SYMMETRIES ARISING FROM X-RAYS

Paolo Dulio
Politecnico di Milano

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The Radon Transform

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For applications, a finite number of X-rays must be employed, and consequently a tomograph works with discrete approximations of the inversion formula.

From CAT to Geometric Tomography.

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Uniqueness Tomographic Problem

Which is the minimum number of radiographies (with parallel or point X-rays) needed to distinguish a set $I \in \mathcal{F}$ from any other member of $\mathcal{F}$?

Switching Components.

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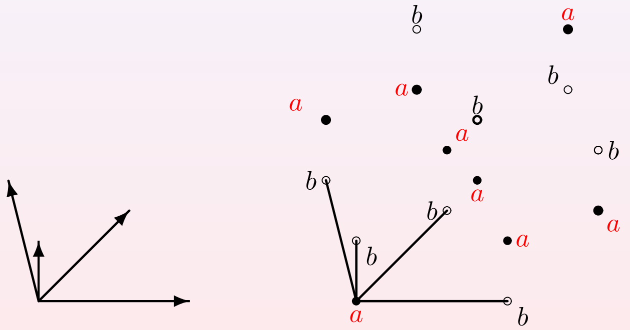
Definition

A union $P = A \cup B$ of disjoint nonempty sets $A, B \subset \mathbb{Z}^2$ is called a U-switching component if $X_u A = X_u B$, for all $u \in U$.

Switching Components.



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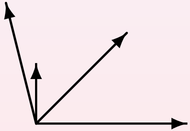

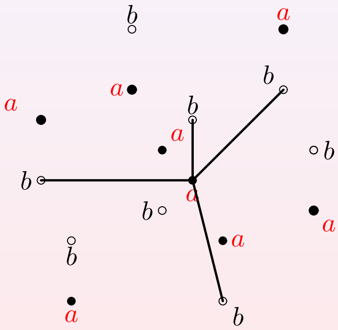

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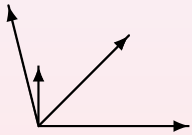

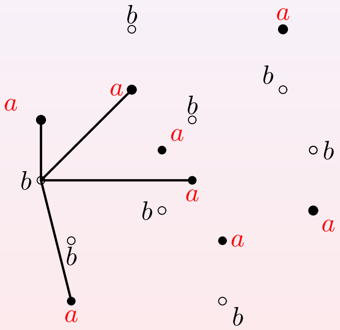

Switching Components.

Problem (R. Gardner)

Characterize the switching components.

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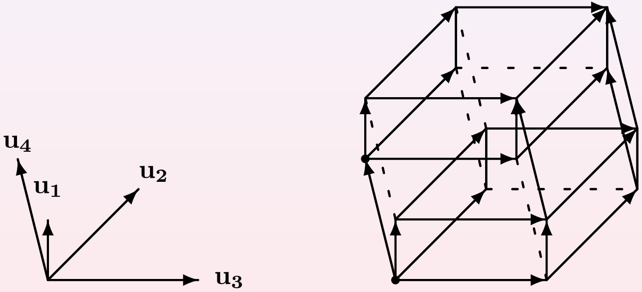
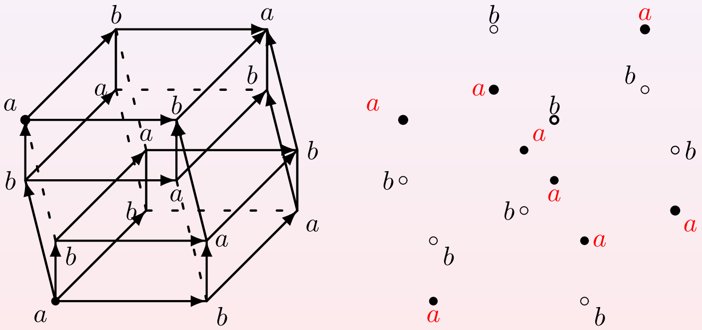
Characterize the switching components.

Theorem (L. Hajdu - R. Tijdeman, 2001)

A switching component is the linear combination of switching elements.

Switching Components.

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The H-T Decomposition.

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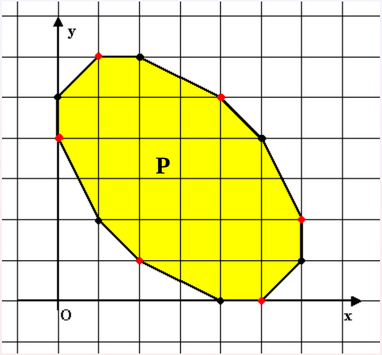
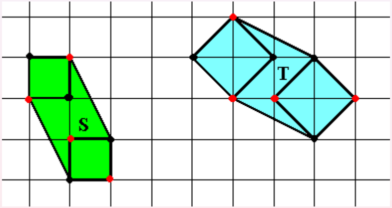
Note that S and T could be non-convex even if P is convex.

The H-T Decomposition for a U-Polygon.





$$P : p(x,y) = x^4 - x^2y + xy^2 - y^4 + y^5 - xy^6 + \\ + x^2y^6 - x^4y^5 + x^5y^4 - x^6y^2 + x^6y - x^5 =$$

$$= \underbrace{(x-1)(y-1)(y^2-x)}_{\mathbf{S}}\underbrace{(y-x)(y-x^2)(xy-1)}_{\mathbf{T}}$$

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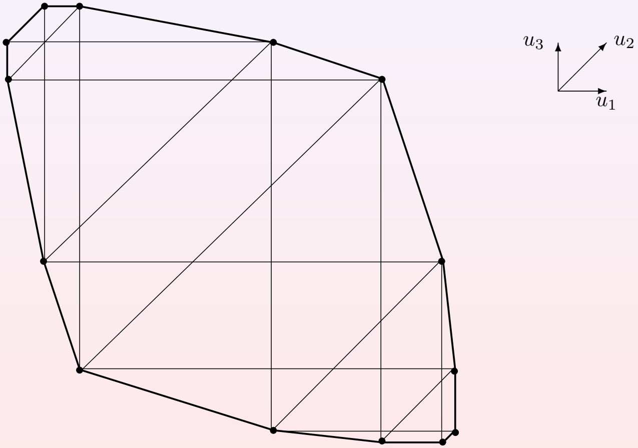
Definition

A non-degenerate convex polygon P is a (weakly) U-polygon if for each vertex $v \in P$ and $u \in U$ the line through v parallel to u meets a vertex v' of P (possibly $v' = v$).

