Minkowski addition of functions and quasi-concavity of solutions to elliptic equations

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work in progress
in collaboration with
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def. A function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is called **quasi-concave** if it has convex super-level sets; i.e. the set

$$\Omega_t = \{x \in \mathbb{R}^n \ s.t. \ u(x) \geq t\},$$

is convex for every $t \in \mathbb{R}$.

- This is equivalent to

$$u\Big((1-\lambda)x_0 + \lambda x_1\Big) \geq \min\{u(x_0), u(x_1)\},$$

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The problem

Here we study the Dirichlet problem

$$\begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_0 \\ u = 1 & \text{on } \partial\Omega_1 \,, \end{cases} \tag{1.1}$$

where $\Omega = \Omega_0 \backslash \overline{\Omega}_1$ is a convex ring of $\mathbb{R}^n$ and F is a

- proper
  i.e. $F(x,u,\vec{q},A) \geq F(x,v,\vec{q},A)$ whenever $u \leq v$,
- continuous and
- (degenerate) elliptic
  i.e. $F(x,u,\vec{q},A) \geq F(x,u,\vec{q},B)$ whenever $A \geq B$

operator.

Problem: since the level sets of u of value 0 and 1 are both convex, we look for assumptions on the operator F such that this property is preserved by every level sets.

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Main ingredients

The main tools of our result are:

- a generalization of a technique used in [CS],

  which is based on the quasi-concave envelope u^* of a function u;

- some formulae recently proved in [CT],[LS],

  regarding the gradient and the Hessian matrix of the Minkowski addition u_λ of quasi-concave functions.

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Quasi-concave envelope of a function u

Given a function $u : \Omega \to \mathbb{R}$,

def. **$\mathbf{u}^*$** is the function whose superlevel sets are the convex hulls of the corresponding superlevel sets of u.

i.e.

$$\{u^* \geq t\} := \Omega_t^* = \mathrm{Conv}\big(\{u \geq t\}\big) \ \forall t.$$

- A function u is quasi-concave if and only if $u = u^*$.

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Minkowski linear combination of functions

Let $u_i : \Omega_i \to \mathbb{R}$ $i = 1, ..., m$,
$\lambda = (\lambda_1, ..., \lambda_m)$ s.t. $\lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1$.

def. $\mathbf{u}_\lambda$ is the function whose super-level sets Ω_t^λ are the Minkowski linear combination of the corresponding super-level sets Ω_t^i of u_i:

$$\Omega_t^\lambda = \sum_{i=1}^m \lambda_i \Omega_t^i, \quad \text{for every } t \in \mathbb{R}.$$

- If $u_1, ..., u_m$ are quasi-concave functions then u_λ is quasi-concave.

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u^* and u_λ

Notice that, taking $u_i = u, \ i = 1, \ldots, m$ we have:

- $$\Omega_t^* = \bigcup_\lambda \Omega_t^\lambda,$$
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The strategy for the problem (1.1)

To show that the solution u is quasi-concave,

we look for conditions that imply $\mathbf{u = u^*}$.

Notice that

- $u^* \geq u$ holds by definition,
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Can we find suitable assumptions on F that force u^* to be a viscosity subsolution of (1.1)?

Some definitions: viscosity solutions

def. Given a function u we say that $\varphi \in C^2$ **touches u from above** (**below**) at a point $\bar{x}$ if

- $\varphi(\bar{x}) = u(\bar{x})$ and,
- $\varphi - u$ has a local minimum (maximum) at $\bar{x}$.

def. A function v is a **viscosity subsolution** (**supersolution**) of $F(x, u, Du, D^2u) = 0$ if

- for every function $\varphi \in C^2$ that touches v from above (below) at x one has

$$F(x, v(x), D\varphi(x), D^2\varphi(x)) \geq (\leq)\ 0,$$

for every $x \in \Omega$.

Some definitions: comparison principle

def. An operator F satisfies a **comparison principle** if, given $u, v : \Omega \to \mathbb{R}$ viscosity subsolution and supersolution respectively,

$$u \le v \text{ on } \mathrm{bd}(\Omega) \quad \text{implies} \quad u \le v \text{ in } \bar{\Omega}.$$

Main theorem

Theorem

Let $\Omega = \Omega_0 \backslash \overline{\Omega}_1$ *be a convex ring and let* $F(x, u, p, A)$ *be a*

- *proper*
- *continuous*
- *elliptic*

operator.

Assume that $G_{t,\theta} : (x, p, A) \to F\left(x, t, \frac{\theta}{p}, \frac{A}{p^3}\right)$ *is quasi-concave for every fixed* $t \in [0, 1], \theta \in S^{n-1}$.

