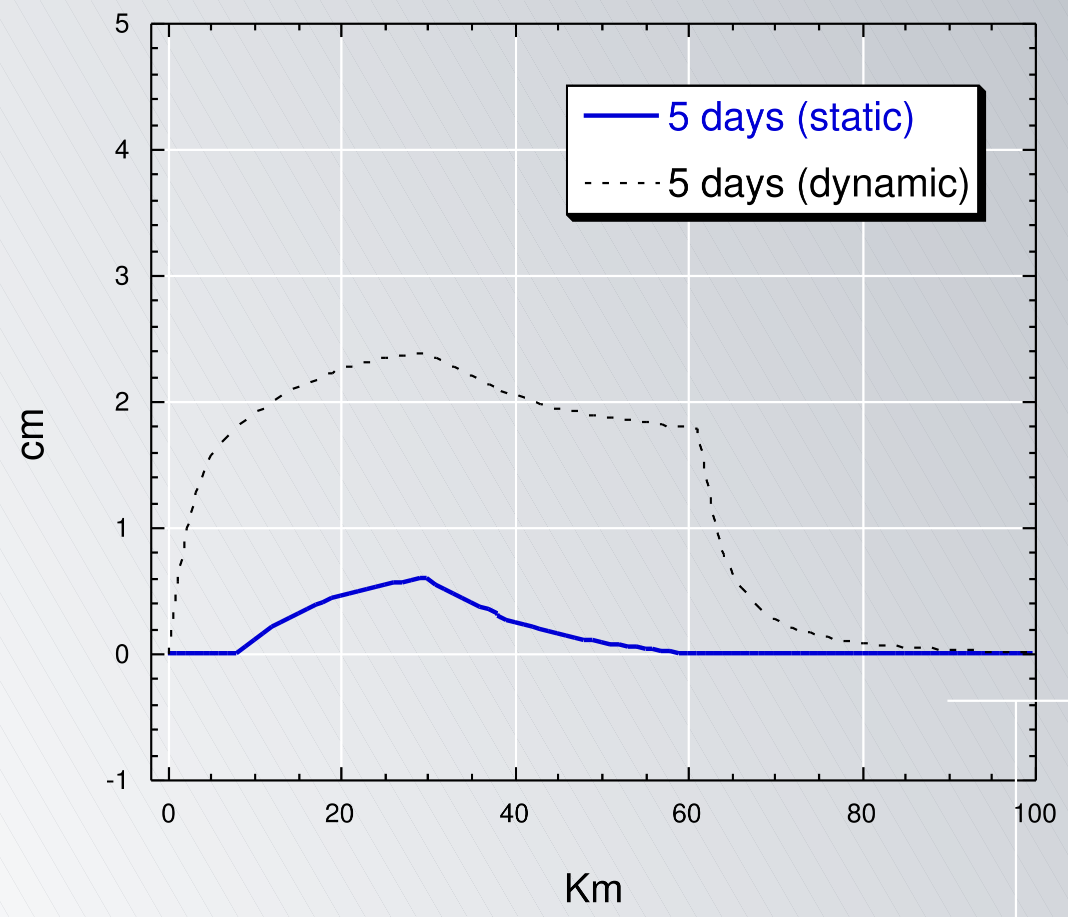
Sedimentation bed profiles

Δ (cm)	t_{cr} ($days$)	x_{cr} (Km)	T_{cr} ($days$)
1.4	0.71	3.57	1.01
1.5	0.97	4.99	1.35
1.6	1.51	7.5	2.08
1.7	3.20	14.2	3.85
1.8	∞	∞	∞
1.9	∞	∞	∞
2.0	∞	∞	∞

Critical times and distances: T_{cr} is the time needed by the system to reach a state in which the thickness of the static layer is above $h_{\mathrm{cr}} = 2R/100$ *over a longitudinal section of the pipe with length* $\approx L/100$

Sedimentation bed profiles

pipe radius $R = 25cm$, discharge $Q = 250\ m^3/h$



Sedimentation bed profiles

pipe radius $R = 25cm$, discharge $Q = 250\ m^3/h$