Radiographies of Convex Bodies.



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Geometric Features of *U*-Polygons.

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- Associate a group structure to a *U*-polygon.

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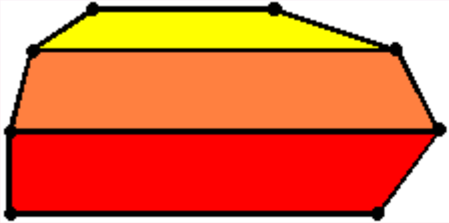

Geometric Features of U-Polygons.

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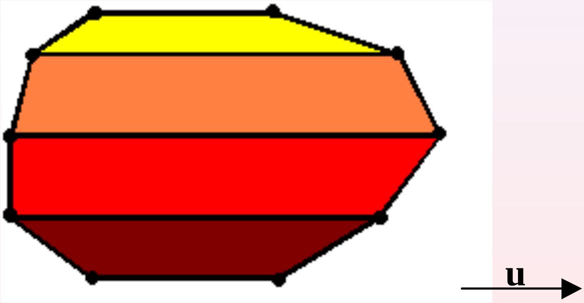

Definitions

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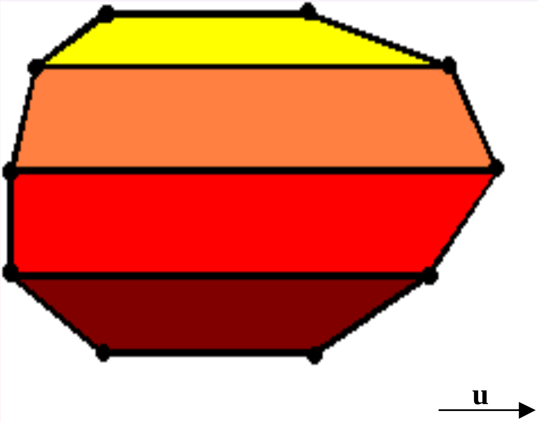
The generalized skew-reflection associated to the pair $(\mathbf{u}, m(\mathbf{u}))$ is defined to be the map switching the endpoints of any segment in direction $\mathbf{u}$ and centered on the mid-line.

Geometric Features of U-Polygons.



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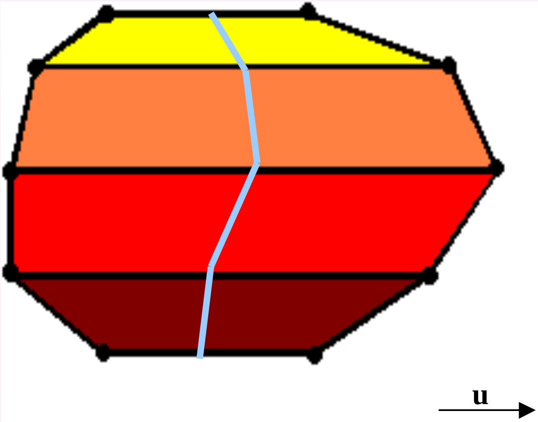

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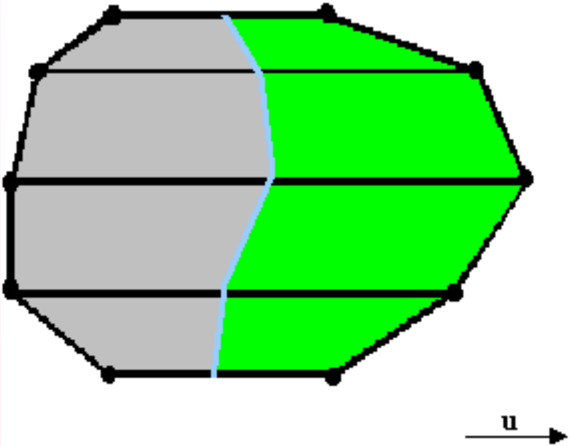
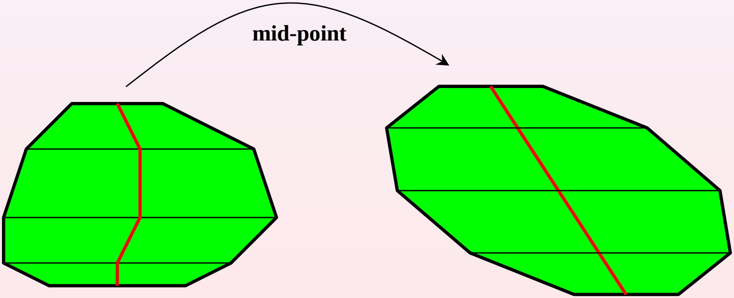


Remark. Note that, via the mid-point construction, the generalized skew-reflection generated by $(\mathbf{u}, m(\mathbf{u}))$ becomes a pair of conjugate skew-reflections.



Decomposition Theorem for U-Polygons.

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Theorem

Let U be a finite set of directions, with $|U| \geq 2$, and let P be a U-polygon. Then $P = P_e \oplus P_o$, where P_e and P_o are convex polygons formed by the even and by the odd edges of P, respectively.