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *is a classical solution of (1.1) such that* $\|Du\| > 0$ *in* Ω, *then*

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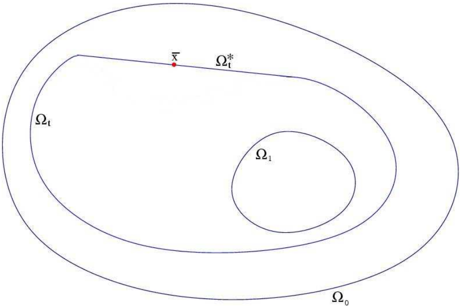
Sketch of the proof

We want to prove that: for every $x \in \Omega$, for every C^2 function φ that touches u^* from above at x, then

$$F(x, u^*(x), D\varphi(x), D^2\varphi(x)) \geq 0,$$

holds.

Let $\bar{x} \in \Omega,\ t = u^*(\bar{x})$



It is known that: $\exists\ x_i \in \mathrm{bd}(\Omega_t),\ i = 1, \dots, n$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ with $0 \le \lambda_i \le 1,\ \sum_{i=1}^{n} \lambda_i = 1$ such that

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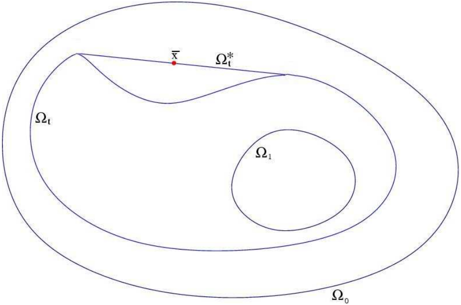

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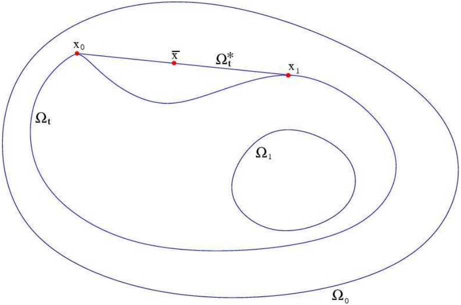

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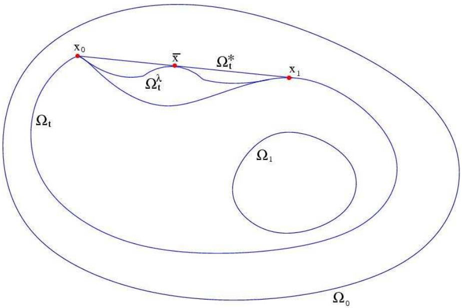

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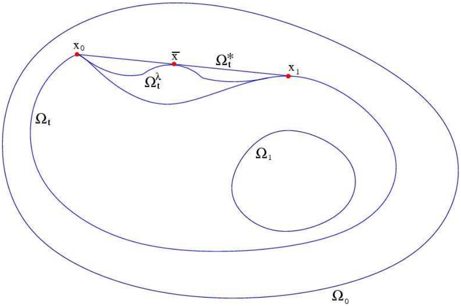

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Then it holds:

$$\begin{aligned}
&u^*(\bar{x}) = u_\lambda(\bar{x}) = u(x_i) = t, && i = 1, ..., n,\\
&u^*(x) \geq u_\lambda(x) \geq u(x) && \text{for every } x \in \Omega,\\
&\frac{Du_\lambda(\bar{x})}{\|Du_\lambda(\bar{x})\|} = \frac{Du(x_i)}{\|Du(x_i)\|} = \theta, && i = 1, ..., n.
\end{aligned}$$

Notice that if φ touches u^* from above at $\bar{x}$, then φ touches u_λ from above at $\bar{x}$; this implies that

$$\begin{aligned}
\varphi(\bar{x}) &= u^*(\bar{x}) = u_\lambda(\bar{x}) = t,\\
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By [CT],[LS] we have:

- $$\frac{1}{\|Du_\lambda(\bar{x})\|} = \sum_{i=1}^{n} \lambda_i \frac{1}{\|Du(x_i)\|},$$
- $$\frac{D^2 u_\lambda(\bar{x})}{\|Du_\lambda(\bar{x})\|^3} \geq \sum_{i=1}^{n} \lambda_i \frac{D^2 u(x_i)}{\|Du(x_i)\|^3}.$$

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&F(\bar{x}, t, D\varphi(\bar{x}), D^2\varphi(\bar{x})) \\
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&\quad \geq F\left(\bar{x}, t, \frac{\theta}{\sum_{i=1}^n \frac{\lambda_i}{\|u(x_i)\|}}, \Big(\sum_{i=1}^n \frac{\lambda_i}{\|Du(x_i)\|}\Big)^{-3}\Big(\sum_{i=1}^n \frac{D^2u(x_i)}{\|Du(x_i)\|^3}\Big)\right) \\
&\quad = G_{t,\theta}\left(\sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i \frac{1}{\|Du(x_i)\|}, \sum_{i=1}^n \lambda_i D^2u(x_i)\right) \\
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This shows that u^* is a viscosity subsolution of the problem (1.1).

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Examples

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