

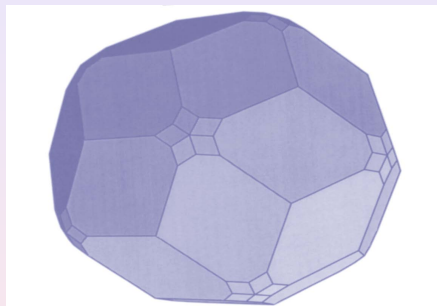
High dimensional sections of centrally symmetric, isotropic convex bodies

Jesús Bastero
bastero@unizar.es

Cortona, June 2007

Introduction

$K \subseteq \mathbb{R}^n$ is a symmetric convex body with volume one



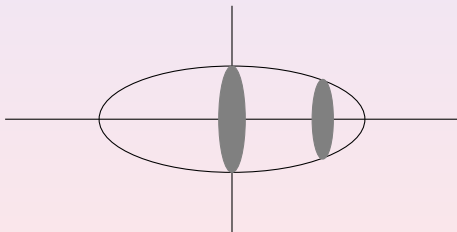
© Richard Gardner

What about the the volume of high dimensional sections?

Brunn-Minkowski theorem and the symmetry imply that

$$|K \cap E|_m \geq |K \cap (y + E)|_m, \quad \forall y \in E^\perp$$

$\dim E = m, 1 \leq m \leq n - 1.$



We say K is *isotropic* if

$$\int_K x_i x_j dx = \delta_{ij} L_K^2$$

for all $1 \leq i, j \leq n$, or equivalently

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 \quad \forall \theta \in S^{n-1}$$

Every centrally symmetric convex body has a unique, up to orthogonal transformation, isotropic position.

A *position* of a convex body is any linear transformation of K , say $T(K)$, with $T \in SL(n)$.

L_K is the isotropy constant of K

It is very easy to prove that

$$L_{D_n} \leq L_K$$

($D_n = B_2^n$ is the euclidean ball) and

$$L_{D_n} \sim C \quad (C > 0)$$

Hyperplane conjecture. Is it true that

$$L_K \leq C?$$

$$L_K \leq Cn^{1/4} \log n \qquad L_K \leq Cn^{1/4}$$

Bourgain 1985

Klartag 2005

Theorem (Hensley, 1979)

Let K be a centrally symmetric, isotropic convex of volume 1 and H_θ the hyperplane orthogonal to the vector $\theta \in S^{n-1}$, then

$$\frac{1}{\sqrt{12} L_K} \leq |H_\theta \cap K|_{n-1} \leq \frac{1}{\sqrt{2} L_K}$$

Hyperplane conjecture. Is it true that

$$C \leq |H_\theta \cap K|_{n-1}$$

for all $\theta \in S^{n-1}$, K isotropic and n ?

Klartag's result implies

$$\frac{C}{n^{1/4}} \leq |H_\theta \cap K|_{n-1}$$

$(n - k)$ -dimensional case

Ball and later Milman and Pajor (1987) extended Hensley's theorem, by using an extension of a result by Busemann and prove that, for every $(n - k)$ -dimensional subspace E of \mathbb{R}^n , $1 \leq k \leq n$,

$$|E \cap K|_{n-k}^{1/k} \sim \frac{L_{\tilde{B}_{k+1}(K, E^\perp)}}{L_K}$$

$B_q(K, E^\perp)$ has the gauge

$$y \in E^\perp \rightarrow |y|^{1+q/(q+1)} \left(\int_{K \cap (y+E)} |\langle x, y \rangle|^q dx \right)^{-1/(q+1)}$$

So,

$$\frac{c_1}{L_K} \leq |E \cap K|_{n-k}^{1/k} \leq \frac{c_2 k^{1/4}}{L_K}$$

whenever K is an isotropic centrally symmetric convex body.

