Star body valued valuations

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Valuations

Valuation

A function Z defined on a family $\mathcal{L}$ of subsets of $\mathbb{R}^n$ with values in an abelian semigroup is called valuation if

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}(K) + \mathrm{Z}(L)$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{L}$.

Hadwiger's theorem

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If $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a valuation which is continuous and invariant under rigid motions, then there exist real constants $c_0, \ldots, c_n$ such that

$$\mathrm{Z}K = c_0 V_0(K) + c_1 V_1(K) + \cdots + c_n V_n(K)$$

for $K \in \mathcal{K}^n$.

(Alesker, Fu, Groemer, Hug, Klain, Kiderlen, Ludwig, McMullen, Reitzner, Schneider, Schuster, Weil,...)

Co- and contravariance

Covariance

Let $G \subset \mathrm{GL}(n)$. An operator Z defined on $\mathcal{L}$ having subsets of $\mathbb{R}^n$ as values is called G covariant if there exists a $q \in \mathbb{R}$ such that

$$\mathrm{Z}\phi K = |\det \phi|^q \phi \mathrm{Z} K, \quad \text{for } \phi \in G,\ K \in \mathcal{L}.$$

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Contravariance

Let $G \subset \mathrm{GL}(n)$. An operator Z defined on $\mathcal{L}$ having subsets of $\mathbb{R}^n$ as values is called G contravariant if there exists a $q \in \mathbb{R}$ such that

$$\mathrm{Z}\phi K = |\det \phi|^q \phi^{-t} \mathrm{Z} K, \quad \text{for } \phi \in G,\ K \in \mathcal{L}.$$

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Centroid body

The centroid body ΓK of a convex body K is defined by

$$h(\Gamma K, u) = \frac{1}{V(K)} \int_K |x \cdot u| \, dx$$

for $u \in S^{n-1}$. (Petty, 1961)

L_p centroid body

For $p \geq 1$, the L_p centroid body $\Gamma_p K$ of a convex body K is defined by

$$h(\Gamma_p K, u)^p = \frac{1}{V(K)} \int_K |x \cdot u|^p \, dx$$

for $u \in S^{n-1}$. (Lutwak and Zhang, 1997)

L_p Minkowski addition

For $p \geq 1$, the L_p Minkowski sum $K +_p L$ of K, $L \in \mathcal{K}_0^n$ is defined by

$$h^p(K +_p L, u) = h^p(K, u) + h^p(L, u)$$

for $u \in S^{n-1}$.

Theorem (Ludwig, 2005)

Let $n \geq 3$, $p \geq 1$ and $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}_0^n, +_p \rangle$ be a nontrivial $\mathrm{GL}(n)^+$ covariant valuation with centrally symmetric images. Then exists a positive constant c such that

$$\mathrm{Z}P = c\, V(P)^{1/p} \Gamma_p P$$

for all polytopes P.

L_p Minkowski valuations

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Remarks:

- For nonsymmetric images, the only example is a slight modification Γ_p^τ of Γ_p.
- In dimension 2 other examples of such valuations occur.

Polar L_p centroid body

For $p \geq 1$, the polar L_p centroid body $\Gamma_p^* K$ has radial function

$$\rho(\Gamma_p^* K, u)^{-p} = \frac{1}{V(K)} \int_K |x \cdot u|^p \, dx$$

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$\Gamma_\infty^* K$

For centrally symmetric bodies we have

$$\Gamma_p^* K \longrightarrow K^*$$

as $p \longrightarrow \infty$.

Polar L_p centroid bodies for $p > -1$, $p \neq 0$

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for $u \in S^{n-1}$.

(Campi, Gardner, Giannopoulos, Grinberg, Gronchi, Kalton, Koldobsky, Ludwig, Lutwak, Milman, Pajor, Paouris, Rubin, Yang, Yaskin, Yaskina, Zhang,...)

L_p intersection bodies

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The nonsymmetric L_p intersection body I_p^+

$$\rho(\mathrm{I}_p^+ K, u)^p = \int_{K \cap u^+} (x \cdot u)^{-p} \, dx$$

The intersection body $\mathrm{I}K$ of a star body $K \in \mathcal{S}^n$ is the star body with radial function

$$\rho(\mathrm{I}K, u) = \mathrm{vol}(K \cap u^{\perp}), \quad u \in S^{n-1}.$$

(Lutwak)

The intersection body and its L_p analogue

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Theorem (H., 2007)

For $K \in \mathcal{K}^n$ with $o \in \mathrm{int}K$ and $p \nearrow 1$

$$(\Gamma(1-p))^{-1/p}\mathrm{I}_p^+ K \to \mathrm{I}K$$

with respect to radial metric.

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Question

Can we characterize L_p intersection bodies for all values of p?

L_p radial addition

For $p \neq 0$, the L_p radial sum $K \tilde{+}_p L$ of two star bodies $K, L \in \mathcal{S}^n$ is defined as

$$\rho(K \tilde{+}_p L, u)^p = \rho(K, u)^p + \rho(L, u)^p$$

for $u \in S^{n-1}$.

Theorem (H., 2007)

Let $n \geq 3$, $p \neq 0$ and $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a $\mathrm{GL}(n)^+$ co- or contravariant valuation. For $p < 1$, the only nontrivial examples are

$$\mathrm{Z}(P) = c_1 \mathrm{I}_p^+(P) \,\tilde{+}_p\, c_2 \mathrm{I}_p^+(-P)$$

with positive constants c_1, c_2. For $p \geq 1$ there exist only trivial examples.

A characterization of L_p intersection bodies

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Let $n \geq 3$, $p \neq 0$ and $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a $\mathrm{GL}(n)^+$ co- or contravariant valuation. For $p < 1$, the only nontrivial examples are

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Remark:

- For dimension 2, all examples of such valuations are rotations of the above one.