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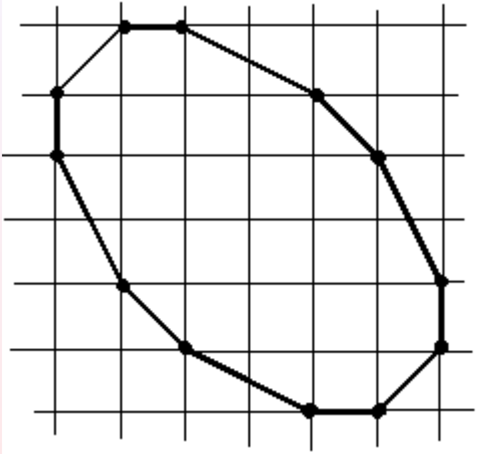
Moreover, one of the following holds

1. U is the set of the directions of edges all belonging to P_e or all belonging to P_o.
2. P_e is a (possibly weakly) V-polygon and P_o is a (possibly weakly) W-polygon , with $U = V \cup W$.

Decomposition Theorem for *U*-Polygons.



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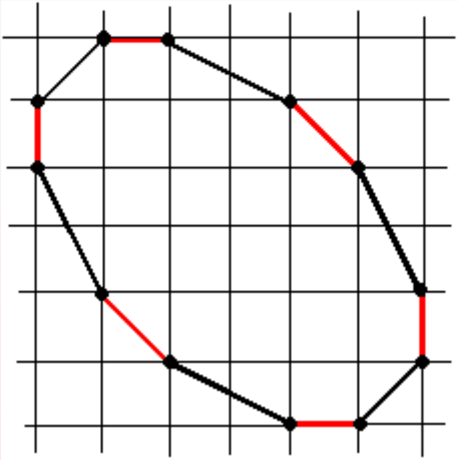

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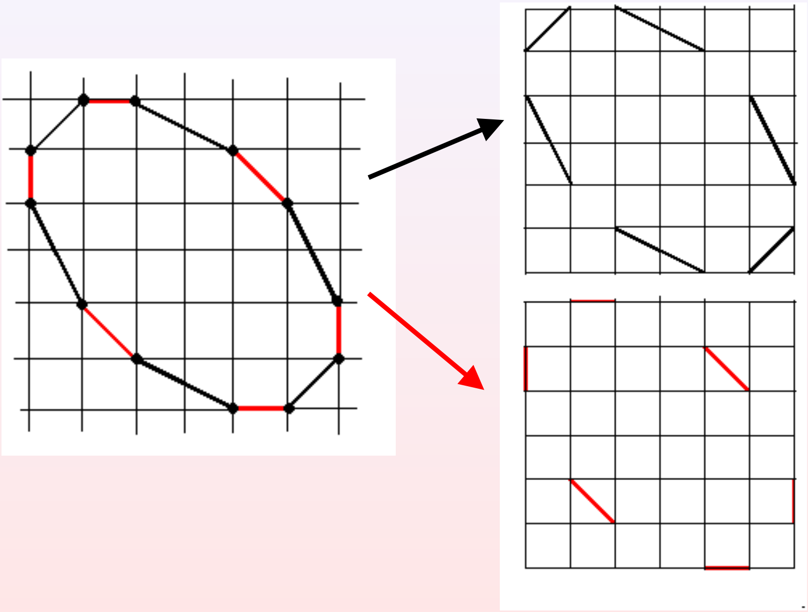

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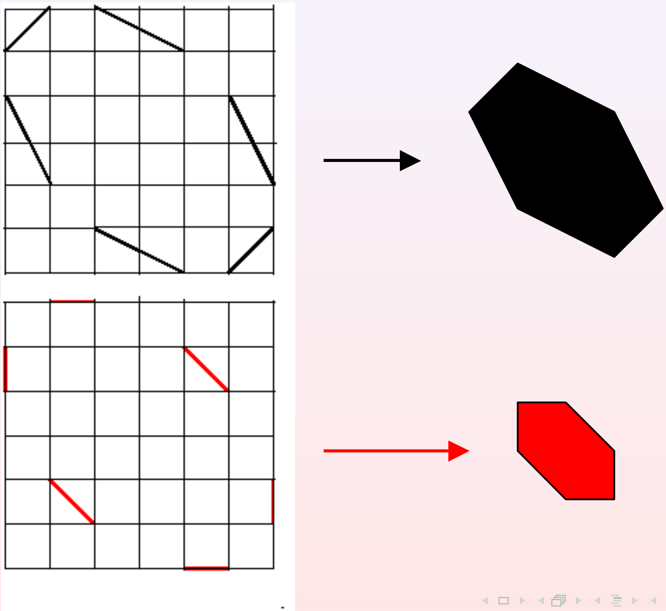