Examples: the cube Q_n

Hadwiger (1972) and Hensley (1979) showed that if $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ then

$$1 \leq |H_\theta \cap Q_n|_{n-1} \leq 5 \quad \forall \theta \in S^{n-1}$$

Hensley also conjectured that

$$|H_\theta \cap Q_n|_{n-1} \leq \sqrt{2} \quad \forall \theta \in S^{n-1}$$

K. Ball did it in 1986 by using Fourier analysis

Left bound is attained by any hyperplane orthogonal to

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \quad 1 \leq i \leq n$$

The upper bound is attained if and only if H contains an $(n - 2)$ -dimensional face of Q_n , for instance, $H = H_\theta$ with

$$\theta = \left(\frac{1}{2}, \frac{1}{2}, 0, \dots, 0 \right)$$

Other codimensions

Let E be a subspace of \mathbb{R}^n , $\dim E = n - k$

$$1 \leq |E \cap Q_n|_{n-k} \quad \text{Vaaler, 1979}$$

Ball in 1989

$$|E \cap Q_n|_{n-k}^{1/k} \leq \sqrt{2}$$

whenever $k < n/2$ and

$$|E \cap Q_n|_{n-k}^{1/k} \leq \left(\frac{n}{n-k} \right)^{(n-k)/2k}$$

whenever $n - k$ divides n .

p -balls

$$B_p^n = \{x \in \mathbb{R}^n; \sum_{i=1}^n |x_i|^p \leq 1\}$$

Meyer and Pajor (1988) proved that

$$\frac{|E \cap B_p^n|_m}{|B_p^m|_m} \geq \frac{|E \cap B_q^n|_m}{|B_q^m|_m}$$

whenever $1 \leq q \leq p \leq \infty$ and E a subspace with $\dim E = m$.

$$|E \cap B_p^n|_m \geq |B_p^m|_m$$

for $2 \leq p \leq \infty$, and

$$|E \cap B_p^n|_m \leq |B_p^m|_m$$

for $1 \leq p \leq 2$ (extended by Caetano and Barthe for $0 < p < 1$)

Minimal central hyperplane section of B_p^n is perpendicular to the main diagonal for $0 < p < 2$: if

$$e = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right)$$

then

$$|H_e \cap B_p^n|_{n-1} \leq |H_\theta \cap B_p^n|_{n-1} \leq |B_p^{n-1}|_{n-1}$$

by Koldobsky in 1998 (Meyer and Pajor proved the case $p = 1$)

Maximal hyperplane sections for $2 < p < \infty$ is open. Oleszkiewicz (2003) proved the answer depends on p and n .

Vaaler conjecture

Vaaler in 1979 conjectured that for every $n \in \mathbb{N}$, and every centrally symmetric convex body of volume one in \mathbb{R}^n and every $m \leq n$ there exists a linear automorphism of \mathbb{R}^n , $T \in SL(n)$ so that for every m -dimensional subspace E of \mathbb{R}^n

$$|T(K) \cap E|_m \geq 1$$

(stronger than the hyperplane conjecture)

For $m = 1$ is true (Ball, using John position and Brascamp-Lieb inequality)

Assume K , symmetric convex body with $|K|_n = 1$ is in John's position

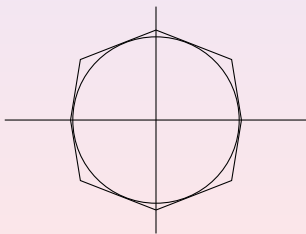
$$rD_n \subset K \quad D_n \subset r^{-1}K$$

for some $r > 0$. By Ball's volume ratio inequality

$$|r^{-1}K|_n \leq |[-1, 1]^n|_n = 2^n$$

so

$$|K \cap [\theta]|_1 \geq 2r \geq 1$$



p -Balls

The conjecture reads

$$|B_p^n \cap E|_m^{1/m} \geq |B_p^n|^{1/n}$$

for every m -dimensional subspace $E \subset \mathbb{R}^n$

The conjecture is true:

- ▶ For $p = \infty$ ($m = 1$ Hadwiger, 1972 and any m Vaaler, 1979),
- ▶ For $p = 1$ or $2 \leq p < \infty$ (Meyer and Pajor, 1988).
- ▶ If $1 < p < 2$ and $m = n - 1$ (Schmuckenslager, B., Galve, Peña, Romance, 2002 and recently Peng, 2007)
- ▶ If $1 < p < 2$ and $1 \leq m \leq (n - 1)/2$ (B., Galve, Peña, Romance, 2002)