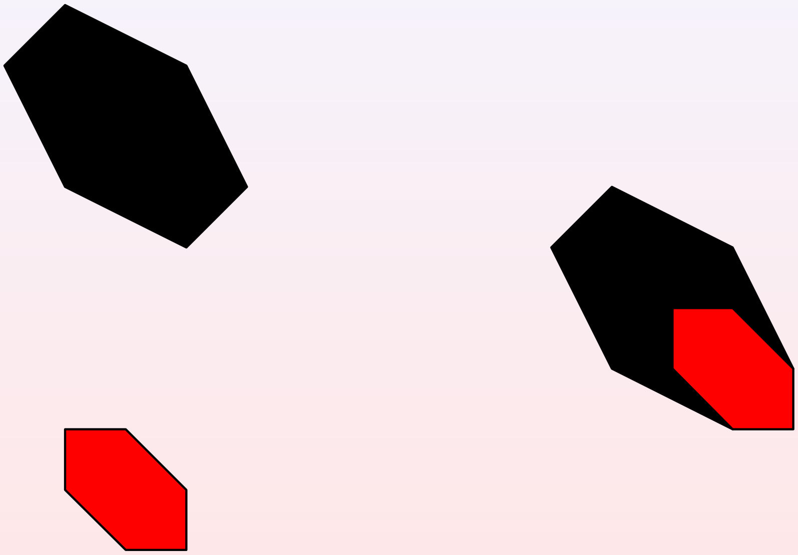

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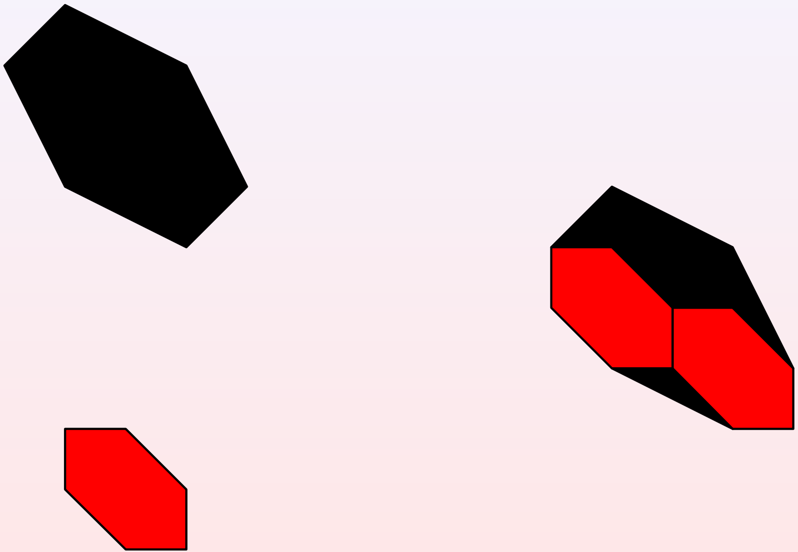

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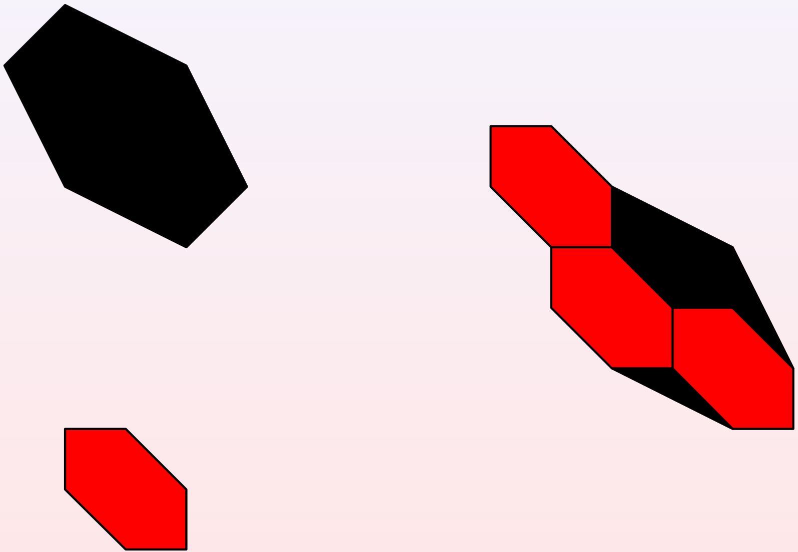

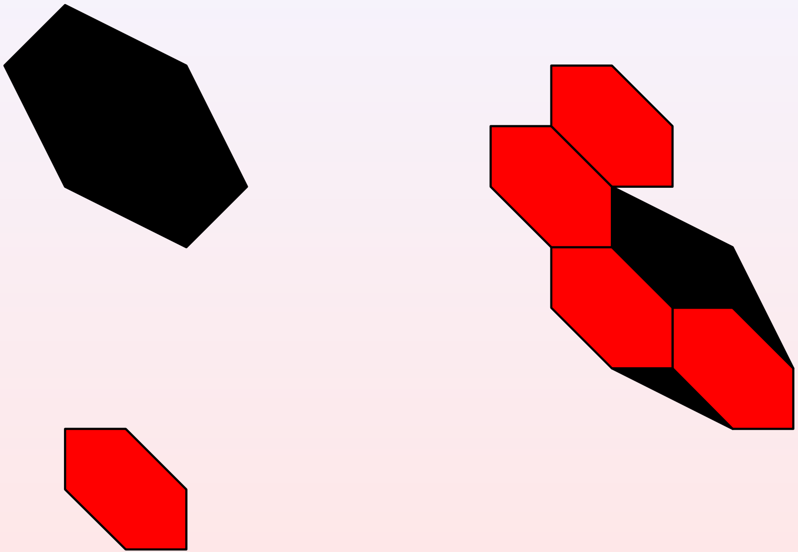

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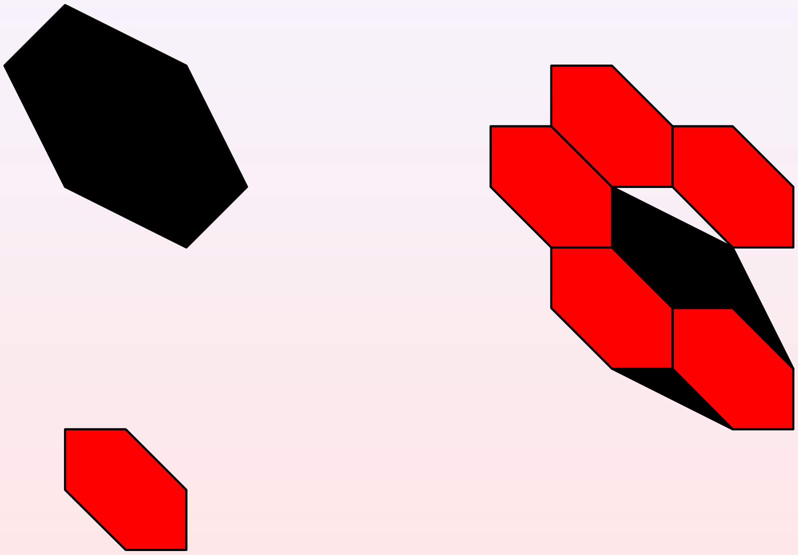

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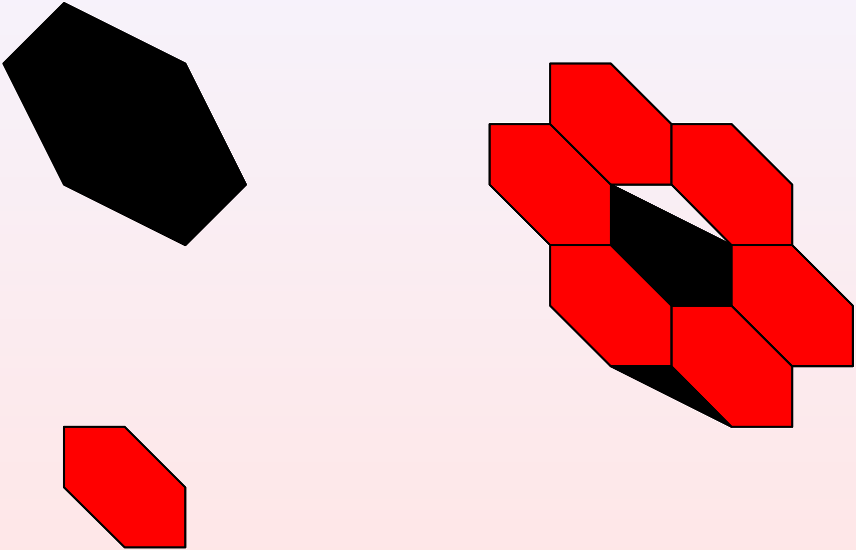

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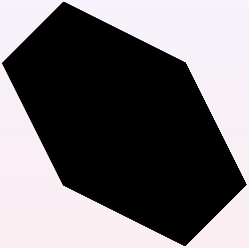

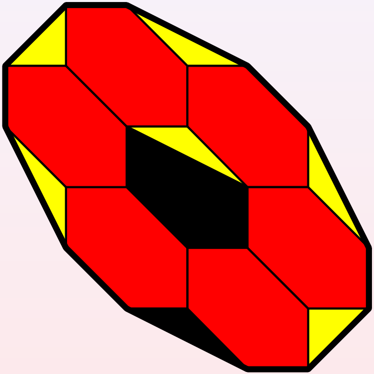

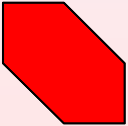

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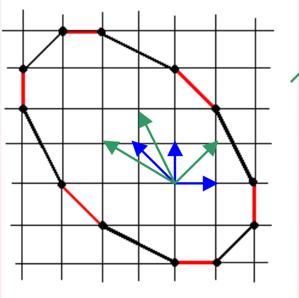

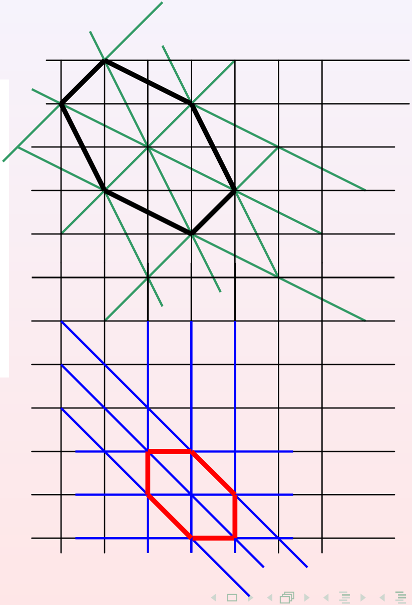

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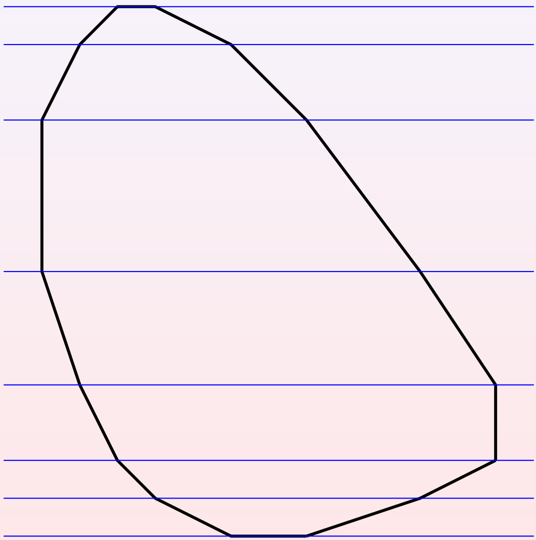

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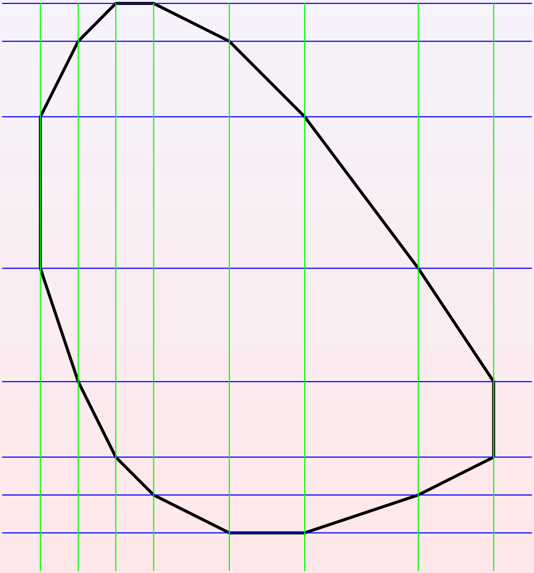