Theorem

(D. Alonso, J.B., J. Bernués and G. Paouris)

Let K be a centrally symmetric isotropic convex body in \mathbb{R}^n with $|K|_n = 1$.

i) The set A of subspaces $E \in G_{n,k}$ such that

$$\frac{c_1}{L_K} \leq |E^\perp \cap K|_{n-k}^{1/k} \leq \frac{c_2}{L_K}$$

holds has probability $\mu(A) \geq 1 - c_3 \exp(-c_4 \sqrt{n})$, whenever $k \leq \sqrt{n}$.

ii) Let $0 < \varepsilon < 1$. The set B of subspaces $E \in G_{n,k}$ such that

$$\frac{1 - \varepsilon}{\sqrt{2\pi} L_K} < |E^\perp \cap K|_{n-k}^{1/k} < \frac{1 + \varepsilon}{\sqrt{2\pi} L_K}$$

holds has probability $\mu(B) \geq 1 - c_5 \exp(-n^{0,9})$, whenever $1 \leq k \leq c_6 \varepsilon \log n / (\log \log n)^2$

All the constants are absolute

Hensley (codimension 1)

$$\frac{1}{\sqrt{12} L_K} \leq |H_\theta \cap K| \leq \frac{1}{\sqrt{2} L_K} \quad \frac{1}{\sqrt{2\pi} L_K}$$

For the cube Q_n (codimension 1)

$$1 \leq |H_\theta \cap Q_n|_{n-1} \leq \sqrt{2} \quad \frac{1}{\sqrt{2\pi} L_K} = \sqrt{\frac{6}{\pi}}$$

For the normalized Euclidean ball (any codimension)

$$|E^\perp \cap \widetilde{D}_n|_{n-k}^{1/k} = \omega_n^{1/n} \left(\frac{\omega_{n-k}}{\omega_n} \right)^{1/k} \sim \omega_n^{1/n} \frac{\sqrt{n-k}}{\sqrt{2\pi}} \quad (k \text{ fixed})$$

and

$$\frac{1}{\sqrt{2\pi} L_{\widetilde{D}_n}} = \omega_n^{1/n} \frac{\sqrt{n+2}}{\sqrt{2\pi}}$$

Skecht of the proof of ii)

We use the concentration of measure phenomena on $G_{n,k}$

Proposition

Let $f: G_{n,k} \rightarrow \mathbb{R}$ Lipschitz. There exist absolute constants $c_1, c_2 > 0$ such that for every $a > 0$,

$$\nu_{n,k} \left\{ E; \left| f(E) - \int_{G_{n,k}} f \right| > \varepsilon \right\} \leq c_1 \exp \left(-\frac{c_2 n \varepsilon^2}{\sigma_f^2} \right)$$

where σ_f is the lipschitz constant of f

We apply it to $f(E) = |E^\perp \cap K|_{n-k}$ by computing:

- the lipschitz constant of f
- $\int_{G_{n,k}} f = \mathbb{E} f$

Proposition

For every $E, F \in G_{n,k}$ we have

$$\begin{aligned} \left| |E^\perp \cap K|_{n-k} - |F^\perp \cap K|_{n-k} \right| &\leq \frac{(c\mathcal{L}_k)^{2k}}{L_K^k} \|P_E - P_F\|_{HS} \\ &\leq \frac{c^k k^{k/2}}{L_K^k} \|P_E - P_F\|_{HS} \end{aligned}$$

where $\mathcal{L}_k := \sup\{L_M \mid M \subset \mathbb{R}^k, \text{ convex body isotropic}\}$.