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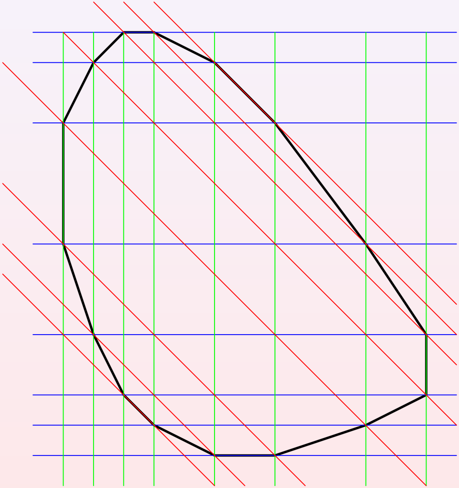

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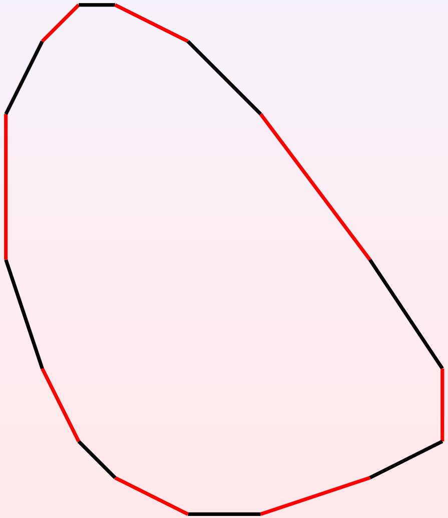

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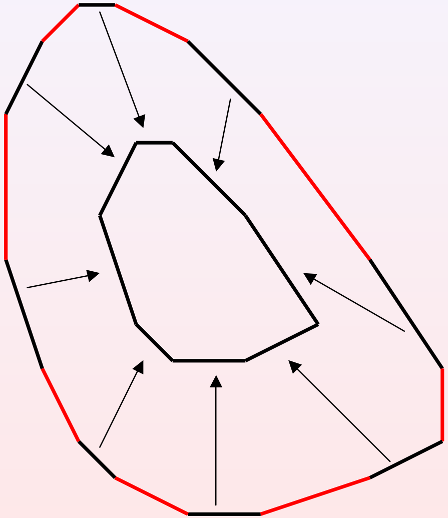

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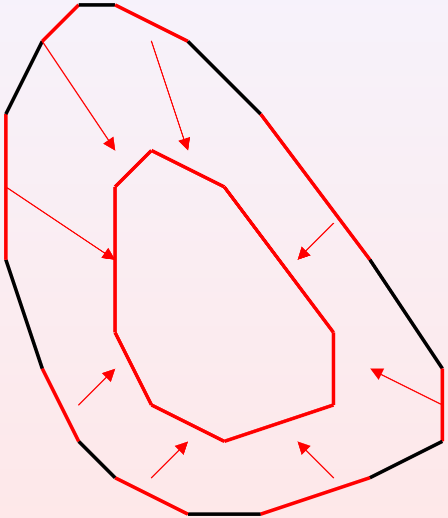

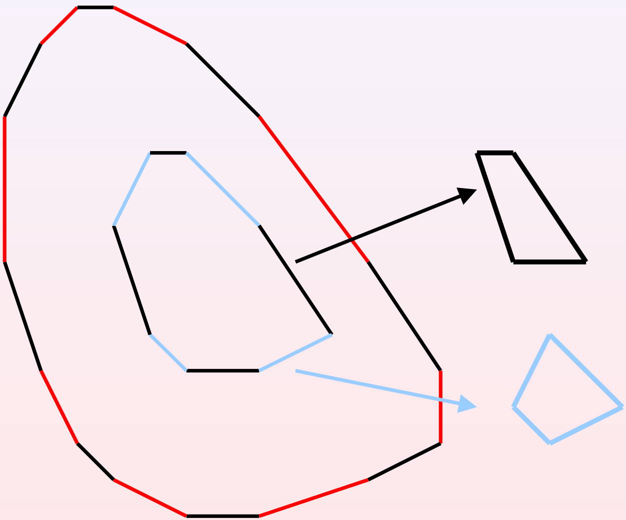

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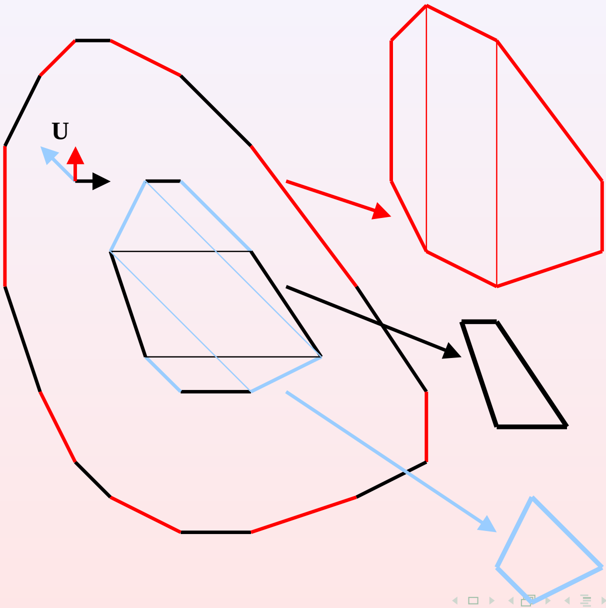

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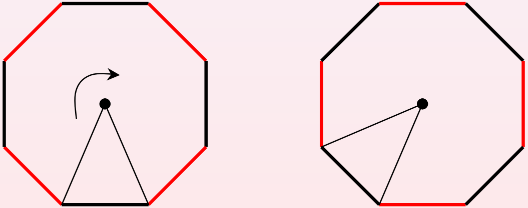
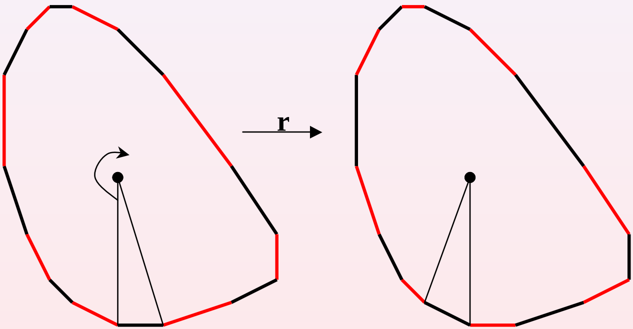
Note that, when a rotation of $\frac{2\pi}{n}$ is performed on a regular n-gon, its even and odd edges exchange.



Remark. Note that r keeps P invariant and fixes its centroid.



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As $|U|$ increases, P could be further decomposed by successive application of the Decomposition Theorem.

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This means more generalized skew-reflections and more generalized rotations, and consequently more symmetry for P.

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Theorem

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Group Structure for U-polygons

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The above property means that, under the assumption, P has a skew-reflection.

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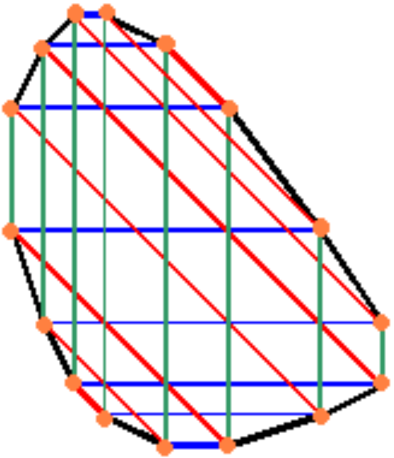
Group Structure for *U*-polygons



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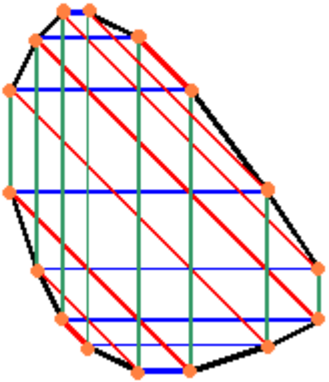

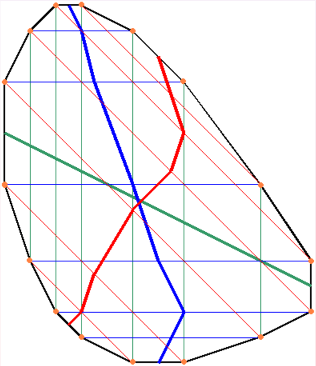

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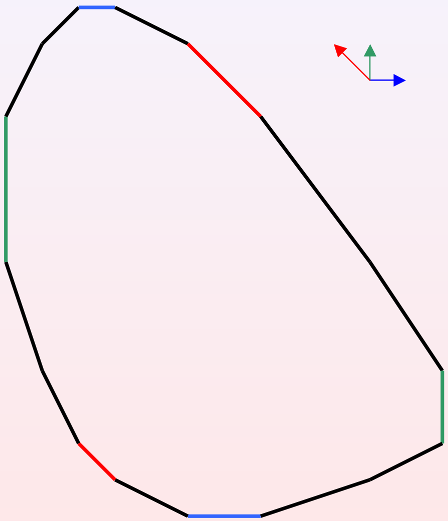
$(w(\mathbf{l}), m(\mathbf{l})) \longrightarrow$ (further extension of generalized skew-reflection).

Group Structure for *U*-polygons



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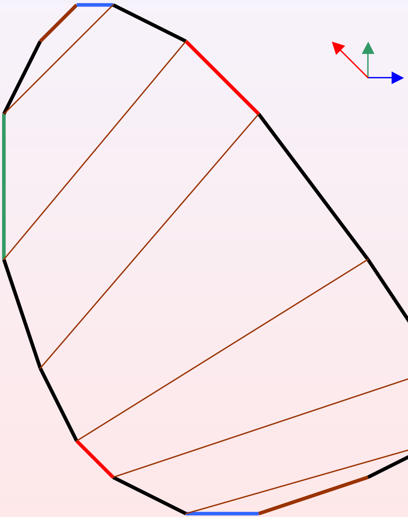

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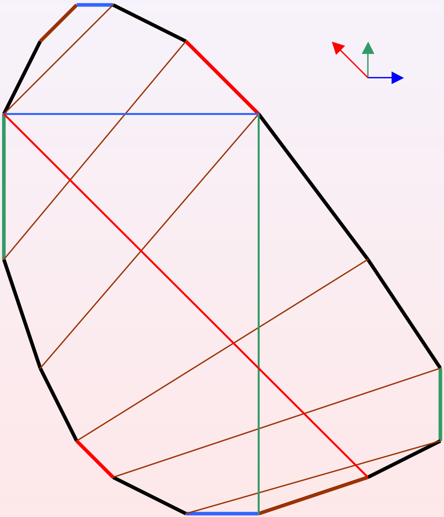

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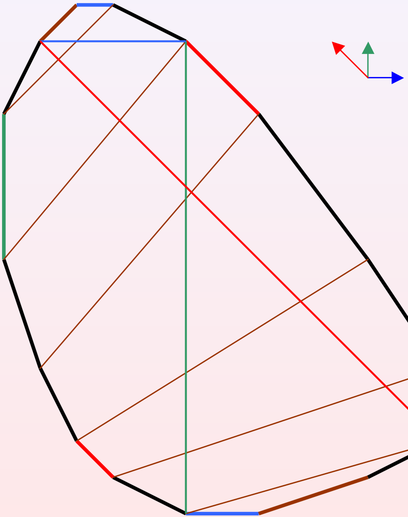

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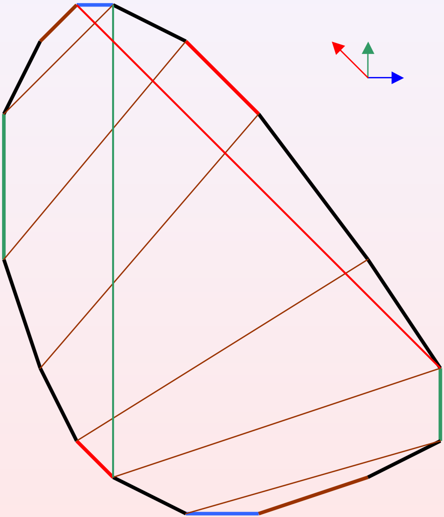