Proof is based in a result by Milman & Pajor

$$E^\perp \ni \theta \rightarrow \|\theta\| := \frac{|\theta|}{|K \cap E(\theta)|}$$

where $E(\theta) = E \oplus \langle \theta \rangle$, is a norm on E^\perp (k -dimensional extension of Busemann's result) and induction by using a suitable orthonormal basis (Giannopoulos & Milman)

Then

$$\begin{aligned}\nu_{n,k} \{E; |f(E) - \mathbb{E}f| > \varepsilon \mathbb{E}f\} &\leq c_1 \exp\left(-\frac{c_2 n \varepsilon^2 (\mathbb{E}f)^2}{\sigma_f^2}\right) \\ &\leq c_1 \exp\left(-\frac{c_2^k \varepsilon^2 n}{k^{k/2}}\right)\end{aligned}$$

since

$$\mathbb{E}f \geq \frac{c_1^k}{L_K^k}$$

by Ball, Milman & Pajor inequality

Computing $\mathbb{E}(f)$

Consider the distribution

$$F_K(t, E) := |\{x \in K : |P_E(x)| \leq t\}|_n, \quad t \geq 0$$

The function $F_K(t, E)$ is the marginal measure (of the uniform measure on K) on E of a t -dilate of the Euclidean unit ball.

$$F_K(t, E) = |S^{k-1}| \int_0^t r^{k-1} \int_{S_E} \left| (r\theta + E^\perp) \cap K \right|_{n-k} d\mu_E(\theta) dr$$

Denote by $\Gamma^k(t)$ the k -dimensional Gaussian measure (centered with variance L_K^2) of $\{s \in \mathbb{R}^k : |s| \leq t\}$.

$$\Gamma^k(t) = \int_{|s| \leq t} \frac{1}{(\sqrt{2\pi}L_K)^k} e^{-|s|^2/2L_K^2} ds, \quad t \geq 0$$

for all $0 \leq t$

J. B. and J. Bernués proved

$$\left| \frac{\int_{G_{n,k}} F_K(t, E) d\mu(E)}{\Gamma_K^k(t)} - 1 \right| \leq \frac{c_1}{n^{0,09}} \quad \forall t > 0$$

whenever $k \leq cn^\gamma$, by assuming the strong **Concentration hypothesis** recently proved by Klartag (2007)

$$\left| \left\{ x \in K : \left| \frac{|x|}{\sqrt{n}L_K} - 1 \right| > t \right\} \right|_n \leq c \exp(-Cn^{0,33}t^{3,33})$$

for $0 < t < 1$.

Since taking limits as $t \rightarrow 0$

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{\int_{G_{n,k}} F_K(t, E) d\mu(E)}{\Gamma_K^k(t)} \\ &= \lim_{t \rightarrow 0} \frac{\int_0^t r^{k-1} \int_{S_E} |(r\theta + E^\perp) \cap K|_{n-k} d\mu_E(\theta) dr}{\int_0^t \frac{r^{k-1}}{(\sqrt{2\pi}L_K)^k} e^{-r^2/2L_K^2} dr} \\ &= \frac{\int_{G_{n,k}} |E^\perp \cap K|_{n-k}}{\frac{1}{(\sqrt{2\pi}L_K)^k}} \end{aligned}$$

we obtain

$$\left| \frac{\mathbb{E}(f)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1 \right| \leq \frac{c_1}{n^{0,09}}$$

whenever $k \leq cn^\gamma$

The hypothesis on k ($\varepsilon < 1$) implies that

$$k \leq cn^\gamma \quad \varepsilon \geq \frac{(\log \log n)^2}{c_1 \log n}$$

$$k^{k/2} \ll c_2 n^{0,1} \quad \frac{c_1}{n^{0,09}} < \frac{\varepsilon}{4}$$

so,

$$\begin{aligned} & \nu_{n,k} \left\{ E \in G_{n,k}; \left| \frac{f(E)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1 \right| \geq \varepsilon \right\} \\ & \leq \nu_{n,k} \left\{ E \in G_{n,k}; |f(E) - \mathbb{E}(f)| \geq \frac{\varepsilon}{2} \mathbb{E}f \right\} \\ & \leq c_1 \exp\left(-\frac{c_2^k \varepsilon^2 n}{k^{k/2}}\right) \leq c_1 \exp(-c_2 n^{0,9}) \end{aligned}$$

Therefore

$$\frac{1 - \epsilon}{\sqrt{2\pi L_K}} < |E^\perp \cap K|_{n-k}^{1/k} < \frac{1 + \epsilon}{\sqrt{2\pi L_K}}$$

holds with probability greater than

$$1 - c_1 \exp(-c_2 n^{0.9})$$