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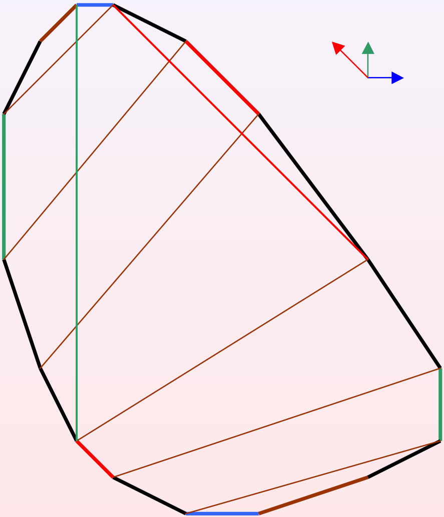

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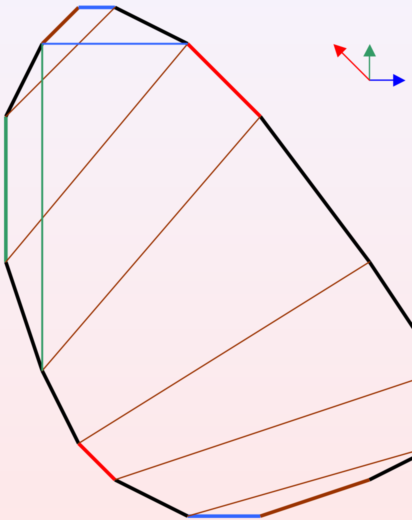

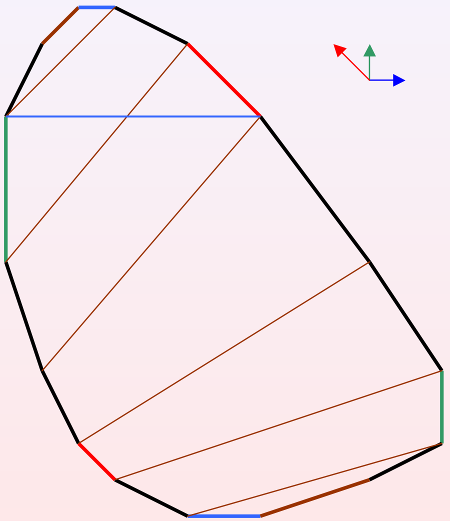

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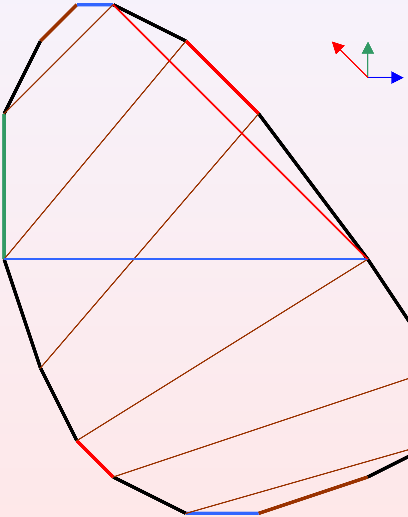

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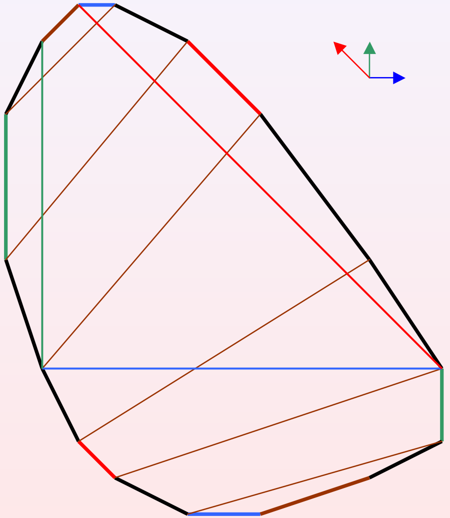

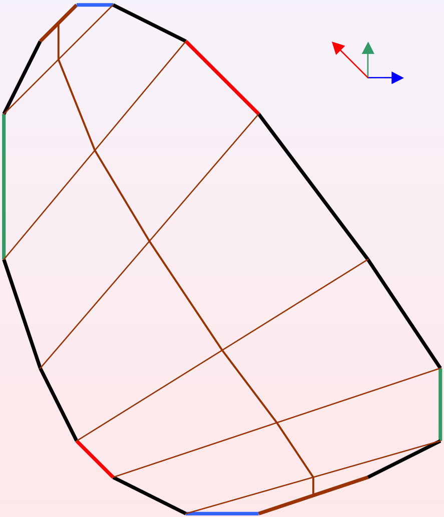

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Generalized skew-reflections $(w(\mathbf{l}), m(\mathbf{l}))$, and generalized rotations allow a group interpretation of a U-polygon, which makes it closer to the dihedral group.

How to go on.

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• The generalized skew-reflections $(\mathbf{u}, m(\mathbf{u}))$ provide for P the same property kept by regular, or affinely regular polygons, to be decomposed in chambers which are, pairwise, of the same area.

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• Extension to U-polygons of the Weyl group associated to a root system Φ. In the case of U-polygons, the root system Φ should be the set of words $w(\mathbf{l})$ for each edge of P.

- A simple system Δ should be identified with U. Consequently, we have a group of generalized skew-reflections $W = \langle U \rangle$, and the pair (W, U) represents a Coxeter system associate to the U-polygon.

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• If P is a lattice U-polygon, then the set U of generators must be the set of generators of a group isomorphic to a crystallographic group